

GEOMETRY OF FREE LOCI AND FACTORIZATION OF NONCOMMUTATIVE POLYNOMIALS

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ABSTRACT. The free singularity locus of a noncommutative polynomial f is defined to be the sequence of hypersurfaces $\mathcal{L}_n(f) = \{X \in M_n(\mathbb{k})^g : \det f(X) = 0\}$. The main theorem of this article shows that f is irreducible if and only if $\mathcal{L}_n(f)$ is eventually irreducible. A key step in the proof is an irreducibility result for linear pencils. Arising from this is a free singularity locus Nullstellensatz for noncommutative polynomials. Apart from consequences to factorization in a free algebra, the paper also discusses its applications to invariant subspaces in perturbation theory and linear matrix inequalities in real algebraic geometry.

1. INTRODUCTION

Algebraic sets, as zero sets of commutative polynomials are called, are basic objects in algebraic geometry and commutative algebra. One of the most fundamental results is Hilbert’s Nullstellensatz, describing polynomials vanishing on an algebraic set. A simple special case of it is the following: if a polynomial h vanishes on a hypersurface given as the zero set of an irreducible polynomial f , then f divides h . Various far-reaching noncommutative versions of algebraic sets and corresponding Nullstellensätze have been introduced and studied by several authors [Ami57, vOV81, RV07, Scm09]. Heavily reliant on these ideas and results are emerging areas of free real algebraic geometry [dOHMP09, HKN14] and free analysis [MS13, K-VV14, AM16, KŠ17]. In the free context there are several natural choices for the “zero set” of a noncommutative polynomial f . For instance, Amitsur proved a Nullstellensatz for the set of tuples of matrices X satisfying $f(X) = 0$ [Ami57], and a conclusion for pairs (X, v) of matrix tuples X and nonzero vectors v such that $f(X)v = 0$ was given by Bergman [HM04]. In contrast with the successes in the preceding two setups, a Nullstellensatz-type analysis for the set of matrix tuples X making $f(X)$ singular (not invertible), which we call the *free singularity locus* of f (free locus for short), is much less developed. In this paper we rectify this. One of our main results

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connects free loci with factorization in free algebra [Coh06, BS15, ARJ15, BHL17, Scr] in the sense of the special case of Hilbert’s Nullstellensatz mentioned above.

A key intermediate step for studying components of free loci is an irreducibility theorem for determinants of linear pencils. To adapt this result for noncommutative polynomials we use a linearization process: given a noncommutative polynomial f , one can “decouple” products of variables using Schur complements to produce a linear matrix pencil L with the same free locus as f . The most effective way of obtaining such an L is via realizations originating in control theory [BGM05] and automata theory [BR11].

The last objective of this paper is to derive consequences of our irreducibility theorem for hermitian pencils. They appear prominently across real algebraic geometry, see e.g. determinantal representations [Brä11, KPV15], the solution of the Lax conjecture [HV07, LPR04] and the solution of the Kadison-Singer paving conjecture [MSS15]. Furthermore, hermitian pencils give rise to linear matrix inequalities (LMIs), the cornerstone of systems engineering [SIG97] and semidefinite optimization [WSV12], and a principal research focus in free convexity [EW97, BPT13, HKM13, DD-OSS17]. Our irreducibility theorem enables us to analyze of the boundary of a free spectrahedron, also known as an LMI domain, associated to a hermitian pencil L . We characterize the smooth points as those X which make $\ker L(X)$ one-dimensional, and then prove the density of these points in the LMI boundary under mild hypotheses.

This paper is of possible interest to functional analysts (especially in free analysis), matrix theorists, those who study semidefinite programming, and to researchers in polynomial identities and representation theory. For a different audience the paper could be written using terminology from invariant theory as was explained to us by Špela Špenko and Claudio Procesi; see [KV17, Appendix A] for a discussion.

Main results. Let \mathbb{k} be an algebraically closed field of characteristic 0, $\mathbf{x} = (x_1, \dots, x_g)$ a tuple of freely noncommuting variables and $\mathbb{k}\langle\mathbf{x}\rangle$ the free \mathbb{k} -algebra over \mathbf{x} . To a noncommutative polynomial $f \in \mathbb{k}\langle\mathbf{x}\rangle$ we assign its *free (singularity) locus*

$$\mathcal{L}(f) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(f), \quad \text{where} \quad \mathcal{L}_n(f) = \{X \in M_n(\mathbb{k})^g : \det f(X) = 0\}.$$

If a noncommutative polynomial f factors as a product of two nonconstant polynomials, $f = f_1 f_2$, then $\mathcal{L}(f) = \mathcal{L}(f_1) \cup \mathcal{L}(f_2)$. It is easy to see that the free locus of a nonconstant polynomial is nonempty, so the hypersurface $\mathcal{L}_n(f)$ has at least two components for large enough $n \in \mathbb{N}$. Our main result is the converse to this simple observation.

Theorem A. *Let $f \in \mathbb{k}\langle\mathbf{x}\rangle$ satisfy $f(\mathbb{k}^g) \neq \{0\}$. Then f is irreducible if and only if there exists $n_0 \in \mathbb{N}$ such that $\mathcal{L}_n(f) \subset M_n(\mathbb{k})^g$ is an irreducible hypersurface for all $n \geq n_0$.*

Note that in general we cannot take $n_0 = 1$ in Theorem A; for example, the noncommutative polynomial $f = (1 - x_1)^2 - x_2^2$ is irreducible, but $f(\omega_1, \omega_2) = (1 - \omega_1 - \omega_2)(1 - \omega_1 + \omega_2)$ for commuting indeterminates ω_1, ω_2 . Theorem A is a corollary of the more general Theorem 4.3 below, which certifies the inclusion of free loci of noncommutative polynomials. For example, consider irreducible polynomials $f_1 = 1 + x_1 x_2$ and $f_2 = 1 + x_2 x_1$. It is well-known that for square matrices A and B of equal size, the eigenvalues of AB and BA

coincide, so $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$ holds. On the other hand, this equality also follows from

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + x_1 x_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -x_2 \end{pmatrix} \begin{pmatrix} 1 + x_2 x_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & x_1 \end{pmatrix}.$$

Such an algebraic condition, called *stable associativity* of (irreducible) polynomials f_1 and f_2 , is necessary for $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$ to hold. More precisely, we obtain the following free locus Nullstellensatz for polynomials:

Theorem B. *Let $f_1, f_2 \in \mathbb{k}\langle \mathbf{x} \rangle$ satisfy $f_i(\mathbb{k}^g) \neq \{0\}$. Then $\mathcal{Z}(f_1) \subseteq \mathcal{Z}(f_2)$ if and only if each irreducible factor of f_1 is (up to stable associativity) an irreducible factor of f_2 .*

Theorems A and B are special cases of Theorem 4.3 below, and both hold for matrix polynomials. The proof of Theorem 4.3 consists of two ingredients. The first one is Cohn's factorization theory for semifirs [Coh06], which deals with the ring-theoretic side of factorization. The second one is Theorem C, an irreducibility result about evaluations of linear matrix pencils, which we discuss next.

Given a *monic linear pencil* $L = I_d - A_1 x_1 - \cdots - A_g x_g$ with $A_j \in M_d(\mathbb{k})$, its evaluation at $X \in M_n(\mathbb{k})^g$ is defined as

$$L(X) = I_{dn} - A_1 \otimes X_1 - \cdots - A_g \otimes X_g \in M_{dn}(\mathbb{k}),$$

where \otimes is the Kronecker product. We say that a monic pencil is *irreducible* if its coefficients generate the whole matrix algebra, or equivalently, they do not admit a non-trivial common invariant subspace. For $k = 1, \dots, g$ let $\Omega_k^{(n)}$ be an $n \times n$ generic matrix, i.e., a matrix of n^2 independent commuting variables ω_{kij} for $1 \leq i, j \leq n$.

Theorem C. *Let L be an irreducible monic pencil. Then there exists $n_0 \in \mathbb{N}$ such that $\det L(\Omega_1^{(n)}, \dots, \Omega_g^{(n)})$ is an irreducible polynomial for all $n \geq n_0$.*

See Theorem 3.4 for the proof, which combines in a novel way invariant theory for the action of $\mathrm{GL}_n(\mathbb{k})$ on $M_n(\mathbb{k})^g$ by simultaneous conjugation and free analysis techniques for transitioning between different sizes of generic matrices. Given a linear pencil L we define its free locus analogously as in the preceding setting of noncommutative polynomials:

$$\mathcal{Z}(L) = \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n(L), \quad \text{where } \mathcal{Z}_n(L) = \{X \in M_n(\mathbb{k})^g : \det L(X) = 0\}.$$

Theorem A is then deduced from Theorem C using a linearization process [Coh06, Section 5.8], which to every noncommutative polynomial f assigns a linear pencil L with $\mathcal{Z}(f) = \mathcal{Z}(L)$.

In the second part of the paper we thus turn our attention to monic pencils L with polynomial free loci, i.e., $\mathcal{Z}(L) = \mathcal{Z}(f)$ for some $f \in \mathbb{k}\langle \mathbf{x} \rangle$. We apply noncommutative Fornasini-Marchesini state space realizations [BV05, BGM05] to prove that the coefficients of such pencils L are of the form $N_j + E_j$, where N_j are jointly nilpotent matrices and E_j are rank-one matrices with coinciding kernels (Corollary 5.5). By connecting this result with Theorem A and minimal factorizations in the sense of realization theory we obtain a curious statement about invariant subspaces. Given a tuple of matrices $A = (A_1, \dots, A_g)$ and $A' = (A'_1, \dots, A'_g)$ we say that A' is a *non-degenerate right rank-one perturbation* of A if $A'_j - A_j = b_j c^\dagger$ for some vectors b_j and c such that $\{b_1, \dots, b_g\}$ is not contained in a proper invariant subspace for A_1, \dots, A_g and c does not lie in a proper invariant subspace for $A_1^\dagger, \dots, A_g^\dagger$. The following is a consequence of Theorem 5.1.

Theorem D. *Let A be a tuple of jointly nilpotent matrices. If a non-degenerate right rank-one perturbation of A has an invariant subspace \mathcal{S} , then there exists a subspace \mathcal{S}^\times that is complementary to \mathcal{S} and invariant under A .*

Lastly we consider consequences of Theorem C in real algebraic geometry. A monic pencil L is hermitian if its coefficients are hermitian matrices. Its *free spectrahedron* or *LMI domain* is defined as

$$\mathcal{D}(L) = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n(L), \quad \text{where } \mathcal{D}_n(L) = \{X \in \mathbb{H}_n(\mathbb{C})^g : L(X) \succeq 0\}.$$

Here $\mathbb{H}_n(\mathbb{C})$ denotes $n \times n$ hermitian matrices and $M \succeq 0$ means that M is positive semidefinite. Then $\mathcal{D}_n(L) \subseteq \mathbb{H}_n(\mathbb{C})^g$ is a convex set and its boundary is contained in $\mathcal{Z}_n(L)$. For $n \in \mathbb{N}$ denote

$$\partial^1 \mathcal{D}_n(L) = \{X \in \mathcal{D}_n(L) : \dim \ker L(X) = 1\}.$$

The points in $\partial^1 \mathcal{D}_n(L)$ are often precisely the smooth points of the boundary of $\mathcal{D}_n(L)$ [Ren06]. The unique-null-vector property makes them vital for optimization [Ren06], Positivstellensätze in free real algebraic geometry [HKN14] and the study of free analytic maps between LMI domains [AHKM+]. Unfortunately, given a fixed $n \in \mathbb{N}$ it can happen that $\partial^1 \mathcal{D}_n(L) = \emptyset$ even for irreducible hermitian pencils, as a consequence of the failure of Kippenhahn’s conjecture [Laf83]. However, we show that this cannot happen for every $n \in \mathbb{N}$ if L is *LMI-minimal*, i.e., of minimal size among all hermitian pencils whose LMI domain equals $\mathcal{D}(L)$.

Theorem E. *Let L be an LMI-minimal hermitian pencil. Then there exists $n_0 \in \mathbb{N}$ such that $\partial^1 \mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L)$ for all $n \geq n_0$.*

Theorem E can be viewed as a quantitative solution of the quantum Kippenhahn conjecture and is proved as Corollary 8.5 below.

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2. PRELIMINARIES

In this section we gather results about free loci and polynomial invariants for the general linear group acting on matrix tuples by conjugation that will be used in the sequel.

Let \mathbb{k} be algebraically closed field of characteristic 0. Throughout the paper we use the following convention. If $\mathcal{S}(n)$ is a statement depending on $n \in \mathbb{N}$, then “ $\mathcal{S}(n)$ holds for large n ” means “there exists $n_0 \in \mathbb{N}$ such that $\mathcal{S}(n)$ holds for all $n \geq n_0$ ”.

2.1. Free loci of matrix pencils. For $g \in \mathbb{N}$ let $\mathbf{x} = (x_1, \dots, x_g)$ be a tuple of freely noncommuting variables. If $A_1, \dots, A_g \in M_d(\mathbb{k})$, then

$$L = I_d - \sum_{j=1}^g A_j x_j$$

is a **monic linear pencil of size d** . For $X \in M_n(\mathbb{k})^g$ let

$$L(X) = I_{dn} - \sum_j A_{j=1}^g \otimes X_j \in M_{dn}(\mathbb{k}),$$

where \otimes denotes the Kronecker product. Let $\mathcal{M}^g = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{k})^g$. The set

$$\mathcal{Z}(L) = \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n(L) \subset \mathcal{M}^g, \quad \text{where } \mathcal{Z}_n(L) = \{X \in M_n(\mathbb{k})^g : \det L(X) = 0\},$$

is the **free locus of L** . We review some terminology and facts about free loci from [KV17] that will be frequently used throughout the paper.

- (1) $\mathcal{Z}(L) = \emptyset$ if and only if A_1, \dots, A_g are jointly nilpotent by [KV17, Corollary 3.4].
- (2) If A_1, \dots, A_g generate $M_d(\mathbb{k})$ as a \mathbb{k} -algebra, we say that L is an **irreducible pencil**. If L_1 and L_2 are irreducible and $\mathcal{Z}(L_1) \subseteq \mathcal{Z}(L_2)$, then $\mathcal{Z}(L_1) = \mathcal{Z}(L_2)$ (follows from [KV17, Theorem 3.6] because a surjective homomorphism from a simple algebra is an isomorphism) and moreover L_1 and L_2 are similar, i.e., they differ only by a basis change on \mathbb{C}^d , by [KV17, Theorem 3.11].
- (3) A free locus is **irreducible** if it is not a union of smaller free loci. By [KV17, Proposition 3.12], a free locus is irreducible if and only if it is a free locus of some irreducible pencil.
- (4) By applying Burnside's theorem [Bre14, Corollary 5.23] on the existence of invariant subspaces of the \mathbb{k} -algebra generated by A_1, \dots, A_g it follows that every monic pencil L is similar to a pencil of the form

$$(2.1) \quad \begin{pmatrix} L_1 & \star & \cdots & \star \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & L_\ell \end{pmatrix},$$

where for every k , $L_k = I$ or L_k is an irreducible pencil.

- (5) Finally, we say that $L \neq I$ is **FL-minimal** (free locus minimal) if it is of minimal size among pencils L' with $\mathcal{Z}(L') = \mathcal{Z}(L)$. An L of the form (2.1) is FL-minimal if and only if the L_k are pairwise non-similar irreducible pencils. Furthermore, the diagonal blocks of an FL-minimal pencil of the form (2.1) are unique up to a basis change by [KV17, Theorem 3.11].

2.2. Simultaneous conjugation of matrices. Consider the action of $GL_n(\mathbb{k})$ on the space $M_n(\mathbb{k})^g$ given by

$$(2.2) \quad \sigma \cdot (X_1, \dots, X_g) = (\sigma X_1 \sigma^{-1}, \dots, \sigma X_g \sigma^{-1}), \quad \sigma \in GL_n(\mathbb{k}).$$

The coordinate ring of $M_n(\mathbb{k})^g$ is the polynomial ring $\mathbb{k}[\boldsymbol{\omega}]$ in gn^2 commuting variables $\omega_{j\iota}$ for $1 \leq j \leq g$ and $1 \leq \iota, j \leq n$. For $1 \leq j \leq g$ let $\Omega_j^{(n)} = (\omega_{j\iota j})_{\iota, j}$ be an $n \times n$ generic matrix and write $\Omega^{(n)} = (\Omega_1^{(n)}, \dots, \Omega_g^{(n)})$. Viewing $p \in \mathbb{k}[\boldsymbol{\omega}]$ as a polynomial in the entries of $\Omega^{(n)}$, the action (2.2) induces an action of $GL_n(\mathbb{k})$ on $\mathbb{k}[\boldsymbol{\omega}]$ defined by

$$(2.3) \quad \sigma \cdot p = p(\sigma^{-1} \cdot \Omega^{(n)}), \quad \sigma \in GL_n(\mathbb{k}), \quad p \in \mathbb{k}[\boldsymbol{\omega}].$$

The subring of $\mathbb{k}[\boldsymbol{\omega}]$ of invariants for this action is denoted $\mathbb{k}[\boldsymbol{\omega}]^{GL_n(\mathbb{k})}$. By [Pro76, Theorem 1.3] it is generated by $\text{tr}(w(\Omega^{(n)}))$ for $w \in \langle \boldsymbol{x} \rangle \setminus \{1\}$. Here tr denotes the usual trace,

and $\langle \mathbf{x} \rangle$ is the free monoid generated by the freely noncommuting variables x_1, \dots, x_g . The elements of $\mathbb{k}[\boldsymbol{\omega}]^{\mathrm{GL}_n(\mathbb{k})}$ are therefore called *pure trace polynomials*. For example, $2 - \mathrm{tr}(\Omega_1^{(n)}) \mathrm{tr}(\Omega_2^{(n)}) + \mathrm{tr}(\Omega_1^{(n)2} \Omega_2^{(n)3} \Omega_1^{(n)} \Omega_2^{(n)})$ is a pure trace polynomial.

In the next section we require the following two lemmas. While they are known to specialists, we provide their proofs to keep the presentation self-contained.

Lemma 2.1. *Every factor of a $\mathrm{GL}_n(\mathbb{k})$ -invariant polynomial is $\mathrm{GL}_n(\mathbb{k})$ -invariant.*

Proof. Let q be an irreducible factor of $p \in \mathbb{k}[\boldsymbol{\omega}]^{\mathrm{GL}_n(\mathbb{k})}$. Let $\sigma \in \mathrm{GL}_n(\mathbb{k})$; since $\sigma \cdot p = p$ and $\sigma \cdot q$ is irreducible, we see that $\sigma \cdot q$ is again an irreducible factor of p . Let \mathcal{S} be the set of all nonzero scalar multiples of irreducible factors of p . Then \mathcal{S} with the Zariski topology is homeomorphic to a disjoint union of copies of $\mathbb{k}^* = \mathbb{k} \setminus \{0\}$. The algebraic group $\mathrm{GL}_n(\mathbb{k})$ is irreducible, and the map

$$f : \mathrm{GL}_n(\mathbb{k}) \rightarrow \mathcal{S}, \quad \sigma \mapsto \sigma \cdot q$$

is regular and $f(\mathrm{id}) = q$. Therefore the image of f lies in the irreducible component of \mathcal{S} containing q , so for every $\sigma \in \mathrm{GL}_n(\mathbb{k})$ we have $\sigma \cdot q = \lambda_\sigma q$ for some $\lambda_\sigma \in \mathbb{k}^*$. Next we observe that the map

$$\lambda : \mathrm{GL}_n(\mathbb{k}) \rightarrow \mathbb{k}^*, \quad \sigma \mapsto \lambda_\sigma$$

is a group homomorphism. Note that $\lambda(\mathbb{k}^* I) = \{1\}$ by the nature of our action; also $\lambda(\mathrm{SL}_n(\mathbb{k})) = \{1\}$ because $\mathrm{SL}_n(\mathbb{k})$ is generated by multiplicative commutators. Since $\mathrm{GL}_n(\mathbb{k})$ is a semidirect product of $\mathrm{SL}_n(\mathbb{k})$ and \mathbb{k}^* , we conclude that λ is constantly equal to 1, so q is a $\mathrm{GL}_n(\mathbb{k})$ -invariant. \square

Lemma 2.2. *Let $X \in M_n(\mathbb{k})^g$ and $X' \in (\mathbb{k}^{n' \times n})^g$. If p is $\mathrm{GL}_{n+n'}(\mathbb{k})$ -invariant, then*

$$p \begin{pmatrix} 0 & X' \\ 0 & X \end{pmatrix} = p \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}.$$

Proof. For every $w \in \langle \mathbf{x} \rangle \setminus \{1\}$ we clearly have

$$w \begin{pmatrix} 0 & X' \\ 0 & X \end{pmatrix} = \begin{pmatrix} 0 & Y_w \\ 0 & w(X) \end{pmatrix}$$

for some $Y_w \in \mathbb{k}^{n' \times n}$ and hence

$$\mathrm{tr} \left(w \begin{pmatrix} 0 & X' \\ 0 & X \end{pmatrix} \right) = \mathrm{tr} \begin{pmatrix} 0 & Y_w \\ 0 & w(X) \end{pmatrix} = \mathrm{tr}(w(X)) = \mathrm{tr} \left(w \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \right).$$

The statement now follows because $\mathbb{k}[\boldsymbol{\omega}]^{\mathrm{GL}_{n+n'}(\mathbb{k})}$ is generated by $\mathrm{tr}(w(\Omega^{(n+n')}))$. \square

3. DETERMINANT OF AN IRREDUCIBLE PENCIL

Let L be a monic pencil and let $\Omega^{(n)}$ be a g -tuple of $n \times n$ generic matrices. In Subsection 3.2 we prove our first main result, Theorem 3.4, which states that for an irreducible pencil L , the commutative polynomial $\det L(\Omega^{(n)})$ is irreducible for large n . The consequences for the locus $\mathcal{X}_n(L)$ are given in Subsection 3.3.

3.1. Degree growth. Let $\text{GM}_n(g) \subseteq \text{M}_n(\mathbb{k}[\boldsymbol{\omega}])$ be the ring of $n \times n$ generic matrices, i.e., the unital \mathbb{k} -algebra generated by $\Omega_1^{(n)}, \dots, \Omega_g^{(n)}$ [For84, Section 5]. Furthermore, let $\text{UD}_n(g) \subseteq \text{M}_n(\mathbb{k}(\boldsymbol{\omega}))$ be the universal division algebra of degree n , which is the ring of central quotients of $\text{GM}_n(g)$.

Lemma 3.1. *Let L be a monic pencil of size d and*

$$f_n = \det L(\Omega^{(n)}) \in \mathbb{k}[\boldsymbol{\omega}]$$

for $n \in \mathbb{N}$. Then there exists $d_L \leq d$ such that $\deg f_n = d_L n$ for all $n \geq d^2 - 1$.

Proof. The case $d = 1$ is clear, so assume $d \geq 2$. Let $\Lambda = I - L$ and define a non-monic matrix pencil \tilde{L} of size d^2 as

$$\tilde{L} = \begin{pmatrix} 0 & -\Lambda & 0 & \cdots & \\ \vdots & I & \ddots & & \vdots \\ & & \ddots & -\Lambda & 0 \\ & & & I & -\Lambda \\ \Lambda & \cdots & 0 & I & \end{pmatrix}.$$

It is easy to check that

$$(3.1) \quad \begin{pmatrix} I & \Lambda & \cdots & \Lambda^{d-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Lambda \\ & & & I \end{pmatrix} \cdot \tilde{L} = \begin{pmatrix} \Lambda^d & & & \\ \Lambda^{d-1} & I & & \\ \vdots & & \ddots & \\ \Lambda & & & I \end{pmatrix}.$$

Next we make a few simple observations. Firstly, if A is an $a \times a$ matrix over a (commutative) field F , then the degree of the univariate polynomial $\det(I - tA) \in F[t]$ equals the rank of A^a over F . Secondly, if B is a $b \times b$ matrix over a skew field, then the rank of B^{b_1} equals the rank of B^b for every $b_1 \geq b$; here the rank of a matrix over a skew field equals the dimension of its range as a linear operator over the skew field. Finally, if C is a $c \times c$ matrix over $\text{UD}_n(g)$ of rank r , then it is equivalent to $I^{\oplus r} \oplus 0^{\oplus(c-r)}$ over $\text{UD}_n(g)$, so C is of rank rn as a $cn \times cn$ matrix over $\mathbb{k}(\boldsymbol{\omega})$.

Let $n \in \mathbb{N}$ be arbitrary. Viewing $\Lambda(\Omega^{(n)})$ as a $d \times d$ matrix over $\text{UD}_n(g)$, the preceding observations and (3.1) imply

$$\begin{aligned} \deg f_n &= \deg_t (I - t\Lambda(\Omega^{(n)})) \\ &= \text{rk}_{\mathbb{k}(\boldsymbol{\omega})} \Lambda(\Omega^{(n)})^{dn} \\ &= \frac{1}{n} \text{rk}_{\text{UD}_n(g)} \Lambda(\Omega^{(n)})^{dn} \\ &= \frac{1}{n} \text{rk}_{\text{UD}_n(g)} \Lambda(\Omega^{(n)})^d \\ &= \text{rk}_{\mathbb{k}(\boldsymbol{\omega})} \Lambda(\Omega^{(n)})^d \\ &= \text{rk}_{\mathbb{k}(\boldsymbol{\omega})} \tilde{L}(\Omega^{(n)}) - (d-1)dn. \end{aligned}$$

By the proof of [DM17, Proposition 2.10], there is $d' \in \mathbb{N}$ such that $\text{rk}_{\mathbb{k}(\boldsymbol{\omega})} \tilde{L}(\Omega^{(n)}) = d'n$ for all $n \geq d^2 - 1$. Hence $d_L = d' - (d-1)d$. \square

Example 3.2. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

One can check that A_1 and A_2 generate $M_3(\mathbb{k})$. If $L = I - A_1x_1 - A_2x_2$, then using Schur complements we see that

$$\det L(\Omega^{(n)}) = \det \begin{pmatrix} I - \Omega_1^{(n)} & -(\Omega_2^{(n)})^2 \\ -\Omega_1^{(n)} & I - \Omega_1^{(n)} - (\Omega_2^{(n)})^2 \end{pmatrix} = \det \left((I - \Omega_1^{(n)})^2 - (\Omega_2^{(n)})^2 \right)$$

is of degree $2n$ for every $n \in \mathbb{N}$. Therefore $d_L = 2 < 3$.

3.2. Eventual irreducibility. Let $\Upsilon^{(n)}$ be an $n \times n$ generic matrix whose entries are independent of the entries in $\Omega_j^{(n)}$ (that is, we introduce n^2 new variables to form $\Upsilon^{(n)}$). The following lemma is a key technical tool for proving Theorem 3.4 below.

Lemma 3.3. *Let $L = I - \sum_{j=1}^g A_j x_j$ be of size d and $n_0 \geq d^2 - 1$. Fix $1 \leq j', j'' \leq g$ and assume that*

$$\det \left(L(\Omega^{(n)}) - A_{j'} A_{j''} \otimes \Upsilon^{(n)} \right)$$

is an irreducible polynomial (in $(g+1)n^2$ variables) for every $n \geq n_0$. Then $\det L(\Omega^{(n)})$ is an irreducible polynomial (in gn^2 variables) for every $n \geq 2n_0$.

Proof. For $n \geq n_0$ denote

$$f_n = \det L(\Omega^{(n)}), \quad \hat{f}_n = \det \left(L(\Omega^{(n)}) - A_{j'} A_{j''} \otimes \Upsilon^{(n)} \right).$$

Suppose that $f_{2n} = pq$ for some nonconstant polynomials p and q . By Lemma 2.1, p and q are $\mathrm{GL}_{2n}(\mathbb{k})$ -invariant. Let $\Omega_j'^{(n)}, \Omega_j''^{(n)}$ be independent $n \times n$ generic matrices and denote

$$\tilde{p} = \tilde{p}(\Omega'^{(n)}, \Omega''^{(n)}) := p(\Omega'^{(n)} \oplus \Omega''^{(n)}), \quad \tilde{q} = \tilde{q}(\Omega'^{(n)}, \Omega''^{(n)}) := q(\Omega'^{(n)} \oplus \Omega''^{(n)}).$$

Note that

$$(3.2) \quad \tilde{p}(\Omega^{(n)}, 0) = \tilde{p}(0, \Omega^{(n)}), \quad \tilde{q}(\Omega^{(n)}, 0) = \tilde{q}(0, \Omega^{(n)})$$

since p and q are $\mathrm{GL}_{2n}(\mathbb{k})$ -invariant. Consider

$$(3.3) \quad f_n(\Omega'^{(n)}) f_n(\Omega''^{(n)}) = f_{2n}(\Omega'^{(n)} \oplus \Omega''^{(n)}) = \tilde{p} \tilde{q}.$$

By Lemma 3.1, the left-hand side of (3.3) has degree $2d_L n$. Since p and q have degree strictly less than $2d_L n$, we conclude that \tilde{p} and \tilde{q} are nonconstant. Moreover, since the left-hand side of (3.3) is a product of two polynomials in disjoint sets of variables, we conclude that

$$\tilde{p} = \tilde{p}_1(\Omega'^{(n)}) \tilde{p}_2(\Omega''^{(n)}), \quad \tilde{q} = \tilde{q}_1(\Omega'^{(n)}) \tilde{q}_2(\Omega''^{(n)})$$

for some polynomials $\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2$. If \tilde{p}_1 were constant, then \tilde{p}_2 would be constant by (3.2), contradicting that \tilde{p} is nonconstant. Hence we conclude that $\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2$ are nonconstant and consequently

$$p \begin{pmatrix} 0 & 0 \\ 0 & \Omega^{(n)} \end{pmatrix} = \tilde{p}(0, \Omega^{(n)}), \quad q \begin{pmatrix} 0 & 0 \\ 0 & \Omega^{(n)} \end{pmatrix} = \tilde{q}(0, \Omega^{(n)})$$

are nonconstant.

Since

$$\det(L(\Omega^{(n)}) - A_{j'}A_{j''} \otimes \Upsilon^{(n)}) = \det \begin{pmatrix} I & A_{j''} \otimes I \\ A_{j'} \otimes \Upsilon^{(n)} & L(\Omega^{(n)}) \end{pmatrix},$$

we see that

$$\hat{f}_n = f_{2n}(\mathcal{Z}),$$

where

$$\mathcal{Z}_j = \begin{pmatrix} 0 & 0 \\ 0 & \Omega_j^{(n)} \end{pmatrix}$$

for $j \notin \{j', j''\}$, and

$$\mathcal{Z}_{j'} = \begin{pmatrix} 0 & 0 \\ \Upsilon^{(n)} & \Omega_{j'}^{(n)} \end{pmatrix}, \quad \mathcal{Z}_{j''} = \begin{pmatrix} 0 & I \\ 0 & \Omega_{j''}^{(n)} \end{pmatrix}$$

if $j' \neq j''$ and

$$\mathcal{Z}_{j'} = \begin{pmatrix} 0 & I \\ \Upsilon^{(n)} & \Omega_{j'}^{(n)} \end{pmatrix}$$

if $j' = j''$. Since $\mathcal{Z}|_{\Upsilon^{(n)}=0}$ is a tuple of block upper triangular matrices, Lemma 2.2 implies

$$p(\mathcal{Z})|_{\Upsilon^{(n)}=0} = p \begin{pmatrix} 0 & 0 \\ 0 & \Omega^{(n)} \end{pmatrix}, \quad q(\mathcal{Z})|_{\Upsilon^{(n)}=0} = q \begin{pmatrix} 0 & 0 \\ 0 & \Omega^{(n)} \end{pmatrix}.$$

In particular, $p(\mathcal{Z})$ and $q(\mathcal{Z})$ are nonconstant and

$$\hat{f}_n = p(\mathcal{Z})q(\mathcal{Z}),$$

a contradiction.

Hence we have proven the statement for every even $n \geq 2n_0$. If $n \geq 2n_0$ is odd, then $n - 1 \geq 2n_0$. If $f_n = pq$, then irreducibility of f_{n-1} implies that $p(0 \oplus \Omega^{(n-1)}) = 1$ and $q(0 \oplus \Omega^{(n-1)}) = f_{n-1}$ (or vice versa). Therefore q is of degree at least $(n - 1)d_L$, so $1 - p$ is of degree at most d_L , is a pure trace identity for $M_{n-1}(\mathbb{k})$ but not for $M_n(\mathbb{k})$. However, this cannot happen by [Pro76, Theorem 4.5] since $n - 1 \geq 2n_0 \geq d_L$. \square

We are now ready to prove the first of our main results, which was announced in [Vol+]. After being informed of Theorem 3.4, Kriel [Kri] independently proved it for hermitian pencils.

Theorem 3.4. *If L is an irreducible pencil, then $\det L(\Omega^{(n)})$ is an irreducible polynomial for large n .*

Proof. Let $L = I - \sum_j A_j x_j$; then $A_1, \dots, A_g \in M_d(\mathbb{k})$ generate $M_d(\mathbb{k})$ as a \mathbb{k} -algebra by assumption. If

$$A_g = \sum_{j=1}^{g-1} \alpha_j A_j,$$

then A_1, \dots, A_{g-1} generate $M_d(\mathbb{k})$ and

$$\det L(\Omega^{(n)}) = \det \left(I - \sum_{j=1}^{g-1} A_j \otimes \left(\Omega_j^{(n)} + \alpha_j \Omega_g^{(n)} \right) \right),$$

so irreducibility of $\det \left(I - \sum_{j < g} A_j \otimes \Omega_j^{(n)} \right)$ implies irreducibility of $\det L(\Omega^{(n)})$. Therefore we can without loss of generality assume that A_1, \dots, A_g are linearly independent.

Let $w_{\iota j} \in \langle \mathbf{x} \rangle$ for $1 \leq \iota, j \leq d$ be such that

$$(3.4) \quad \{x_1, \dots, x_g\} \subseteq \{w_{\iota j}: 1 \leq \iota, j \leq d\}, \quad \text{span} \{w_{\iota j}(A): 1 \leq \iota, j \leq d\} = M_d(\mathbb{k}).$$

For $1 \leq \iota, j \leq d$ let $E_{\iota j} \in M_d(\mathbb{k})$ be the standard matrix units. If $\Omega_{\iota j}^{(n)}$ for $1 \leq \iota, j \leq d$ are $n \times n$ generic matrices, then

$$\sum_{\iota, j} E_{\iota j} \otimes \Omega_{\iota j}^{(n)}$$

is a $(dn) \times (dn)$ generic matrix, so its determinant is irreducible for every $n \in \mathbb{N}$ [GW09, Lemma B.2.10]. The polynomial

$$(3.5) \quad \det \left(I - \sum_{\iota, j} w_{\iota j}(A) \otimes \Omega_{\iota j}^{(n)} \right)$$

is a composition of the determinant of a generic matrix and an affine map on variables (this map is invertible by (3.4)), and therefore irreducible. Starting with polynomial (3.5) we $N = \sum_{\iota, j} |w_{\iota j}| - g$ times recursively apply Lemma 3.3 to get rid of $w_{\iota j}(A)$ for $|w_{\iota j}| > 1$ and conclude that $\det L(\Omega^{(n)})$ is an irreducible polynomial for every $n \geq (d^2 - 1)2^N$. \square

Remark 3.5. From the proof of Theorem 3.4 one can derive a deterministic bound on n for checking the irreducibility of $\det L(\Omega^{(n)})$ that is exponential in the size of L .

3.3. Irreducible free loci. Recall that a free locus is irreducible if it is not a union of smaller free loci. If \mathcal{Z} is a free locus, then \mathcal{Z}_n is either an empty set or a hypersurface for every $n \in \mathbb{N}$.

Corollary 3.6. *If \mathcal{Z} is an irreducible free locus, then \mathcal{Z}_n is an irreducible hypersurface for large n .*

Proof. Every irreducible free locus is a free locus of an irreducible pencil, so Theorem 3.4 applies. \square

Example 3.7. Let $A_1, A_2 \in M_3(\mathbb{k})$ be as in Example 3.2. Note that

$$\det(I - \omega_1 A_1 - \omega_2 A_2) = (1 - \omega_1 + \omega_2)(1 - \omega_1 - \omega_2).$$

Hence $L = I - A_1 x_1 - A_2 x_2$ is an irreducible pencil and $\mathcal{Z}_1(L)$ is a union of two lines. On the other hand one can check that $\mathcal{Z}_2(L)$ is irreducible.

Together with the block form (2.1) of a monic pencil L , Corollary 3.6 shows that components of $\mathcal{Z}_n(L)$ for large n arise from the global decomposition of $\mathcal{Z}(L)$.

Corollary 3.8. *Let L be a monic pencil and let L_1, \dots, L_k be pairwise non-similar irreducible pencils appearing in (2.1). For large n ,*

$$\mathcal{Z}_n(L) = \mathcal{Z}_n(L_1) \cup \dots \cup \mathcal{Z}_n(L_k)$$

is the decomposition of $\mathcal{Z}_n(L)$ into distinct irreducible hypersurfaces.

Remark 3.9. Let L be an FL-minimal pencil. Let L_1, \dots, L_ℓ be pairwise non-similar irreducible pencils appearing in the decomposition of L as in (2.1). Since $\mathcal{Z}(L_i) \neq \mathcal{Z}(L_{i'})$

for $i \neq i'$, Theorem 3.4 implies that $\det L_i(\Omega^{(n)})$ are distinct irreducible polynomials for large n . Therefore

$$\det L(\Omega^{(n)}) = \prod_{i=1}^{\ell} \det L_i(\Omega^{(n)})$$

is square-free for large n . Hence $\det L(\Omega^{(n)})$ is a minimum degree defining polynomial for $\mathcal{Z}_n(L)$ and it generates the vanishing (radical) ideal of $\mathcal{Z}_n(L)$.

4. IRREDUCIBILITY OF MATRICES OVER A FREE ALGEBRA

In this section we extend Theorem 3.4 to matrices over a free algebra. Let $\mathbb{k}\langle \mathbf{x} \rangle$ be the free \mathbb{k} -algebra generated by $\mathbf{x} = (x_1, \dots, x_g)$. Its elements are **noncommutative polynomials**. For $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ let

$$\mathcal{Z}(f) = \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n(f) \subseteq \mathcal{M}^g, \quad \text{where } \mathcal{Z}_n(f) = \{X \in M_n(\mathbb{k})^g : \det f(X) = 0\},$$

be the **free locus** of f . In Theorem 4.3 below we prove that if f does not factor in $M_d(\mathbb{k}\langle \mathbf{x} \rangle)$, then $\mathcal{Z}_n(f)$ is an irreducible hypersurface for large n .

A fundamental finding of Cohn is that $\mathbb{k}\langle \mathbf{x} \rangle$ is a free ideal ring, abbreviated fir, and that even rings with the weaker ‘‘semifir’’ property exhibit excellent behavior when it comes to factorizations. We list few definitions and facts about factorization of matrices over semifirs extracted from [Coh06, Chapter 3].

- (1) If $f \in \text{GL}_d(\mathbb{k}\langle \mathbf{x} \rangle)$, then $\det f(0) \neq 0$; moreover, $\det f(\Omega^n)$ is a polynomial without zeros and hence constant, so $\det f(X) = \det f(0)^n$ for every $X \in M_n(\mathbb{k})^g$. In particular, irreducible monic pencils are non-invertible by [KV17, Corollary 3.4].
- (2) A matrix $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ is **regular** if f is not a zero divisor in $M_d(\mathbb{k}\langle \mathbf{x} \rangle)$. In particular, if $f(0) = I$, then f is regular. A regular non-invertible matrix is an **atom** [Coh06, Section 3.2] if it is not a product of two non-invertible matrices in $M_d(\mathbb{k}\langle \mathbf{x} \rangle)$. In the special case $d = 1$, a nonconstant noncommutative polynomial is an atom if it is not a product of two nonconstant noncommutative polynomials. We use this terminology to avoid confusion with the notion of irreducibility for monic pencils.
- (3) We say that $f_1 \in M_{d_1}(\mathbb{k}\langle \mathbf{x} \rangle)$ and $f_2 \in M_{d_2}(\mathbb{k}\langle \mathbf{x} \rangle)$ are **stably associated** if there exist $e_1, e_2 \in \mathbb{N}$ with $d_1 + e_1 = d_2 + e_2$ and $P, Q \in \text{GL}_{d_1+e_1}(\mathbb{k}\langle \mathbf{x} \rangle)$ such that

$$f_1 \oplus I_{e_1} = P(f_2 \oplus I_{e_2})Q.$$

Stable associativity is clearly an equivalence relation for regular (square) matrices over $\mathbb{k}\langle \mathbf{x} \rangle$. By [Coh06, Corollary 0.5.5], f_1 and f_2 are stably associated if and only if

$$\mathbb{k}\langle \mathbf{x} \rangle^{d_1} / f_1 \cdot \mathbb{k}\langle \mathbf{x} \rangle^{d_1} \cong \mathbb{k}\langle \mathbf{x} \rangle^{d_2} / f_2 \cdot \mathbb{k}\langle \mathbf{x} \rangle^{d_2}$$

as right $\mathbb{k}\langle \mathbf{x} \rangle$ -modules.

- (4) Let $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ be regular. By the definition of torsion modules [Coh06, Section 3.2] and their relations to factorization [Coh06, Propositions 0.5.2 and 3.2.1], it follows that f is an atom if and only if $\mathbb{k}\langle \mathbf{x} \rangle^d / f \cdot \mathbb{k}\langle \mathbf{x} \rangle^d$ has no non-trivial torsion submodules. In particular, if regular matrices f_1 and f_2 are stably associated, then f_1 is an atom if and only if f_2 is an atom.

Remark 4.1. If $h_1, h_2 \in \mathbb{k}\langle \mathbf{x} \rangle$ are homogeneous and stably associated, then there exists $\lambda \in \mathbb{k}^* = \mathbb{k} \setminus \{0\}$ such that $h_2 = \lambda h_1$. We now prove this. Let $d \in \mathbb{N}$ be such that

$$(4.1) \quad P(h_1 \oplus I) = (h_2 \oplus I)Q$$

for some $P, Q \in \mathrm{GL}_d(\mathbb{k}\langle \mathbf{x} \rangle)$. Then it is easy to see that h_1 and h_2 are of the same degree δ . Let P_i and Q_i be homogeneous parts of P and Q , respectively, of degree i . By looking at the constant part of (4.1) we obtain $P_0(0 \oplus I) = (0 \oplus I)Q_0$, so the first row of P_0 equals $(\alpha 0 \dots 0)$ for some $\alpha \in \mathbb{k}$ and the first column of Q_0 equals $(\beta 0 \dots 0)^t$ for some $\beta \in \mathbb{k}$. Moreover, $\alpha, \beta \in \mathbb{k}^*$ since $P_0, Q_0 \in \mathrm{GL}_d(\mathbb{k})$. Next, the homogeneous part of (4.1) of degree δ equals

$$P_0(h_1 \oplus 0) + P_\delta(0 \oplus I) = (h_2 \oplus 0)Q_0 + (0 \oplus I)Q_\delta.$$

Note that the first column of $P_\delta(0 \oplus I)$ and the first row of $(0 \oplus I)Q_\delta$ are zero. Since the $(1, 1)$ -entries of $P_0(h_1 \oplus 0)$ and $(h_2 \oplus 0)Q_0$ are αh_1 and βh_2 , respectively, we can take $\lambda = \alpha\beta^{-1}$.

Lemma 4.2. *Let $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ and $f(0) = I$. If f is an atom, then f is stably associated to an irreducible monic pencil L .*

Proof. By linearization, also known as Higman's trick [Coh06, Section 8.5], we have

$$(4.2) \quad \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 + a_1 a_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_2 & 1 \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ -a_2 & 1 \end{pmatrix}$$

for all square matrices a_0, a_1, a_2 of equal sizes. Using (4.2) we can step-by-step “decouple” products appearing in f to obtain

$$(4.3) \quad P(f \oplus I)Q = L$$

for some monic linear pencil L of size d' and $P, Q \in \mathrm{GL}_{d'}(\mathbb{k}\langle \mathbf{x} \rangle)$. We remark that P (resp. Q) is upper (resp. lower) unitriangular. Hence f and L are stably associated, so L is an atom in $M_{d'}(\mathbb{k}\langle \mathbf{x} \rangle)$. As in (2.1), there exists $U \in \mathrm{GL}_{d'}(\mathbb{k})$ such that

$$(4.4) \quad ULU^{-1} = \begin{pmatrix} L_1 & \star & \cdots & \star \\ & \ddots & \ddots & \vdots \\ & & \ddots & \star \\ & & & L_\ell \end{pmatrix},$$

where each L_k is either I or an irreducible monic pencil. Since f is not invertible, at least one of L_k is irreducible (i.e., $L_k \neq I$); let ℓ_0 be the largest such k . By multiplying ULU^{-1} on the left-hand side with an appropriate invertible matrix (note that the block, which is below and to the right of L_{ℓ_0} , is invertible) we see that L (and thus f) is stably associated to

$$L' = \begin{pmatrix} L_1 & \cdots & \star \\ & \ddots & \vdots \\ & & L_{\ell_0} \end{pmatrix}.$$

If $L_k \neq I$ for some $k < \ell_0$, then

$$L' = \begin{pmatrix} I & & \\ & \ddots & \\ & & L_{\ell_0} \end{pmatrix} \begin{pmatrix} L_1 & \cdots & \star \\ & \ddots & \vdots \\ & & I \end{pmatrix}$$

would be a product of non-invertible matrices, contradicting that L' is an atom. Therefore

$$L' = \begin{pmatrix} I & \cdots & \star \\ & \ddots & \vdots \\ & & L_{\ell_0} \end{pmatrix} = \begin{pmatrix} I & & \\ & \ddots & \\ & & L_{\ell_0} \end{pmatrix} \begin{pmatrix} I & \cdots & \star \\ & \ddots & \vdots \\ & & I \end{pmatrix},$$

so L' (and thus f) is stably associated to L_{ℓ_0} . \square

Let $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ be regular. Then f admits a factorization $f = f_1 \cdots f_\ell$ into atoms $f_i \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ that are unique up to stable associativity by [Coh06, Proposition 3.2.9], and thus

$$\mathcal{Z}(f) = \mathcal{Z}(f_1) \cup \cdots \cup \mathcal{Z}(f_\ell).$$

If we change the order of atomic factors in a factorization of f , we possibly obtain a different noncommutative polynomial with the same free locus as f . Also, if an atomic factor is replaced by a (power of a) stably associated element in $M_d(\mathbb{k}\langle \mathbf{x} \rangle)$, the polynomial can change but its free locus still equals $\mathcal{Z}(f)$.

We are now ready for our main result. Recall that $\Omega^{(n)}$ is a g -tuple of $n \times n$ generic matrices.

Theorem 4.3 (Polynomial Singularitatstellensatz). *For $i \in \{1, 2\}$ let $f_i \in M_{d_i}(\mathbb{k}\langle \mathbf{x} \rangle)$ satisfy $f_i(0) = I$.*

- (1) *If f_1 is an atom, then $\det f_1(\Omega^{(n)})$ is an irreducible polynomial for large n .*
- (2) *If f_1 and f_2 are atoms and $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$, then f_1 and f_2 are stably associated.*
- (3) *$\mathcal{Z}(f_1) \subseteq \mathcal{Z}(f_2)$ if and only if each atomic factor of f_1 is up to stable associativity an atomic factor of f_2 .*

Proof. (1) This is a direct consequence of Lemma 4.2 and Theorem 3.4.

(2) By Lemma 4.2, f_i is stably associated to an irreducible pencil L_i for $i = 1, 2$. Since $\mathcal{Z}(L_1) = \mathcal{Z}(L_2)$, L_1 and L_2 are similar, so f_1 and f_2 are stably associated.

(3) Follows by (2) and a factorization of f_i into atoms. \square

Remark 4.4. The conclusions of Theorem 4.3 hold more generally for $f \in M_d(\mathbb{k}\langle \mathbf{x} \rangle)$ satisfying $f(\mathbb{k}^g) \cap \mathrm{GL}_d(\mathbb{k}) \neq \emptyset$, which readily follows from translating \mathbf{x} by a scalar point and multiplying f with an invertible matrix.

For scalar noncommutative polynomials and linear matrix pencils we also give an effective converse to Theorem 4.3(1).

Proposition 4.5. *For $\delta > 1$ and $d > 1$ set*

$$n_1 = \left\lceil \frac{\delta}{2} \right\rceil, \quad n_2 = \begin{cases} 1 & d = 2, \\ \left\lceil (d-1) \sqrt{\frac{2(d-1)^2}{d-2} + \frac{1}{4}} + \frac{d-1}{2} - 2 \right\rceil & d \geq 3. \end{cases}$$

- (1) *Let $f \in \mathbb{k}\langle \mathbf{x} \rangle$ be of degree δ and $f(0) = 1$. If $\det f(\Omega^{(n)})$ is irreducible for some $n \geq n_1$, then f is an atom.*
- (2) *Let L be a monic pencil of size d . If $\det L(\Omega^{(n)})$ is irreducible for some $n \geq n_2$, then L is an atom.*

Proof. (1) If $f = f_1 f_2$ for non-constant $f_1, f_2 \in \mathbb{k}\langle \mathbf{x} \rangle$, then $\deg f_i \leq \delta - 1$, so it suffices to show that $\det f_i(\Omega^{(n)})$ is not constant for $i = 1, 2$. Suppose $\det f_i(\Omega^{(n)}) = 1$. By the Cayley-Hamilton theorem we have

$$(4.5) \quad f_1(\Omega^{(n)}) \operatorname{adj} f_1(\Omega^{(n)}) = 1,$$

where $\operatorname{adj} M$ denotes the adjugate of a square matrix M [MN88, Section 1.9]. Note that $f_1(\Omega^{(n)})$ and $\operatorname{adj} f_1(\Omega^{(n)})$ both belong to the *trace ring* of $n \times n$ matrices, a unital \mathbb{k} -algebra R generated by $\Omega_j^{(n)}$ and $\operatorname{tr}(w(\Omega^{(n)}))$ for $w \in \langle \mathbf{x} \rangle$. By [For84, Section 5], R is a graded domain, where the grading is imposed by the total degree in $M_n(\mathbb{k}[\boldsymbol{\omega}])$. Therefore (4.5) implies $\deg f_1(\Omega^{(n)}) = 0$ and so $f_1(\Omega^{(n)}) = 1$. Hence $1 - f_1$ is a polynomial identity for $n \times n$ matrices of degree at most $\delta - 1$, which contradicts $2n > \delta - 1$.

(2) If L is not an atom, then $\det L(\Omega^{(n)}) = \det L_1(\Omega^{(n)}) \det L_2(\Omega^{(n)})$ for monic pencils L_i of size at most $d - 1$ whose coefficients are not jointly nilpotent. Then $\det L_i(\Omega^{(n)})$ is not constant for $i = 1, 2$ by [KV17, Proposition 3.3]. \square

5. PENCILS WITH POLYNOMIAL FREE LOCUS AND MATRIX PERTURBATIONS

The purpose of this section is twofold. In Subsection 5.3 we characterize monic pencils whose free locus is the free locus of a noncommutative polynomial using state space realization theory. This leads to an efficient algorithm for checking the equality of free loci presented in Subsection 6.1. Using properties of minimal factorizations of realizations we then obtain an intriguing statement about invariant subspaces of rank-one perturbations of jointly nilpotent matrices (Theorem 5.1).

5.1. Perturbations and invariant subspaces. While perturbations of matrices is a classical theory [Kat95], our path takes a different direction than the classical ones. Given a g -tuple $A = (A_1, \dots, A_g)$ of matrices in $M_d(\mathbb{k})$ we shall consider perturbations called **right rank-one perturbations**, namely, ones of the form

$$A_j + b_j c^t \quad j = 1, \dots, g$$

for $b_j, c \in \mathbb{k}^d$. The perturbation is called **non-degenerate** if

- (i) $\{b_1, \dots, b_g\}$ is not contained in a non-trivial invariant subspace for A ,
- (ii) c does not lie in a non-trivial invariant subspace for A^t .

One consequence of Theorem 4.3 relates invariant subspaces for a matrix tuple to invariant subspaces for its right rank-one perturbations.

Theorem 5.1. *Let $A \in M_d(\mathbb{k})^g$ be a tuple of jointly nilpotent matrices. If \mathcal{S} is an invariant subspace of a non-degenerate right rank-one perturbation of A , then there is a complementary space \mathcal{S}^\times , $\mathbb{k}^d = \mathcal{S} \dot{+} \mathcal{S}^\times$, which is invariant under A .*

To prove this we shall use state space systems realizations, a technique closely related to the “linearizations” introduced in the proof of Lemma 4.2. The proof of Theorem 5.1 concludes in Subsection 5.4.1.

5.2. State space realizations. Next we introduce some essential background. Let $\mathbb{k}\langle \mathbf{x} \rangle$ be the free skew field [Coh06] and $\mathbb{k}\langle \mathbf{x} \rangle_0 \subset \mathbb{k}\langle \mathbf{x} \rangle$ the subring of noncommutative rational functions that are regular at the origin [K-VV09, Vol18]. Each $\mathfrak{r} \in \mathbb{k}\langle \mathbf{x} \rangle_0$ admits a **noncommutative Fornasini–Marchesini state space realization** (shortly an FM-realization)

$$(5.1) \quad \mathfrak{r} = \delta + c^t L^{-1} b,$$

where $\delta \in \mathbb{k}$, $c \in \mathbb{k}^d$, $b = \sum_j b_j x_j$ for $b_j \in \mathbb{k}^d$ and $L = I - \sum_j A_j x_j$ for $A_j \in M_d(\mathbb{k})$; see e.g. [BGM05, Section 2.1]. Here $d \in \mathbb{N}$ is the size of realization (5.1).

For future use we recall some well-known facts about minimal FM-realizations.

- (1) We say that the realization (5.1) is **controllable** if

$$\text{span} \{w(A)b_j : w \in \langle \mathbf{x} \rangle, 1 \leq j \leq g\} = \mathbb{k}^d$$

and **observable** if

$$\text{span} \{w(A)^t c : w \in \langle \mathbf{x} \rangle\} = \mathbb{k}^d.$$

By [BGM05, Theorem 9.1], the realization (5.1) is minimal if and only if it is observable and controllable.

- (2) A minimal realization of \mathfrak{r} is unique up to similarity [BGM05, Theorem 8.2].
 (3) The domain of \mathfrak{r} is precisely the complement of $\mathcal{Z}(L)$ if (5.1) is minimal by [K-VV09, Theorem 3.1] and [Vol17, Theorem 3.10].
 (4) Lastly, if $\mathfrak{r}(0) = \delta \neq 0$ (equivalently, $\mathfrak{r}^{-1} \in \mathbb{k}\langle \mathbf{x} \rangle_0$), then by [BGM05, Theorem 4.3] we have an FM-realization

$$(5.2) \quad \mathfrak{r}^{-1} = \delta^{-1} + (-\delta^{-1} c^t) (L^\times)^{-1} (\delta^{-1} b),$$

of \mathfrak{r}^{-1} , where $L^\times = I - \sum_j A_j^\times x_j$ and

$$(5.3) \quad A_j^\times = A_j - \delta^{-1} b_j c^t.$$

Because the realizations (5.1) and (5.2) are of the same size, we see that (5.1) is minimal for \mathfrak{r} if and only if (5.2) is minimal for \mathfrak{r}^{-1} .

Remark 5.2. The linearization trick (4.3) when inverted gives a type of a realization: if $e^t = (1, 0, \dots, 0)$, then

$$f^{-1} = e^t (f^{-1} \oplus I) e = e^t Q^{-1} L^{-1} P^{-1} e = e^t L^{-1} e$$

using that P^{-1} (resp. Q^{-1}) is upper (resp. lower) unitriangular. Recall that L is a monic linear pencil. This is often called a descriptor realization of f^{-1} .

To convert to an FM-realization we use the assumption $f(0) = 1$ which makes $\delta = 1$. Then

$$e^t L^{-1} e - 1 = e^t L^{-1} (I - L) e,$$

so f^{-1} has an FM-realization with $L = I - \sum_j A_j x_j$, $c = e$, $b_j = A_j e$ and $\delta = 1$. Most importantly, the representing pencils for the FM and descriptor realizations are the same.

5.3. Flip-poly pencils. Next we define the pencils to which we shall associate polynomial free loci. A monic pencil $L = I - \sum_j A_j x_j$ is **flip-poly** if $A_j = N_j + E_j$, where N_j are jointly nilpotent matrices and $\text{codim}(\bigcap_j \ker E_j) \leq 1$.

Lemma 5.3. *Let $f \in \mathbb{k}\langle \mathbf{x} \rangle$ and $f(0) = 1$. If L is a monic pencil appearing in a minimal realization of f^{-1} , then L is flip-poly, the intersection of kernels of its coefficients is trivial and $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ for all $n \in \mathbb{N}$.*

Proof. Let $f = 1 - c^t L_0^{-1} b$ with $L_0 = I - \sum_j N_j x_j$ be a minimal realization. Since $\text{dom } f = \mathcal{M}^g$, L_0 is invertible at every matrix point, so $\mathcal{Z}(L_0) = \emptyset$ and hence N_j are jointly nilpotent matrices by [KV17, Corollary 3.4]. Since minimal realizations are unique up to similarity, we can assume that $L = I - \sum_j A_j x_j$ where $A_j = N_j + b_j c^t$. Then L is flip-poly and $f^{-1} = 1 + c^t L^{-1} b$ is a minimal realization by (5.2).

Since

$$\begin{pmatrix} I & 0 \\ c^t L_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} L_0 & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} I & L_0^{-1} b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L_0 & b \\ c^t & 1 \end{pmatrix} = \begin{pmatrix} I & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ c^t & 1 \end{pmatrix}$$

and N_j are jointly nilpotent, we have

$$\det f(\Omega^{(n)}) = \det L_0(\Omega^{(n)}) \det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$$

for all $n \in \mathbb{N}$.

Lastly suppose there exists a nonzero $v \in \bigcap_j \ker A_j$. Since the realization $1 + c^t L^{-1} b$ is observable, it follows that $c^t v \neq 0$. Because $N_j v + (c^t v) b_j = A_j v = 0$ holds for all j , we conclude that

$$\text{span} \{w(N) b_j : w \in \langle \mathbf{x} \rangle, 1 \leq j \leq g\} \subseteq \sum_j \text{ran } N_j,$$

which contradicts controllability of $1 - c^t L_0^{-1} b$. Hence $\bigcap_j \ker A_j = \{0\}$. \square

Proposition 5.4. *For every flip-poly pencil L there exists a flip-poly pencil L_0 such that $\mathcal{Z}(L) = \mathcal{Z}(L_0)$ and L_0 appears in a minimal realization of f^{-1} for some $f \in \mathbb{k}\langle \mathbf{x} \rangle$ with $f(0) = 1$.*

Proof. By assumption we have $L = I - \sum_j (N_j - b_j c^t) x_j$ for some $b_j, c \in \mathbb{k}^d$ and jointly nilpotent $N_j \in M_d(\mathbb{k})$. Let

$$(5.4) \quad f = 1 + c^t \left(I - \sum_j N_j x_j \right)^{-1} \left(\sum_j b_j x_j \right).$$

Since N_j are jointly nilpotent, f is a noncommutative polynomial. By (5.2) we have

$$(5.5) \quad f^{-1} = 1 - c^t L^{-1} \left(\sum_j b_j x_j \right).$$

While the realization (5.5) is not necessarily minimal, it suffices to prove that its minimization results in a realization with a flip-poly pencil L_0 satisfying $\mathcal{Z}(L_0) = \mathcal{Z}(L)$.

Recall that the minimization algorithm comprises of two steps. A starting realization $c^t (I - \sum_j A_j x_j)^{-1} b$ is first restricted to the invariant subspace $\text{span}\{w(A) b_j\}_{w,j}$. The resulting realization $c'^t (I - \sum_j A'_j x_j)^{-1} b'$ is then restricted to the invariant subspace $\text{span}\{w(A')^t c'\}_w$, which yields a minimal realization.

Denote $A_j = N_j - b_j c^t$ and assume $\mathcal{S}_1 = \text{span}\{w(A)b_j\}_{w,j} \neq \mathbb{k}^d$. Let $\mathcal{S}_2 \subset \mathbb{k}^d$ be a complementary space of \mathcal{S}_1 , i.e., $\mathcal{S}_1 \dot{+} \mathcal{S}_2 = \mathbb{k}^d$. For $i = 1, 2$ let $\iota_i : \mathcal{S}_i \rightarrow \mathbb{k}^d$ and $\pi_i : \mathbb{k}^d \rightarrow \mathcal{S}_i$ be the corresponding embeddings and projections, respectively. Then $\pi_1 \circ L \circ \iota_1$ is the pencil produced after the first step of minimization and

$$\mathcal{Z}(L) = \mathcal{Z}(\pi_1 \circ L \circ \iota_1) \cup \mathcal{Z}(\pi_2 \circ L \circ \iota_2).$$

Since $A_j v = N_j v - (c^t v) b_j$ for every $v \in \mathbb{k}^d$ and $b_j \in \mathcal{S}_1$, we see that \mathcal{S}_1 is also invariant under N_j . Therefore $\pi_1 \circ L \circ \iota_1$ is a flip-poly pencil. On the other hand we have $A_j v - N_j v \in \mathcal{S}_1$ for every $v \in \mathbb{k}^d$, so $\pi_2 \circ A_j \circ \iota_2$ are jointly nilpotent and hence $\mathcal{Z}(\pi_2 \circ L \circ \iota_2) = \emptyset$.

Analogous reasoning holds for the second step of minimization, so the pencil appearing in a minimal realization of f^{-1} is flip-poly and its free locus equals $\mathcal{Z}(L)$. \square

Corollary 5.5. *The set of free loci of noncommutative polynomials coincides with the set of free loci of flip-poly pencils.*

Proof. Direct consequence of Lemma 5.3 and Proposition 5.4. \square

Proposition 5.6. *Let L be an irreducible pencil. If L is not flip-poly, then $\mathcal{Z}(L) \neq \mathcal{Z}(f)$ for all $f \in \mathbb{k}\langle \mathbf{x} \rangle$.*

Proof. Suppose $\mathcal{Z}(L) = \mathcal{Z}(f)$ for some $f \in \mathbb{k}\langle \mathbf{x} \rangle$. Since L is irreducible, we can assume that $f(0) = 1$ and f is irreducible by Theorems 3.4 and 4.3. Let L' be a monic pencil appearing in the realization of f^{-1} . By Lemma 5.3 we have

$$L(\Omega^{(n)}) = \det f(\Omega^{(n)}) = \det L'(\Omega^{(n)})$$

for all $n \in \mathbb{N}$. Since the intersection of kernels of coefficients of L' is trivial, we deduce that L' is irreducible by writing it in the form (2.1). Therefore L and L' are similar and hence L is flip-poly. \square

Example 5.7. Assume $A \in M_d(\mathbb{k})$ has an eigenvalue $\lambda \neq 0$ with geometric multiplicity at least 2 and let $b, c \in \mathbb{k}^d$ be arbitrary. Then an easy calculation shows that λ is also an eigenvalue of $A + bc^t$, so $A + bc^t$ is not nilpotent. Therefore a monic pencil having A as one of its coefficients is not flip-poly.

In particular, if $d \geq 3$, then there exist $A_1, A_2 \in M_d(\mathbb{k})$ such that A_1 has a nonzero eigenvalue with geometric multiplicity at least 2 and A_1, A_2 generate $M_d(\mathbb{k})$. For example, one can choose A_2 to be the permutation matrix corresponding to the cycle $(1 \ 2 \ \dots \ d)$ and $A_1 = \text{diag}(1, \dots, 1, -1)$. Then $L = I - A_1 x_1 - A_2 x_2$ is an irreducible pencil that is not flip-poly, so $\mathcal{Z}(L) \neq \mathcal{Z}(f)$ for all $f \in \mathbb{k}\langle \mathbf{x} \rangle$ by Proposition 5.6.

Example 5.8. Let $L = I - A_1 x_1 - A_2 x_2$ for $A, A_2 \in M_2(\mathbb{k})$; we claim $\mathcal{Z}(L) = \mathcal{Z}(f)$ for some $f \in \mathbb{k}\langle \mathbf{x} \rangle$ of degree at most 2.

Looking at the zeros of the polynomial $\det(\omega_1 A_1 + \omega_2 A_2) \in \mathbb{k}[\omega_1, \omega_2]$ we see that there exists a nonzero $u \in \mathbb{k}^2$ such that $A_1 u$ and $A_2 u$ are linearly dependent. If $A_1 u, A_2 u \in \mathbb{k} \cdot u$, then A_1 and A_2 have a common eigenvector, so clearly $\mathcal{Z}(L) = \mathcal{Z}(\ell_1 \ell_2)$ for some affine linear $\ell_i \in \mathbb{k}\langle \mathbf{x} \rangle$. Otherwise we have $A_1 u, A_2 u \in \mathbb{k} \cdot v$ for some $v \in \mathbb{k}^2 \setminus \mathbb{k} \cdot u$. With respect to the basis $\{u, v\}$ of \mathbb{k}^2 we have

$$A_j = \begin{pmatrix} 0 & \alpha_{j1} \\ \alpha_{j2} & \alpha_{j3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha_{j2} & 0 \end{pmatrix} + \begin{pmatrix} \alpha_{j1} \\ \alpha_{j3} \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix},$$

so L is flip-poly and hence $L = \mathcal{Z}(f)$ for some $f \in \mathbb{k}\langle \mathbf{x} \rangle$ of degree at most 2 by (5.2).

5.4. Minimal factorizations. A factorization $\mathfrak{r} = \mathfrak{r}_1 \mathfrak{r}_2$ for $\mathfrak{r}, \mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{k}\langle \mathbf{x} \rangle_0$ is **minimal** if the size of the minimal realization of \mathfrak{r} equals the sum of the sizes of minimal realizations of \mathfrak{r}_1 and \mathfrak{r}_2 . That is, if $\mathfrak{r}_i = 1 + c_i^t L_i^{-1} b_i$ is a minimal realization for $i = 1, 2$, then $\mathfrak{r} = \mathfrak{r}_1 \mathfrak{r}_2$ is a minimal factorization if and only if

$$(5.6) \quad \mathfrak{r} = 1 + \begin{pmatrix} c_1^t & c_2^t \end{pmatrix} \begin{pmatrix} L_1 & b_1 c_2^t \\ 0 & L_2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

is a minimal realization by [BGM05, Theorem 4.1].

Let $\mathfrak{r} = \delta + c^t L^{-1} b$ be a minimal realization of size d and $\delta \neq 0$. In [K-VV09, Section 4] it is explained that by the multivariable noncommutative version of [BGKR08, Theorem 9.3], minimal factorizations of \mathfrak{r} are in one-to-one correspondence with pairs $(\mathcal{S}, \mathcal{S}^\times)$ of subspaces in \mathbb{k}^d such that

- (a) \mathcal{S} is invariant under A_1, \dots, A_g ,
- (b) \mathcal{S}^\times is invariant under $A_1^\times, \dots, A_g^\times$,
- (c) $\mathcal{S} \dot{+} \mathcal{S}^\times = \mathbb{k}^d$.

Proposition 5.9. *Let $f \in \mathbb{k}\langle \mathbf{x} \rangle$ and $f(0) \neq 0$. Minimal factorizations of f are precisely polynomial factorizations of f .*

Proof. Let $f = \mathfrak{r}' \mathfrak{r}''$ be a minimal factorization. Then $\text{dom } f = \text{dom } \mathfrak{r}' \cap \text{dom } \mathfrak{r}''$ by [K-VV09, Theorem 4.2]. Consequently $\text{dom } f = \mathcal{M}^g$ implies $\text{dom } \mathfrak{r}' = \text{dom } \mathfrak{r}'' = \mathcal{M}^g$, so $\mathfrak{r}', \mathfrak{r}'' \in \mathbb{k}\langle \mathbf{x} \rangle$ by [KV17, Theorem 4.2].

Let $f = f' f''$ be a polynomial factorization; without loss of generality let $f(0) = f'(0) = f''(0) = 1$. As already mentioned in the proof of Lemma 5.3, the coefficients of the pencil appearing in a minimal realization of a noncommutative polynomial are jointly nilpotent. By (5.6) it thus suffices to prove the following: if

$$(5.7) \quad 1 + c'^t \left(I - \sum_j N'_j x_j \right)^{-1} \left(\sum_j b'_j x_j \right), \quad 1 + c''^t \left(I - \sum_j N''_j x_j \right)^{-1} \left(\sum_j b''_j x_j \right)$$

are minimal realizations of size d' and d'' , respectively, where N'_j are jointly nilpotent and N''_j are jointly nilpotent, then the “product realization”

$$(5.8) \quad 1 + c^t \left(I - \sum_j N_j x_j \right)^{-1} \left(\sum_j b_j x_j \right),$$

where

$$(5.9) \quad c = \begin{pmatrix} c' \\ c'' \end{pmatrix}, \quad N_j = \begin{pmatrix} N'_j & b'_j c''^t \\ 0 & N''_j \end{pmatrix}, \quad b_j = \begin{pmatrix} b'_j \\ b''_j \end{pmatrix},$$

is minimal.

We now prove that the product system (5.8) is observable. Let $w_0 \in \langle \mathbf{x} \rangle$ and $1 \leq j_0 \leq g$ be such that $\alpha = c'^t w_0 (N') b'_{j_0} \neq 0$ and $c'^t w (N') b'_j = 0$ for all $|w| > |w_0|$ and $1 \leq j \leq g$. Since $\text{span}\{w (N') b'_j\}_{w,j} = \mathbb{k}^{d'}$, it follows that $c'^t w (N') = 0$ for all $|w| > |w_0|$. Denote $w_1 = w_0 x_{j_0}$. We claim that

$$(5.10) \quad \text{span} \left\{ \alpha w (N'')^t c'' + \sum_{i=0}^{|w_1|-1} \beta_i (w_1^i w) (N'')^t c'' : w \in \langle \mathbf{x} \rangle \right\} = \mathbb{k}^{d''}$$

for arbitrary choice of $\beta_1, \dots, \beta_{|w_1|} \in \mathbb{k}$. Here w_1^i is obtained by removing the first i letters in w . Indeed, by induction on $k = 1, \dots, d''$ we show that the sets of rows

$$\left\{ \alpha c^{m_t} w(N'') + \sum_{i=0}^{|w_1|-1} \beta_i c^{m_t}(w_1^i w)(N'') : |w| \geq d'' - k \right\}, \quad \{c^{m_t} w(N'') : |w| \geq d'' - k\}$$

span the same subspace and then (5.10) follows by $\text{span}\{w(N'')^t c'' : w \in \langle \mathbf{x} \rangle\} = \mathbb{k}^{d''}$, which holds by the minimality assumption.

A routine computation next shows that for $w = x_{j_1} \cdots x_{j_\ell}$ we have

$$(5.11) \quad w(N) = \begin{pmatrix} w(N') & \sum_{w=ux_j v} u(N') b'_j c^{m_t} v(N'') \\ 0 & w(N'') \end{pmatrix}.$$

Hence

$$(5.12) \quad c^t w(N) = (c^t w(N') \quad \star)$$

holds for all $w \in \langle \mathbf{x} \rangle$. On the other hand, (5.11) also implies

$$(5.13) \quad c^t(w_1 w)(N) = \left(0 \quad \alpha c^{m_t} w(N'') + \sum_{i=0}^{|w_1|-1} \beta_i c^{m_t}(w_1^i w)(N'') \right)$$

for all $w \in \langle \mathbf{x} \rangle \setminus \{1\}$, where $\beta_i \in \mathbb{k}$ depend on w . Finally, since the first realization in (5.7) is observable and (5.10) holds, (5.12) and (5.13) imply

$$\text{span}\{w(N)^t c : w \in \langle \mathbf{x} \rangle\} = \mathbb{k}^{d'+d''},$$

hence (5.7) is an observable realization. By an analogous argument we check controllability, so (5.7) is a minimal FM-realization. \square

5.4.1. *Proof of Theorem 5.1.* Let $B_j = A_j + b_j c^t$ for $j = 1, \dots, g$ and

$$L_0 = I - \sum_j A_j x_j, \quad L = I - \sum_j B_j x_j, \quad b = \sum_j b_j x_j.$$

In the language of FM-realizations, A_j being nilpotent and $(B_j)_j$ being a non-degenerate perturbation of $(A_j)_j$ means that $1 - c^t L_0^{-1} b$ is a minimal realization of a nonconstant $f \in \mathbb{k}\langle \mathbf{x} \rangle$. By (5.2), $1 + c^t L^{-1} b$ is a minimal realization of f^{-1} .

If \mathcal{S} is a non-trivial invariant subspace for the B_j , then L is similar to

$$\begin{pmatrix} L' & \star \\ 0 & L'' \end{pmatrix}$$

for monic pencils L' and L'' . We claim that the coefficients of L' and L'' are not jointly nilpotent. For this to hold we need to show that $B_j|_{\mathcal{S}}$ are not jointly nilpotent and that the induced operators $\tilde{B}_j : \mathbb{k}^d/\mathcal{S} \rightarrow \mathbb{k}^d/\mathcal{S}$ are not jointly nilpotent.

If $B_j|_{\mathcal{S}}$ are jointly nilpotent, there exists $v \in \cap_j \ker B_j \setminus \{0\}$. Since $\{w(B)^t c\}_w = \mathbb{k}^d$, we have $c^t v \neq 0$. Then $A_j v = B_j v - (c^t v) b_j$ implies $b_j \in \sum_j \text{ran } A_j$, so

$$\{w(A) b_j\}_{w,j} \subseteq \sum_j \text{ran } A_j \neq \mathbb{k}^d$$

because A_j are jointly nilpotent, which contradicts minimality.

If \tilde{B}_j are jointly nilpotent, then $\sum_j \text{ran } \tilde{B}_j \neq \mathbb{k}^d/\mathcal{S}$ and hence $\sum_j \text{ran } B_j \neq \mathbb{k}^d$. Since A_j are jointly nilpotent, there exists $v \in \bigcap_j \ker A_j \setminus \{0\}$ and $c^t v \neq 0$ because $\{w(A)^t c\}_w = \mathbb{k}^d$. Therefore $B_j v = (c^t v) b_j$ implies

$$\{w(B)b_j\}_{w,j} \subseteq \sum_j \text{ran } B_j \neq \mathbb{k}^d,$$

which contradicts minimality.

By Lemma 5.3 we have

$$\det f(\Omega^n) = \det L(\Omega^n) = \det L'(\Omega^n) \det L''(\Omega^n)$$

for all $n \in \mathbb{N}$. This factorization is non-trivial for large n because the coefficients of L' and L'' are not jointly nilpotent. Therefore f is not an atom in $\mathbb{k}\langle \mathbf{x} \rangle$ by Theorem 4.3(1).

For a moment assume that \mathcal{S} is an irreducible invariant subspace for B_j (that is, \mathcal{S} does not contain a smaller nonzero invariant subspace). If $f = f_\ell \cdots f_1$ is a factorization of f into atoms, then it is a minimal factorization by Proposition 5.9, so $f^{-1} = f_1^{-1} \cdots f_\ell^{-1}$ is also a minimal factorization. If $f_i^{-1} = 1 + c_i L_i^{-1} b_i$ is a minimal realization, then

$$(5.14) \quad 1 + \begin{pmatrix} c_1 & c_2 & \cdots & c_\ell \end{pmatrix} \begin{pmatrix} L_1 & b_1 c_2^t & \cdots & b_1 c_\ell^t \\ & L_2 & \cdots & b_2 c_\ell^t \\ & & \ddots & \vdots \\ & & & L_\ell \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_\ell \end{pmatrix}$$

is a minimal realization of f^{-1} by (5.6). The block structure of (5.14) gives us a chain of invariant subspaces for B_j

$$\{0\} = \mathcal{V}_0 \subsetneq \mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_\ell = \mathbb{k}^d$$

such that $\mathcal{V}_{i+1}/\mathcal{V}_i$ are irreducible (for the action of linear maps on quotient spaces induced by B_j). We claim that after a basis change preserving the structure of (5.14) we can assume that $\mathcal{S} = \mathcal{V}_1$. Indeed, if $\mathcal{V}_{i-1} \subsetneq \mathcal{S} \subseteq \mathcal{V}_i$, then $\mathcal{V}_i = \mathcal{V}_{i-1} \dot{+} \mathcal{S}$ and therefore $b_{i'} c_i^t = 0$ for all $i' < i$; applying the basis change corresponding to switching \mathcal{V}_{i-1} and \mathcal{S} thus preserves the structure of (5.14) and results in replacing \mathcal{V}_1 by \mathcal{S} . Hence \mathcal{S} determines a minimal factorization $f^{-1} = f_1^{-1} h^{-1}$, where f_1 is an atom and h is a nonconstant polynomial (since f is not an atom). Due to the correspondence between minimal factorizations and pairs of invariant subspaces it follows that there exists an invariant subspace \mathcal{S}^\times for $B_j^\times = B_j - b_j c^t = A_j$ that is complementary to \mathcal{S} .

Now let \mathcal{S} be an arbitrary non-trivial invariant subspace for B_j . Then we can find a chain of invariant subspaces

$$0 = \mathcal{S}_0 \subsetneq \mathcal{S}_1 \subsetneq \cdots \subsetneq \mathcal{S}_m = \mathcal{S}$$

such that $\mathcal{S}_{i+1}/\mathcal{S}_i$ are irreducible. Inductively applying the reasoning from the previous paragraph we see that the sequence of (quotient) spaces $\mathcal{S}_1, \mathcal{S}_2/\mathcal{S}_1, \dots, \mathcal{S}_m/\mathcal{S}_{m-1}$ yields a minimal factorization $f^{-1} = f_1^{-1} \cdots f_m^{-1} h^{-1}$. Hence \mathcal{S} determines a minimal factorization $f^{-1} = (f_m \cdots f_1)^{-1} h^{-1}$ and we obtain \mathcal{S}^\times as in the previous paragraph. \square

Remark 5.10. Observe that Theorem 4.3(1) (at least for scalar noncommutative polynomials) can be deduced from Theorem 5.1 without using Cohn's semifir factorization theory. Indeed, let $f \in \mathbb{k}\langle \mathbf{x} \rangle$ satisfy $f(0) = 1$ and let $f^{-1} = 1 + c^t L^{-1} b$ be a minimal FM-realization. By Lemma 5.3 we have $\det f(\Omega^n) = \det L(\Omega^n)$, so if $\det f(\Omega^n)$ is not

irreducible for large n , the monic pencil L is not irreducible by Theorem 4.3. Therefore the coefficients of L have a non-trivial invariant subspace \mathcal{S} , so the assumptions of Theorem 5.1 are met and from it we obtain an invariant subspace \mathcal{S}^\times which yields a minimal factorization of f . By Proposition 5.9, f is not an atom, so Theorem 4.3 holds.

5.5. A factorization result with missing variables. Let $\mathbf{y} = (y_1, \dots, y_h)$ be another tuple of freely noncommuting variables. For every $n \in \mathbb{N}$ and $1 \leq j \leq h$ let $\Upsilon_j^{(n)} = (\nu_{j\ell})_{\ell j}$ be an $n \times n$ generic matrix. For every $f \in \mathbb{k}\langle \mathbf{x}, \mathbf{y} \rangle$ we have $\det f(\Omega^{(n)}, \Upsilon^{(n)}) \in \mathbb{k}\langle \boldsymbol{\omega}, \boldsymbol{\nu} \rangle$.

Proposition 5.11. *Let $f \in \mathbb{k}\langle \mathbf{x}, \mathbf{y} \rangle$ and $f(0) = 1$. If $\det f(\Omega^{(n)}, \Upsilon^{(n)})$ is independent of $\Upsilon^{(n)}$ for every $n \in \mathbb{N}$, then $f \in \mathbb{k}\langle \mathbf{x} \rangle$.*

Proof. Without loss of generality let f be an atom. Let $f^{-1} = 1 + c^\dagger L^{-1} b$ be a minimal FM-realization, where

$$b = \sum_{i=1}^g b_i x_i + \sum_{j=1}^h \tilde{b}_j y_j, \quad L = I - \sum_{i=1}^g A_i x_i - \sum_{j=1}^h \tilde{A}_j y_j$$

for $c, b_i, \tilde{b}_j \in \mathbb{k}^d$ and $A_i, \tilde{A}_j \in M_d(\mathbb{k})$. Since $f = 1 - c^\dagger (L + bc^\dagger)^{-1} b$, it suffices to show that $\tilde{A}_j = 0$ and $\tilde{b}_j = 0$ for $j = 1, \dots, h$.

By the assumption we have

$$\det L(\Omega^{(n)}, \Upsilon^{(n)}) = \det f(\Omega^{(n)}, \Upsilon^{(n)}) = \det f(\Omega^{(n)}, 0) = \det L(\Omega^{(n)}, 0)$$

for all $n \in \mathbb{N}$. By [KV17, Proposition 3.3], $\tilde{A}_1, \dots, \tilde{A}_h$ generate a nilpotent ideal in the \mathbb{k} -algebra generated by $A_1, \dots, A_g, \tilde{A}_1, \dots, \tilde{A}_h$. Since f is an atom, L is irreducible by Lemma 5.3 and Proposition 4.5. Then $A_1, \dots, A_g, \tilde{A}_1, \dots, \tilde{A}_h$ generate $M_d(\mathbb{k})$, which is a simple algebra and therefore $\tilde{A}_j = 0$ for all j .

To prove $\tilde{b}_j = 0$ for a fixed $1 \leq j \leq h$ it therefore suffices to show that $c^\dagger w(A) \tilde{b}_j = 0$ for all $w \in \langle \mathbf{x} \rangle$, which we prove by induction on $|w|$. Note that

$$A_1 - b_1 c^\dagger, \dots, A_g - b_g c^\dagger, \tilde{b}_1 c^\dagger, \dots, \tilde{b}_h c^\dagger$$

are jointly nilpotent because $1 - c^\dagger (L + bc^\dagger)^{-1} b$ is a minimal realization of a noncommutative polynomial. Firstly, $c^\dagger \tilde{b}_j = \text{tr}(\tilde{b}_j c^\dagger) = 0$ because $\tilde{b}_j c^\dagger$ is nilpotent. Now suppose that $c^\dagger w(A) \tilde{b}_j = 0$ holds for all w with $|w| \leq \ell$; then also $c^\dagger w(A - bc^\dagger) \tilde{b}_j = 0$ for all w with $|w| \leq \ell$. Hence for every $i = 1, \dots, g$ and $w \in \ell$ we have

$$\begin{aligned} c^\dagger (x_i w)(A) \tilde{b}_j &= c^\dagger (A_i - b_i c^\dagger) w(A) \tilde{b}_j \\ &= c^\dagger (A_i - b_i c^\dagger) w(A - bc^\dagger) \tilde{b}_j \\ &= \text{tr} \left((x_i w)(A - bc^\dagger) \tilde{b}_j c^\dagger \right) \\ &= 0 \end{aligned}$$

because $(x_i w)(A - bc^\dagger) \tilde{b}_j c^\dagger$ is nilpotent. □

Corollary 5.12. *Let $f \in \mathbb{k}\langle \mathbf{x}, \mathbf{y} \rangle$ and $f(0) \neq 0$. If for large n , the polynomial $\det f(\Omega^{(n)}, \Upsilon^{(n)})$ has a factor independent of $\Upsilon^{(n)}$, then f has a factor in $\mathbb{k}\langle \mathbf{x} \rangle$.*

Proof. Immediate consequence of Theorem 4.3 and Proposition 5.11. □

6. ALGORITHMS

In this section we present algorithms based off the results presented above. In Subsection 6.1 we present a simple algorithm for comparing free loci, and in Subsection 6.2 we give an algorithm for factorization of noncommutative polynomials.

6.1. Comparing polynomial free loci. While Theorem 4.3 characterizes the equality of free loci, stable associativity seems evasive to check directly. Thus we now describe a procedure that is more practical for checking equality $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$ for noncommutative polynomials f_1 and f_2 .

Let $f_1, f_2 \in \mathbb{k}\langle x \rangle$ be such that $f_i(0) \neq 0$. Then we can test for $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$ as follows. First we compute minimal FM-realizations for f_1^{-1} and f_2^{-1} , which can be effectively done using algorithms based on linear algebra [BGM05]. Let L_1 and L_2 be the monic pencils appearing in these realizations. Since $\mathcal{Z}(f_1) = \mathcal{Z}(f_2)$ is equivalent to $\mathcal{Z}(L_1) = \mathcal{Z}(L_2)$, it then suffices to compare irreducible blocks in the invariant subspace decomposition for the coefficients of L_1 and L_2 , which can be done using probabilistic algorithms with polynomial complexity [Ebe91, CIW97].

Example 6.1. Let

$$\begin{aligned} f_1 &= 1 + x_1 + x_2 + x_1^2 x_2, \\ f_2 &= 1 + x_1 + x_2 + x_1 x_2 x_1, \\ f_3 &= 1 + x_1 + x_2 + x_2 x_1^2. \end{aligned}$$

Since every affine linear polynomial clearly has an FM-realization of size 1 (note that b in (5.1) is linear), we can use the constructions of FM-realizations associated with the sum, product and inverse [BGM05, Section 4] to build FM-realizations for f_i^{-1} of size 5. After applying the minimization algorithm we obtain minimal realizations $f_i^{-1} = 1 + c^t L_i^{-1} b_i$, where $c^t = (1, 0, 0)$ and

$$\begin{aligned} L_1 &= \begin{pmatrix} 1 + x_1 + x_2 & x_1 & 0 \\ 0 & 1 & -x_1 \\ -x_2 & 0 & 1 \end{pmatrix}, & b_1 &= \begin{pmatrix} -x_1 - x_2 \\ 0 \\ x_2 \end{pmatrix}, \\ L_2 &= \begin{pmatrix} 1 + x_1 + x_2 & x_1 & 0 \\ 0 & 1 & -x_2 \\ -x_1 & 0 & 1 \end{pmatrix}, & b_2 &= \begin{pmatrix} -x_1 - x_2 \\ 0 \\ x_1 \end{pmatrix}, \\ L_3 &= \begin{pmatrix} 1 + x_1 + x_2 & x_2 & 0 \\ 0 & 1 & -x_1 \\ -x_1 & 0 & 1 \end{pmatrix}, & b_3 &= \begin{pmatrix} -x_1 - x_2 \\ 0 \\ x_1 \end{pmatrix}. \end{aligned}$$

It is easy to check that the coefficients of L_1 generate $M_3(\mathbb{k})$, so L_1 is irreducible. Next we consider two homogeneous linear systems $PL_1 = L_2P$ and $PL_1 = L_3P$, where P is a 3×3 matrix of scalar indeterminates. While the second system admits only the trivial solution $P = 0$, the first system has a one-dimensional solution space which intersects $GL_3(\mathbb{k})$. Therefore L_1 and L_3 are similar but L_2 is not similar to L_1 . Consequently,

$$\mathcal{Z}(f_1) = \mathcal{Z}(f_3) \neq \mathcal{Z}(f_2).$$

Alternatively, one can use a computer algebra system to check $\mathcal{L}_n(f_1) \setminus \mathcal{L}_n(f_2) \neq \emptyset$ for a fixed n . For example, we have

$$\left(\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right) \in \mathcal{L}_2(f_1) \setminus \mathcal{L}_2(f_2).$$

6.2. Factorization via state space realizations. We now describe an algorithm for factorization of noncommutative polynomials based on their FM-realizations (cf. [Scr]). As with comparing free loci, the only computational expenses of the algorithm arise from construction and minimization of FM-realizations, and finding an invariant subspace of a tuple of matrices.

Given $f \in \mathbb{k}\langle x \rangle$ with $f(0) = 1$ we first find a minimal realization $f^{-1} = 1 + c^t L^{-1} b$; note that $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ by Lemma 5.3.

- (1) If the coefficients of L do not admit a non-trivial invariant subspace, L is an irreducible pencil, so $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ is irreducible for large n and hence f is an atom.
- (2) If the coefficients of L admit a non-trivial invariant subspace, then we find an irreducible one, \mathcal{S} . With respect to it we have

$$(6.1) \quad f^{-1} = 1 + \begin{pmatrix} c_1^t & c_2^t \end{pmatrix} \begin{pmatrix} L_1 & \star \\ 0 & L_2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for monic pencils L_1 and L_2 . Following the proof of Theorem 5.1 (Subsection 5.4.1), their coefficients are not jointly nilpotent, so f factors. Moreover, since \mathcal{S} is irreducible, the same reasoning as in Subsection 5.4.1 implies that the realization (6.1) necessarily yields a minimal factorization $f^{-1} = f_1^{-1} f_2^{-1}$ and from (5.6) we read off $f_i^{-1} = 1 + c_i^t L_i^{-1} b_i$. Therefore $f = f_2 f_1$ is a polynomial factorization by Proposition 5.9 and f_1 is an atom.

Hence we obtain a factorization of f into atomic factors by repeating (2) until (1) applies. Note that the obtained factorization depends on the choice of irreducible invariant subspaces in (2).

Example 6.2. Let

$$f = 1 + \frac{3}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}(x_1^2 + x_1x_2 + x_2x_1) + \frac{1}{2}x_1x_2x_1.$$

Then f^{-1} admits a minimal realization

$$(6.2) \quad 1 + \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 + \frac{3}{2}x_1 + \frac{1}{2}x_2 & x_1 & -x_1 + \frac{1}{2}x_2 \\ -\frac{3}{2}x_1 - \frac{1}{2}x_2 & 1 & -\frac{1}{2}x_2 \\ -x_1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{3}{2}x_1 - \frac{1}{2}x_2 \\ \frac{3}{2}x_1 + \frac{1}{2}x_2 \\ x_1 \end{pmatrix}.$$

Let L be the monic pencil in (6.2). The coefficients of L have a common eigenvector $(-1, \frac{3}{2}, 1)^t$ and one can check that (2) yields

$$f = (1 + \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_1x_2)(1 + x_1).$$

However, the linear span of $(1, 0, 1)^t$ and $(-1, 1, 0)^t$ is also an irreducible invariant subspace for the coefficients of L , and in this case (2) results in

$$f = (1 + x_1)(1 + \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_2x_1).$$

Let L^\times be the monic pencil appearing in the inverse of the realization (6.2). Its coefficients generate the algebra of strictly upper triangular 3×3 matrices, so they admit exactly two non-trivial invariant subspaces, namely $\text{span}\{(1, 0, 0)^t\}$ and $\text{span}\{(1, 0, 0)^t, (0, 1, 0)^t\}$. From the perspective of Theorem 5.1, the first one is complementary to $\text{span}\{(-1, \frac{3}{2}, 1)^t\}$ and the second one is complementary to $\text{span}\{(1, 0, 1)^t, (-1, 1, 0)^t\}$.

7. SMOOTH POINTS ON A FREE LOCUS

Let L be a monic pencil. In this section we study the relation between smooth points of the free locus $\mathcal{Z}(L)$ and one-dimensional kernels of evaluations of the pencil L . Let us define

$$\begin{aligned} \mathcal{Z}^1(L) &= \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n^1(L), & \mathcal{Z}_n^1(L) &= \{X \in \mathcal{Z}_n(L) : \dim \ker L(X) = 1\}, \\ \mathcal{Z}^{1a}(L) &= \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n^{1a}(L), & \mathcal{Z}_n^{1a}(L) &= \{X \in \mathcal{Z}_n(L) : \dim \ker L(X)^2 = 1\}. \end{aligned}$$

That is, $X \in \mathcal{Z}^1(L)$ if the geometric multiplicity of the zero eigenvalue in $L(X)$ is 1 and $X \in \mathcal{Z}^{1a}(L)$ if the algebraic multiplicity of the zero eigenvalue in $L(X)$ is 1. Note that $\mathcal{Z}_n^{1a}(L) \subseteq \mathcal{Z}_n^1(L)$ are Zariski open subsets of $\mathcal{Z}_n(L)$ for every $n \in \mathbb{N}$. However, the set $\mathcal{Z}_n^1(L)$ can be empty for a fixed n even if L is an irreducible pencil; see Laffey's counterexample to Kippenhahn's conjecture [Laf83].

Recall that $E_{ij} \in M_n(\mathbb{k})$ denote the standard matrix units.

Lemma 7.1. *Let $L = I - \sum_j A_j x_j$ and $X \in \mathcal{Z}_n(L)$.*

- (1) $X \notin \mathcal{Z}_n^1(L)$ if and only if $\text{adj } L(X) = 0$.
- (2) If $\det L(\Omega^{(n)})$ is a minimum degree defining polynomial for $\mathcal{Z}_n(L)$, then $\mathcal{Z}_n(L)$ is singular at X if and only if $\text{tr}(\text{adj } L(X)(A_j \otimes E_{ij})) = 0$ for all $1 \leq j \leq g$ and $1 \leq i, j \leq n$.
- (3) $X \notin \mathcal{Z}_n^{1a}(L)$ if and only if $\text{tr}(\text{adj } L(X)(\sum_j A_j \otimes X_j)) = 0$.

Proof. Firstly, (1) is clear by the definition of the adjugate. Denote $f = \det L(\Omega^{(n)})$ and $p_X = \det L(tX)$ for $X \in M_n(\mathbb{k})^g$. Then Jacobi's formula [MN88, Theorem 8.3.1] for the derivative of a determinant implies

$$\begin{aligned} \nabla f &= - \left(\text{tr}(\text{adj } L(\Omega^{(n)})(A_j \otimes E_{ij})) \right)_{j,i,j}, \\ \frac{dp_X}{dt} &= - \text{tr} \left(\text{adj } L(tX) \left(\sum_j A_j \otimes X_j \right) \right). \end{aligned}$$

Now (2) and (3) follow because $\mathcal{Z}_n(L)$ is singular at X if and only if $(\nabla f)(X) = 0$, and $X \notin \mathcal{Z}_n^{1a}(L)$ if and only if $\frac{dp_X}{dt}(1) = 0$. \square

Theorem 7.2. *If L is an FL-minimal pencil, then*

$$\emptyset \neq \mathcal{Z}_n^{1a}(L) \subseteq \{\text{smooth points of } \mathcal{Z}_n(L)\} \subseteq \mathcal{Z}_n^1(L)$$

for large n .

Proof. By Remark 3.9, $\det L(\Omega^{(n)})$ is a minimum degree defining polynomial for $\mathcal{Z}_n(L)$ for large n . Hence the inclusions hold by Lemma 7.1. Note that

$$\frac{d}{dt} \det L(t\Omega^{(n)}) = -\operatorname{tr} \left(\operatorname{adj} L(t\Omega^{(n)}) \left(\sum_j A_j \otimes \Omega_j^{(n)} \right) \right)$$

and hence

$$\begin{aligned} \deg \det L(\Omega^{(n)}) &= \deg_t \det L(t\Omega^{(n)}) \\ &= 1 + \deg_t \frac{d}{dt} \det L(t\Omega^{(n)}) \\ &= 1 + \deg_t \operatorname{tr} \left(\operatorname{adj} L(t\Omega^{(n)}) \left(\sum_j A_j \otimes \Omega_j^{(n)} \right) \right) \\ &= \deg \operatorname{tr} \left(\operatorname{adj} L(\Omega^{(n)}) \left(\sum_j A_j \otimes \Omega_j^{(n)} \right) \right). \end{aligned}$$

Note that the constant terms of $\operatorname{tr}(\operatorname{adj} L(\Omega^{(n)})(\sum_j A_j \otimes \Omega_j^{(n)}))$ and $\det L(\Omega^{(n)})$ equal 0 and 1, respectively. Since these two polynomials have the same degree, we conclude that $\operatorname{tr}(\operatorname{adj} L(\Omega^{(n)})(\sum_j A_j \otimes \Omega_j^{(n)}))$ is not a multiple of $\det L(\Omega^{(n)})$. Therefore $\mathcal{Z}_n^{1a}(L) \neq \emptyset$ for large n by Theorem 3.4 and Lemma 7.1(2). \square

Example 7.3. Let $L = I - A_1x_1 - A_2x_2$ be as in Example 3.2 and

$$X = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \right), \quad Y = \left(\begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \right).$$

One can check that $\mathcal{Z}_2(L)$ is singular at $X \in \mathcal{Z}_2^1(L)$ and smooth at $Y \in \mathcal{Z}_2(L) \setminus \mathcal{Z}_2^{1a}(L)$. Hence the inclusions in Proposition 7.4 are strict in general.

The next proposition describes the behavior of smooth points when moving between different levels of a free locus.

Proposition 7.4. *Let L be a monic pencil, $X \in \mathcal{Z}(L)$ and $Y \in M_n(\mathbb{k})^g$.*

- (1) *If $Y \in \mathcal{Z}(L)$, then $\mathcal{Z}(L)$ is singular at $X \oplus Y$.*
- (2) *If $Y \notin \mathcal{Z}(L)$ and n is large enough, then $\mathcal{Z}(L)$ is smooth at $X \oplus Y$ if and only if $\mathcal{Z}(L)$ is smooth at X .*

Proof. Since the statement is about the free locus and not L directly, we can assume that L is FL-minimal.

(1) If $X, Y \in \mathcal{Z}(L)$, then $X \oplus Y \notin \mathcal{Z}^1(L)$ and hence $X \oplus Y$ is not a smooth point of $\mathcal{Z}(L)$ by Theorem 7.2.

(2) Observe that $\operatorname{adj}(M_1 \oplus M_2) = (\det M_1 \operatorname{adj} M_2) \oplus (\det M_2 \operatorname{adj} M_1)$ for arbitrary M_1, M_2 . Hence

$$\operatorname{adj} L(X \oplus Y) = (\det L(Y) \operatorname{adj} L(X)) \oplus 0$$

and the equivalence follows by Lemma 7.1(2). \square

The quasi-affine variety $\mathcal{Z}_n^1(L)$ comes equipped with a natural line bundle: to each $X \in \mathcal{Z}^1(L)$ we assign the line $\ker L(X)$. Define $\pi : \mathbb{k}^{dn} \rightarrow \mathbb{k}^d$ by setting $\pi(v) = u_1$ for

$v = \sum_{i=1}^n u_i \otimes e_i \in \mathbb{k}^d \otimes \mathbb{k}^n$. For a monic pencil L define

$$\text{hair}(L) = \bigcup_{X \in \mathcal{Z}^1(L)} \pi(\ker L(X)) \subseteq \mathbb{k}^d.$$

Since $\mathcal{Z}_n^1(L)$ is closed under $\text{GL}_n(\mathbb{k})$ -conjugation, we have

$$\text{hair}(L) = \bigcup_{X \in \mathcal{Z}^1(L)} \left\{ u_i \in \mathbb{k}^d : \sum_i u_i \otimes e_i \in \ker L(X) \right\}.$$

Proposition 7.5. *If L is irreducible of size d , then $\text{span hair}(L) = \mathbb{k}^d$.*

Proof. Let $L = I - \sum_j A_j x_j$ be irreducible and suppose $\text{span hair}(L) \neq \mathbb{k}^d$. Let $P \in \text{M}_d(\mathbb{k})$ be a projection onto $\text{span hair}(L)$ and $L' = I - \sum_j A_j P x_j$. Let $X \in \mathcal{Z}^1(L)$ be arbitrary. If $L(X)v = 0$, then $v = (P \otimes I)v$ and

$$L'(X)v = \left(I - \sum_j A_j P \otimes X_j \right) v = v - \sum_j (A_j P \otimes X_j) v = v - \sum_j (A_j \otimes X_j) v = 0,$$

so $X \in \mathcal{Z}(L')$. Since $\mathcal{Z}_n^1(L)$ contains all the smooth points of $\mathcal{Z}_n(L)$ for large $n \in \mathbb{N}$ by Theorem 7.2 and is therefore dense in $\mathcal{Z}_n(L)$, we conclude that $\mathcal{Z}(L) \subseteq \mathcal{Z}(L')$. By [KV17, Theorem 3.6] there exists a surjective homomorphism $\mathcal{A}' \rightarrow \text{M}_d(\mathbb{k})$ given by $A_j P \mapsto A_j$, where \mathcal{A}' is the \mathbb{k} -algebra generated by $A_1 P, \dots, A_g P$. But $\mathcal{A}' \subsetneq \text{M}_d(\mathbb{k})$, a contradiction. \square

The above results can be also applied to free loci of more general matrices of noncommutative polynomials. For example, we obtain the following.

Corollary 7.6. *Let $f \in \text{M}_d(\mathbb{k}\langle \mathbf{x} \rangle)$ be an atom with $f(\mathbb{k}^g) \cap \text{GL}_d(\mathbb{k}) \neq \emptyset$. Then there exists $X \in \mathcal{Z}(f)$ such that $\dim \ker f(X) = 1$.*

Proof. By Remark 4.4 we can assume $f(0) = I$. By Lemma 4.2, f is stably associated to an irreducible monic pencil L , so there exists $X \in \mathcal{Z}(L)$ satisfying $\dim \ker L(X) = 1$ by Theorem 7.2. By the definition of stable associativity we then have $\dim \ker f(X) = 1$. \square

8. APPLICATIONS TO REAL ALGEBRAIC GEOMETRY

In this section we present two applications of our results to real and convex algebraic geometry. In Corollary 8.5 we prove a density result for points X on the boundary of a free spectrahedron determined by a hermitian pencil L , where the kernel of $L(X)$ is one-dimensional. As a consequence we obtain Corollary 8.7, which improves upon the main result of [HKN14].

8.1. Boundaries of free spectrahedra. Let $\text{H}_n(\mathbb{C})$ denote the \mathbb{R} -space of $n \times n$ hermitian matrices. A monic pencil $L = I - \sum_j A_j x_j$ is **hermitian** if $A_j \in \text{H}_n(\mathbb{C})$ for $1 \leq j \leq g$. Its **free spectrahedron** (also called **LMI domain**) [HKM13] is the set

$$\mathcal{D}(L) = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n(L), \quad \mathcal{D}_n(L) = \{X \in \text{H}_n(\mathbb{C})^g : L(X) \succeq 0\}.$$

Also denote

$$\begin{aligned}\mathcal{Z}^h(L) &= \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n^h(L), & \mathcal{Z}_n^h(L) &= \mathcal{Z}_n(L) \cap \mathbb{H}_n(\mathbb{C})^g, \\ \partial\mathcal{D}(L) &= \bigcup_{n \in \mathbb{N}} \partial\mathcal{D}_n(L), & \partial\mathcal{D}_n(L) &= \mathcal{D}_n(L) \cap \mathcal{Z}_n^h(L), \\ \partial^1\mathcal{D}(L) &= \bigcup_{n \in \mathbb{N}} \partial^1\mathcal{D}_n(L), & \partial^1\mathcal{D}_n(L) &= \partial\mathcal{D}_n(L) \cap \mathcal{Z}_n^1(L).\end{aligned}$$

The set $\mathcal{Z}^h(L)$ is the **free real locus** of L . A hermitian monic pencil L is **LMI-minimal** if it is of minimal size among all hermitian pencils L' satisfying $\mathcal{D}(L') = \mathcal{D}(L)$. Note that if L_1 and L_2 are hermitian pencils, then $\mathcal{Z}(L_1) = \mathcal{Z}(L_2)$ implies $\mathcal{D}(L_1) = \mathcal{D}(L_2)$. Using Burnside's theorem and the hermitian structure of an LMI-minimal L it is then easy to deduce that L is unitarily equivalent to $L_1 \oplus \cdots \oplus L_\ell$, where L_k are pairwise non-similar irreducible hermitian pencils. In particular, every LMI-minimal pencil is also FL-minimal.

Remark 8.1. Instead of hermitian monic pencils in hermitian variables \mathbf{x} as above, one can also consider hermitian monic pencils in non-hermitian variables \mathbf{x} and \mathbf{x}^* , i.e., pencils of the form $L = I - \sum_j A_j x_j - \sum_j A_j^* x_j^*$ for $A_j \in M_d(\mathbb{C})$, with evaluations

$$L(X) = I - \sum_j A_j \otimes X_j - \sum_j A_j^* \otimes X_j^*$$

for $X \in M_n(\mathbb{C})^g$. However, by introducing new hermitian variables $y_j = \frac{1}{2}(x_j + x_j^*)$ and $z_j = \frac{1}{2i}(x_j - x_j^*)$ we observe that results about free real loci and LMI domains of pencils in hermitian variables, for instance those in [KV17], readily translate into results about free real loci and LMI domains of pencils in non-hermitian variables. More concretely, let

$$L' = I - \sum_j (A_j + A_j^*) y_j - \sum_j i(A_j - A_j^*) z_j.$$

Involution-free properties are the same for L and L' ; for example, L is irreducible if and only if L' is irreducible, and $L(\Omega^{(n)} + i\Upsilon^{(n)}, \Omega^{(n)} - i\Upsilon^{(n)})$ is irreducible if and only if $L'(\Omega^{(n)}, \Upsilon^{(n)})$ is irreducible. Likewise, topological relations (e.g., density) among the sets

$$(8.1) \quad \partial^1\mathcal{D}(L) \subset \partial\mathcal{D}(L) \subset \mathcal{Z}^h(L) \subset \mathcal{Z}(L)$$

are the same as those among

$$(8.2) \quad \partial^1\mathcal{D}(L') \subset \partial\mathcal{D}(L') \subset \mathcal{Z}^h(L') \subset \mathcal{Z}(L')$$

because one passes between (8.1) and (8.2) via \mathbb{R} -linear transformations. Using these two observations it becomes clear that the following results (Lemma 8.2, Proposition 8.3, and Corollaries 8.5 and 8.7) also hold in the $(\mathbf{x}, \mathbf{x}^*)$ setup.

Lemma 8.2. *Let L be an LMI-minimal hermitian pencil. Then $\partial^1\mathcal{D}_n(L)$ are precisely the smooth points of $\partial\mathcal{D}_n(L)$ for large n .*

Proof. The polynomial $\det L(\Omega^{(n)})$ is square-free for large n by Remark 3.9. Let us consider $M_n(\mathbb{C})^g = \mathbb{H}_n(\mathbb{C})^g + i\mathbb{H}_n(\mathbb{C})^g$ as the decomposition of the affine space $M_n(\mathbb{C})^g$ into its real and imaginary part. Since L is hermitian, $\det L(\Omega^{(n)})$ is a complex analytic polynomial with real coefficients. Let F be the homogenization of $\det L(\Omega^{(n)})$. Viewed as a real polynomial, F is hyperbolic with respect to the direction of the homogenizing variable

because L is a monic and hermitian. Since $\det L(\Omega^{(n)})$ is square-free, F is also square-free. Thus it follows by [Ren06, Lemma 7] that $\partial^1 \mathcal{D}_n(L)$ are precisely the smooth points of $\partial \mathcal{D}_n(L)$ for large n . \square

Given an LMI-minimal pencil L , smooth points of the boundary of the free spectrahedron of L are (at least for large sizes) characterized as the points where the kernel of L attains minimal dimension. On the other hand, the points where L has maximal kernel are (Euclidean) extreme points of the spectrahedron [RG95, EHKM+, DD-OSS17, ANT+, Kri].

Proposition 8.3. *Let L be an LMI-minimal hermitian pencil. Then $\partial \mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L)$ for large n .*

Proof. Let $L = L_1 \oplus \cdots \oplus L_\ell$, where L_k are pairwise non-similar irreducible hermitian pencils. Fix $1 \leq k \leq \ell$. By the minimality of L we have

$$(8.3) \quad \bigcap_{k' \neq k} \mathcal{D}(L_{k'}) \not\subseteq \mathcal{D}(L_k).$$

Since

$$\bigcup_{k=1}^{\ell} (\partial \mathcal{D}(L_k) \cap \partial \mathcal{D}(L)) \subseteq \partial \mathcal{D}(L) \subseteq \mathcal{Z}(L) = \bigcup_{k=1}^{\ell} \mathcal{Z}(L_k),$$

it suffices to prove that $\partial \mathcal{D}_n(L_k) \cap \partial \mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L_k)$ for large n .

Since the LMI domain of a hermitian monic pencil is a convex set with nonempty interior, then by (8.3) for large n there exists $X_0 \in \mathbb{H}_n(\mathbb{C})^g$ such that $L_k(X_0) \not\prec 0$ and $L_{k'}(X_0) \succ 0$ for $k' \neq k$. Since this is an open condition in Euclidean topology, there exists $\varepsilon > 0$ such that for every $X \in B(X_0, \varepsilon)$ we have $L_k(X) \not\prec 0$ and $L_{k'}(X) \succ 0$ for $k' \neq k$, where $B(X_0, \varepsilon) \subset \mathbb{H}_n(\mathbb{C})^g$ is the closed ball about X_0 with radius ε in Euclidean norm. Let \mathcal{C} be the convex hull of the origin and $B(X_0, \varepsilon)$. By convexity we have $\mathcal{C} \subset \bigcap_{k' \neq k} \mathcal{D}(L_{k'})$ and thus

$$\mathcal{Z}_n^{\text{h}}(L_k) \cap \mathcal{C} \subseteq \partial \mathcal{D}_n(L_k) \cap \partial \mathcal{D}_n(L).$$

Observe that for every $X \in B(X_0, \varepsilon)$ there exists $t \in (0, 1)$ such that $\det L_k(tX) = 0$ by the choice of X_0 and ε . Therefore $\mathcal{Z}_n^{\text{h}}(L_k) \cap \mathcal{C} \subseteq \mathbb{H}_n(\mathbb{C})^g$ is a semialgebraic set of (real) dimension $gn^2 - 1$ by [BCR98, Theorem 2.8.8]. Therefore its Zariski closure in $\mathbb{M}_n(\mathbb{C})^g = \mathbb{H}_n(\mathbb{C})^g + i\mathbb{H}_n(\mathbb{C})^g$ is a hypersurface by [BCR98, Proposition 2.8.2]. Since the latter is contained in $\mathcal{Z}_n(L_k)$, which is an irreducible hypersurface for large n , we conclude that $\mathcal{Z}_n^{\text{h}}(L_k) \cap \mathcal{C}$ is Zariski dense in $\mathcal{Z}_n(L_k)$ and therefore $\partial \mathcal{D}_n(L_k) \cap \partial \mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L_k)$ for large n . \square

Remark 8.4. The essence of the last proof is that $\partial^1 \mathcal{D}_n(L)$ has a nonempty interior with respect to the Euclidean topology on $\mathcal{Z}_n^{\text{h}}(L)$ for large n if L is an irreducible hermitian pencil.

The following statement is a spectrahedral version of the quantum Kippenhahn conjecture (cf. [KV17, Corollary 5.7]).

Corollary 8.5. *Let L be an LMI-minimal hermitian pencil. Then $\partial^1 \mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L)$ for large n .*

Proof. For large n , $\mathcal{Z}_n^1(L)$ is Zariski dense and open in $\mathcal{Z}_n(L)$ by Theorem 7.2 and $\partial^1\mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L)$ by Proposition 8.3. Therefore $\partial^1\mathcal{D}_n(L)$ is Zariski dense in $\mathcal{Z}_n(L)$ for large n . \square

Remark 8.6. Let us consider the real symmetric setup, where the coefficients of a monic pencil L are real symmetric matrices and we are only interested in evaluations of L on tuples of real symmetric matrices. Then the analog of Corollary 8.5 fails in general. For example, let L be a monic symmetric pencil of size $4d$ whose coefficients generate the algebra of $d \times d$ matrices over quaternions; then L is irreducible over \mathbb{R} and thus LMI-minimal as a pencil over \mathbb{R} . However, L is unitarily equivalent to $L' \oplus L'$ for an irreducible hermitian pencil L' , so $\mathcal{Z}^1(L) = \emptyset$. On the positive side, the real version of quantum Kippenhahn’s conjecture holds: if L is a symmetric irreducible pencil over \mathbb{R} , then by [KV17, Corollary 5.8] there exists a tuple of symmetric matrices X such that $\dim \ker L(X) = 2$.

8.2. Randstellensatz. An important result in free real algebraic geometry is the Randstellensatz [HKN14, Theorem 1.1] which describes noncommutative polynomials defining a given LMI domain and its boundary. It holds for monic pencils L satisfying the “zero determining property” [HKN14, Subsection 5.2]. Without going into technical details we assert that every LMI-minimal hermitian pencil L satisfies the zero determining property because $\det L(\Omega^{(n)})$ is a minimum degree defining polynomial for $\mathcal{Z}_n(L)$ by Remark 3.9 and thus also for the Zariski closure of $\partial^1\mathcal{D}_n(L)$ in $\mathbb{H}_n(\mathbb{C})^g$ (which equals $\mathcal{Z}_n^h(L)$) by Corollary 8.5. Thus we obtain the following improvement of [HKN14, Theorem 1.1].

Corollary 8.7. *Let L be an LMI-minimal hermitian pencil of size d and $f \in M_d(\mathbb{C}\langle \mathbf{x} \rangle)$. Then*

$$f|_{\mathcal{D}(L)} \succeq 0 \quad \text{and} \quad \ker L(X) \subseteq \ker f(X) \quad \forall X \in \mathcal{D}(L)$$

if and only if

$$f = L \left(\sum_i q_i^* q_i \right) L + \sum_j (r_j L + C_j)^* L (r_j L + C_j)$$

for $q_i \in \mathbb{C}\langle \mathbf{x} \rangle^d$, $r_j \in M_d(\mathbb{C}\langle \mathbf{x} \rangle)$ and $C_j \in M_d(\mathbb{C})$ satisfying $C_j L = L C_j$.

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