Geometric Hermite interpolation by cubic G^1 splines

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Abstract

In this paper, geometric Hermite interpolation by planar cubic G^1 splines is studied. Three data points and three tangent directions are interpolated per each polynomial segment. Sufficient conditions for the existence of such G^1 spline are determined that cover most of the cases encountered in practical applications. The existence requirements are based only upon geometric properties of data and can easily be verified in advance. The optimal approximation order six is confirmed, too.

Key words: Cubic spline curve, Hermite geometric interpolation, G^1 continuity, Nonlinear equations, Existence, Approximation order.

1 Introduction

Geometric interpolation by parametric polynomial curves has gained a lot of attention since it was introduced in [1]. The interpolants depend only on geometric quantities such as data points, tangent directions, curvatures, etc. But the parameters at which the interpolant should pass through a point, magnitudes of tangents or curvatures may not be prescribed in advance. This often results in a higher approximation order, and in a fact that geometric interpolants please human eye more than their linear counterparts. But, what makes these schemes somewhat difficult are the nonlinear problems included, so questions like the existence of the solution, the approximation order and an efficient implementation need to be considered. This is the reason why most of the results are obtained by asymptotic analysis ([2], [3], [4], [5], [6], [7], etc.). But only a few papers deal with geometric conditions for the existence of the solution ([8], [9], [10], [11], etc.). An excellent recent overview of the results on planar Hermite geometric interpolation is given in [12].

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In this paper, geometric interpolation by planar Hermite cubic G^1 spline curves that interpolate three points and three tangent directions at every segment is studied. Entirely geometric conditions that imply the existence of the interpolant are given, and optimal approximation order is confirmed. The problem considered is the following. Suppose that $2m+1$ points and tangent directions

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad \boldsymbol{d}_i \in \mathbb{R}^2, \qquad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \quad \|\boldsymbol{d}_i\|_2 = 1, \quad i = 0,1,\ldots,2m,$$

are prescribed. Find a regular cubic G^1 spline curve $\boldsymbol{P} : [0,1] \to \mathbb{R}^2$ with breakpoints $(t_{2i})_{i=0}^m$,

$$0 =: t_0 < t_1 < \cdots < t_{2m-1} < t_{2m} := 1,$$

that interpolates the data points $\boldsymbol{T}_i$ and tangent directions $\boldsymbol{d}_i$ at parameters t_i,

$$\boldsymbol{P}(t_i) = \boldsymbol{T}_i, \quad \frac{1}{\|\boldsymbol{P}'(t_i)\|_2}\boldsymbol{P}'(t_i) = \boldsymbol{d}_i, \quad i = 0,1,\ldots,2m, \tag{1}$$

where $(t_{2i-1})_{i=1}^m$ are the unknowns. Note that (1) makes sense even if $\boldsymbol{P}'$ jumps at a breakpoint t_{2i} since the tangent direction is continuous. This interpolation scheme is quite clearly local. Namely, the change of one point or one tangent direction effects only those segments that the point or the direction belongs to. So all the analysis and estimations can be done locally.

For a motivation, let us consider some numerical examples. As the first one suppose that the data are sampled from an exponential and logarithmic spiral

$$\boldsymbol{f}_1(t) := \exp\left(\frac{t}{4}\right)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \; t \in [0,3\pi], \quad \boldsymbol{f}_2(t) := \log(1+t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \; t \in [0,4\pi],$$

at equidistantly chosen parameters in the parameter domain. In Fig 1 interpolating G^1 spline curves composed of five segments, i.e. $m = 5$, are shown for each curve $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. The parametric error estimates ([13]) between $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and their interpolants are

$$\mathrm{dist}(\boldsymbol{P}, \boldsymbol{f}_1) = 0.007915, \quad \mathrm{dist}(\boldsymbol{P}, \boldsymbol{f}_2) = 0.051094.$$

Considering a single segment case, Table 1 numerically suggests that the approximation order is optimal, i.e., 6. However, the data do not need to be sampled from smooth curves only, they can be provided in some other way, maybe given by the user for design purposes, obtained from some other application, etc. The data do not need to be convex either. Fig 2 shows some more examples and, as one can see, the spline follows the shape of the data quite nicely.

The main results of the paper are given in the next two theorems.

Theorem 1 *If on every segment* $[t_{2\ell-2}, t_{2\ell}]$, $\ell = 1,2,\ldots,m$, *one of the sufficient conditions prescribed by Theorem 10, 11, 13 or 15 is fulfilled, then the* G^1 *spline curve* $\boldsymbol{P}$ *exists.*

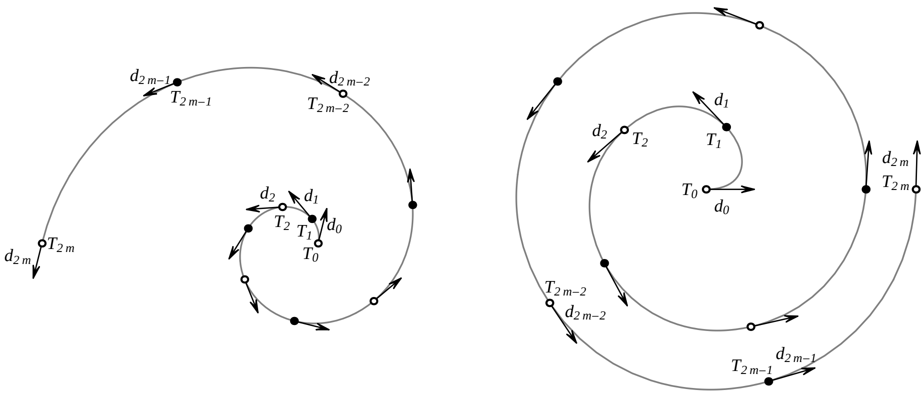

Fig. 1. The interpolating G^1 spline curves $\boldsymbol{P}$ for data obtained from curves $\boldsymbol{f}_1$ (left) and $\boldsymbol{f}_2$ (right).

Table 1
The errors between curves $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ and their polynomial geometric interpolants.

Interval	Approximation error		Decay exponent	
	$\boldsymbol{f}_1$	$\boldsymbol{f}_2$	$\boldsymbol{f}_1$	$\boldsymbol{f}_2$
$[0,\pi]$	3.3754×10^{-2}	2.2251×10^{-1}	/	/
$[0,\frac{9\pi}{10}]$	1.6644×10^{-2}	1.0817×10^{-1}	6.71	6.85
$[0,\frac{8\pi}{10}]$	7.6630×10^{-3}	5.1067×10^{-2}	6.59	6.37
$[0,\frac{7\pi}{10}]$	3.2233×10^{-3}	2.2485×10^{-2}	6.49	6.14
$[0,\frac{6\pi}{10}]$	1.2017×10^{-3}	8.8151×10^{-3}	6.40	6.07
$[0,\frac{5\pi}{10}]$	3.7917×10^{-4}	2.8939×10^{-3}	6.33	6.11
$[0,\frac{4\pi}{10}]$	9.3807×10^{-5}	7.2252×10^{-4}	6.26	6.22
$[0,\frac{3\pi}{10}]$	1.5773×10^{-5}	1.4194×10^{-4}	6.20	6.39

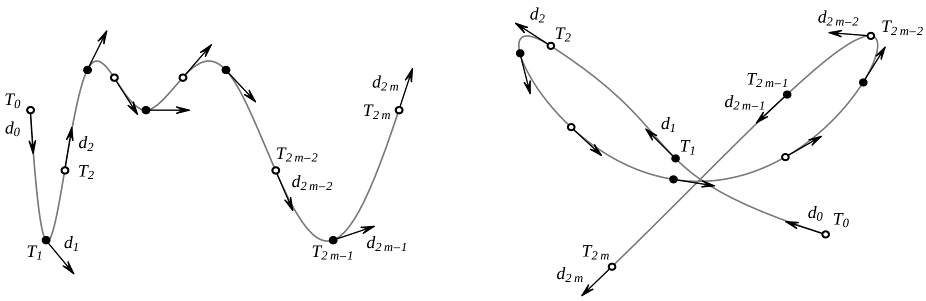

Fig. 2. Cubic G^1 spline curves $\boldsymbol{P}$ for given data points and tangent directions.

Theorem 2 *Suppose that the data are sampled from a smooth convex regular parametric curve* $\boldsymbol{f}:[a,b]\to\mathbb{R}^2$,

$$\boldsymbol{T}_i=\boldsymbol{f}(s_i),\ \ \boldsymbol{d}_i=\frac{1}{\|\boldsymbol{f}'(s_i)\|_2}\boldsymbol{f}'(s_i),\ \ a=s_0<s_1<\cdots<s_{2m}=b,\ i=0,1,\ldots,2m,$$

and let $h := \max\{\Delta s_i:\ i = 1,2,\ldots,2m-1\}$*. Then one can find a constant* $h_0 > 0$ *such that for all h,* $0 < h \le h_0$*, a cubic* G^1 *spline curve* $\boldsymbol{P}$ *that satisfies* (1) *exists, and approximates* $\boldsymbol{f}$ *with the optimal approximation order six.*

The outline of the paper is the following. In Section 2 a system of equations is derived for a single segment case, and in Section 3 the conditions that imply its solution to attain the values that are not allowed are given. These results together with the number of solutions for some particular data lead to the existence theorems of Section 4. They are proved in Section 5 by the help of convex homotopy and Brouwer's degree argument. The last section deals with asymptotic analysis and contains the proof of Theorem 2.

2 Single segment case

Since the G^1 interpolation scheme (1) is local, all the properties can be determined from the simplest case $m = 1$. So from now on to the end of the paper we assume $m = 1$. The equations (1) simplify to

$$\boldsymbol{P}(t_i) = \boldsymbol{T}_i, \qquad \boldsymbol{P}'(t_i) = \alpha_i \boldsymbol{d}_i, \quad \alpha_i > 0, \qquad i = 0,1,2, \tag{2}$$

with $0 := t_0 < t_1 < t_2 := 1$. One is thus left with twelve equations for eight unknown coefficients of $\boldsymbol{P}$, and four unknown parameters t_1, α_0, α_1 and α_2.

Remark 3 *If the tangent directions are not normalized the existence of the solution of* (2) *is not affected. Only the magnitudes of* α_i*,* $i = 0,1,2$*, change.*

The first step is to separate the unknown coefficients from the rest of the unknowns. For any t_1, α_0, α_1 and α_2 there exists a unique polynomial $\boldsymbol{P}_5$ of degree ≤ 5 that solves the interpolation problem (2). But this $\boldsymbol{P}_5$ will be of degree three, i.e., $\boldsymbol{P}_5 = \boldsymbol{P}$, iff the coefficients at powers 4 and 5 are zero. This is true iff

$$[t_0,t_0,t_1,t_1,t_2]\boldsymbol{P}_5 = 0, \quad [t_0,t_1,t_1,t_2,t_2]\boldsymbol{P}_5 = 0, \tag{3}$$

which gives the system of four equations for four unknowns t_1, α_0, α_1, α_2, that must lie in an open set

$$\mathscr{D} := \{t_1;\ 0 < t_1 < 1\} \times \{(\alpha_0,\alpha_1,\alpha_2);\ \alpha_i > 0,\ i = 0,1,2\}.$$

Establishing these parameters is the only nonlinear part of the problem. The coefficients of $\boldsymbol{P}$ are then obtained by using any standard interpolation scheme componentwise. Since $t_0 = 0$ and $t_2 = 1$ the equations (3) simplify to

$$\begin{aligned} \frac{\alpha_0}{t_1^2}\boldsymbol{d}_0 + \frac{\alpha_1}{(1-t_1)\,t_1^2}\boldsymbol{d}_1 - \frac{(2+t_1)}{t_1^3}\Delta\boldsymbol{T}_0 - \frac{1}{(1-t_1)^2}\Delta\boldsymbol{T}_1 = 0, \\ \frac{\alpha_1}{(1-t_1)^2\,t_1}\boldsymbol{d}_1 + \frac{\alpha_2}{(1-t_1)^2}\boldsymbol{d}_2 - \frac{1}{t_1^2}\Delta\boldsymbol{T}_0 + \frac{(t_1-3)}{(1-t_1)^3}\Delta\boldsymbol{T}_1 = 0. \end{aligned} \tag{4}$$

To simplify the analysis it will be assumed from now on that the points $\boldsymbol{T}_0$, $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are not collinear. Using $\det(\cdot, \Delta\boldsymbol{T}_0)$ and $\det(\cdot, \Delta\boldsymbol{T}_1)$ on (4) one obtains

$$\begin{aligned} \frac{\alpha_0}{t_1^2}\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_0) - \frac{\alpha_1}{(1-t_1)\,t_1^2}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1) + \frac{1}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0, \\ \frac{\alpha_0}{t_1^2}\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_1) + \frac{\alpha_1}{(1-t_1)\,t_1^2}\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1) - \frac{(2+t_1)}{t_1^3}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0, \\ \frac{\alpha_1}{(1-t_1)^2\,t_1}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1) + \frac{\alpha_2}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_2) + \frac{(t_1-3)}{(1-t_1)^3}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0, \\ \frac{\alpha_1}{(1-t_1)^2\,t_1}\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1) - \frac{\alpha_2}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_1, \boldsymbol{d}_2) - \frac{1}{t_1^2}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0. \end{aligned} \tag{5}$$

Let us define the constants that are determined by the data as

$$\begin{aligned} \lambda_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_0)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \lambda_2 := \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \\ \lambda_3 &:= \frac{\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \lambda_4 := \frac{\det(\Delta\boldsymbol{T}_1, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \\ \mu_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \mu_2 := \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}. \end{aligned} \tag{6}$$

Their signs have a clear geometric interpretation as one can see in Fig 3. With these

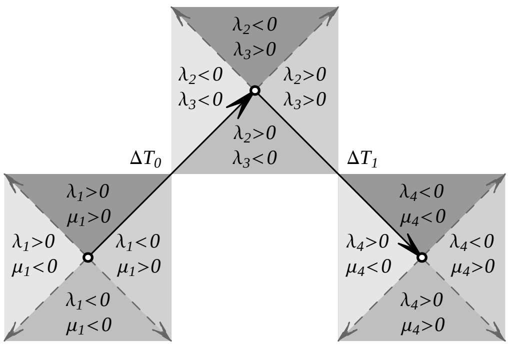

Fig. 3. Geometric interpretation of signs of the constants λ_i and μ_i.

constants, the equations (5) become

$$\boldsymbol{F}(t_1, \boldsymbol{\alpha}) := \boldsymbol{F}(t_1, \boldsymbol{\alpha}; \boldsymbol{\lambda}, \mu_1, \mu_2) := (F_i(t_1, \boldsymbol{\alpha}))_{i=1}^4 = \boldsymbol{0},$$

where

$$
\begin{aligned}
F_1(t_1,\boldsymbol{\alpha}) &:= \lambda_1 \frac{\alpha_0}{t_1^2} - \lambda_2 \frac{\alpha_1}{t_1^2\,(1-t_1)} + \frac{1}{(1-t_1)^2} = 0,\\
F_2(t_1,\boldsymbol{\alpha}) &:= \mu_1 \frac{\alpha_0}{t_1^2} + \lambda_3 \frac{\alpha_1}{t_1^2\,(1-t_1)} - \frac{(2+t_1)}{t_1^3} = 0,\\
F_3(t_1,\boldsymbol{\alpha}) &:= \mu_2 \frac{\alpha_2}{(1-t_1)^2} + \lambda_2 \frac{\alpha_1}{(1-t_1)^2\,t_1} - \frac{(3-t_1)}{(1-t_1)^3} = 0,\\
F_4(t_1,\boldsymbol{\alpha}) &:= \lambda_4 \frac{\alpha_2}{(1-t_1)^2} - \lambda_3 \frac{\alpha_1}{(1-t_1)^2\,t_1} + \frac{1}{t_1^2} = 0.
\end{aligned}
\tag{7}
$$

Here $\boldsymbol{\alpha} := (\alpha_i)_{i=0}^2$, $\boldsymbol{\lambda} := (\lambda_i)_{i=1}^4$. Moreover, if

$$
\lambda_1\lambda_3 + \lambda_2\mu_1 \neq 0, \quad \lambda_2\lambda_4 + \lambda_3\mu_2 \neq 0,
\tag{8}
$$

equations (7) can be rewritten as

$$
\begin{aligned}
\alpha_0 &= \frac{\left(t_1^3 - 3t_1 + 2\right)\lambda_2 - t_1^3\lambda_3}{(t_1-1)^2\,t_1\,(\lambda_1\lambda_3 + \lambda_2\mu_1)},\\
\alpha_1 &= -\frac{\mu_1 t_1^3 + \left(t_1^3 - 3t_1 + 2\right)\lambda_1}{(t_1-1)\,t_1\,(\lambda_1\lambda_3 + \lambda_2\mu_1)},\\
\alpha_2 &= \frac{(t_1-3)\,t_1^2\lambda_3 - (t_1-1)^3\lambda_2}{(t_1-1)\,t_1^2\,(\lambda_2\lambda_4 + \lambda_3\mu_2)},
\end{aligned}
\tag{9}
$$

and

$$
\frac{\mu_1 t_1^3 + \left(t_1^3 - 3t_1 + 2\right)\lambda_1}{\lambda_1\lambda_3 + \lambda_2\mu_1} + \frac{\mu_2\,(t_1-1)^3 + (t_1-3)\,t_1^2\lambda_4}{\lambda_2\lambda_4 + \lambda_3\mu_2} = 0.
\tag{10}
$$

The only nonlinear part remained is (10) which is a cubic equation for t_1 that can easily be solved numerically. The next lemma follows immediately.

Lemma 4 *A cubic polynomial curve* $\boldsymbol{P}$ *that satisfies* (2) *does not exist in any of the following cases:*

$$
\begin{aligned}
&1.\ \lambda_2 \leq 0 \quad \textit{and} \quad \lambda_1 \geq 0, \qquad && 2.\ \lambda_3 \leq 0 \quad \textit{and} \quad \lambda_4 \geq 0,\\
&3.\ \lambda_3 \leq 0 \quad \textit{and} \quad \mu_1 \leq 0, \qquad && 4.\ \lambda_2 \leq 0 \quad \textit{and} \quad \mu_2 \leq 0.
\end{aligned}
$$

As can be seen in Fig 3 cases where one of λ_i is equal to zero are very exceptional and for the sake of simplicity it will be assumed from now on that $\lambda_i \neq 0$, $i = 1,2,3,4$. The first step to the existence of $\boldsymbol{P}$ is to find the relations between the data that force the solution of (7) to approach the boundary $\partial\mathscr{D}$. This analysis is given in the next section.

3 Relations, implying the solution to approach the boundary

If the solution $(t_1, \boldsymbol{\alpha})$ touches the boundary $\partial\mathscr{D}$, it attains the values that are not allowed. As it turns out this implies certain relations between data that could be used to avoid the parameter choices that are not admissible. The next two lemmas reveal the relations for $t_1 \to 0, 1$.

Lemma 5 *Suppose that $\lambda_i \neq 0$, $i = 1,2,3,4$. Parameter t_1 tends to 0 if*

$$\mu_2 \to \varphi_2(\boldsymbol{\lambda}, \mu_1) := \frac{2\lambda_1\lambda_2\lambda_4}{\lambda_2\mu_1 - \lambda_1\lambda_3},$$

$\lambda_1\lambda_2 > 0$ *and*

$$\begin{aligned}
&\lambda_3 > 0, \quad \lambda_4 > 0, \quad -\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2}, \quad \textit{or}\\
&\lambda_3 > 0, \quad \lambda_4 < 0, \quad \frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1, \quad \textit{or}\\
&\lambda_3 < 0, \quad \lambda_4 < 0, \quad -\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1.
\end{aligned}$$

Under these conditions $\lambda_1\varphi_2(\boldsymbol{\lambda}, \mu_1) < 0$.

Lemma 6 *Suppose that $\lambda_i \neq 0$, $i = 1,2,3,4$. Parameter t_1 tends to 1 if*

$$\mu_1 \to \varphi_1(\boldsymbol{\lambda}, \mu_2) := \frac{2\lambda_1\lambda_3\lambda_4}{\lambda_3\mu_2 - \lambda_2\lambda_4},$$

$\lambda_3\lambda_4 > 0$ *and*

$$\begin{aligned}
&\lambda_1 > 0, \quad \lambda_2 > 0, \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3}, \quad \textit{or}\\
&\lambda_1 < 0, \quad \lambda_2 > 0, \quad \frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2, \quad \textit{or}\\
&\lambda_1 < 0, \quad \lambda_2 < 0, \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2.
\end{aligned}$$

Under these conditions $\lambda_4\varphi_1(\boldsymbol{\lambda}, \mu_2) < 0$.

PROOF. From the symmetry between λ_i, λ_{5-i}, $i = 1,2$, and μ_1, μ_2, and the symmetry in equations (7) it is enough to prove Lemma 5 only. If $\lambda_1\lambda_3 + \lambda_2\mu_1 = 0$, the first two equations in (7) become

$$\alpha_0 = \frac{-t_1^2 + (1-t_1)\,\alpha_1\lambda_2}{(1-t_1)^2\,\lambda_1}, \qquad \frac{\lambda_3}{(1-t_1)^2\,\lambda_2} - \frac{t_1+2}{t_1^3} = 0.$$

Therefore it is clear that t_1 cannot approach zero or one. The same is true when $\lambda_2\lambda_4+\lambda_3\mu_2=0$. Thus let us assume that conditions (8) hold. Equation (10) can be rewritten as

$$\frac{2\lambda_1}{\lambda_1\lambda_3+\lambda_2\mu_1}-\frac{\mu_2}{\lambda_2\lambda_4+\lambda_3\mu_2}+\mathscr{O}(t_1)=0,$$

and $t_1\to 0$ implies $\mu_2\to\varphi_2(\boldsymbol{\lambda},\mu_1)$. Moreover,

$$\lim_{t_1\to 0}(t_1\alpha_0)=\frac{2\lambda_2}{\lambda_1\lambda_3+\lambda_2\mu_1},\quad \lim_{t_1\to 0}(t_1\alpha_1)=\frac{2\lambda_1}{\lambda_1\lambda_3+\lambda_2\mu_1},$$
$$\lim_{t_1\to 0}\left(t_1^2\alpha_2\right)=-\frac{\lambda_2}{\lambda_2\lambda_4+\lambda_3\mu_2}.$$

It is now easy to check that $\boldsymbol{\alpha}$ is positive iff the conditions in lemma are fulfilled, and further that $\lambda_1\varphi_2(\boldsymbol{\lambda},\mu_1)<0$, which completes the proof.

The relations implied by $\alpha_i=0$ are given in the next lemma.

Lemma 7 *Parameter* $\alpha_0=0$ *if* $\lambda_i>0$ *for* $i=2,3,4$*, and* $\mu_2=\phi_2(\boldsymbol{\lambda})$*, where*

$$\phi_2(\boldsymbol{\lambda}):=\lambda_4\frac{\tau(\lambda_2,\lambda_3)^2(3-2\tau(\lambda_2,\lambda_3))}{(1-\tau(\lambda_2,\lambda_3))^2(1+2\tau(\lambda_2,\lambda_3))}>0,$$

and $\tau(\lambda_2,\lambda_3)$ *is defined as a unique solution* t_1 *of the problem*

$$g(t_1;\lambda_2,\lambda_3):=\frac{1}{\lambda_2}\frac{t_1^2}{1-t_1}-\frac{1}{\lambda_3}\frac{(1-t_1)(2+t_1)}{t_1}=0,\quad 0<t_1<1. \tag{11}$$

Similarly, parameter $\alpha_2=0$ *if* $\lambda_i>0$ *for* $i=1,2,3$*, and* $\mu_1=\phi_1(\boldsymbol{\lambda})$*, where*

$$\phi_1(\boldsymbol{\lambda}):=\lambda_1\frac{\tau(\lambda_3,\lambda_2)^2(3-2\tau(\lambda_3,\lambda_2))}{(1-\tau(\lambda_3,\lambda_2))^2(1+2\tau(\lambda_3,\lambda_2))}>0.$$

Moreover, parameter $\alpha_1=0$ *if* $\lambda_1<0$*,* $\lambda_4<0$*,* $\mu_1>0$ *and*

$$\mu_2=\psi_2(\lambda_1,\lambda_4,\mu_1):=-\lambda_4\frac{(3-\tau(-\lambda_1,\mu_1))\,\tau(-\lambda_1,\mu_1)^2}{(1-\tau(-\lambda_1,\mu_1))^3}>0.$$

PROOF. First let us prove that for $\lambda_2>0$ and $\lambda_3>0$, the equation (11) has a unique solution. Since

$$\lim_{t_1\downarrow 0}g(t_1;\lambda_2,\lambda_3)=-\text{sign}\left(\frac{1}{\lambda_3}\right)\cdot\infty=-\infty,\quad \lim_{t_1\uparrow 1}g(t_1;\lambda_2,\lambda_3)=\text{sign}\left(\frac{1}{\lambda_2}\right)\cdot\infty=\infty, \tag{12}$$
$$\lim_{t_1\to-\infty}g(t_1;\lambda_2,\lambda_3)=\text{sign}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)\cdot\infty,\quad \lim_{t_1\to\infty}g(t_1;\lambda_2,\lambda_3)=-\text{sign}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_3}\right)\cdot\infty,$$

there exists at least one $t_1 \in (0,1)$ that solves (11). It is straightforward to compute that when $\lambda_2 = \lambda_3$ a solution is unique, i.e., $\tau(\lambda_2, \lambda_3) := 2/3$. Now, for $\lambda_2 \neq \lambda_3$ the only possible solutions are

$$\tilde{t}_1 = \frac{\lambda_2(\lambda_3 - \lambda_2) - \sigma^{\frac{2}{3}}}{(\lambda_3 - \lambda_2)\sigma^{\frac{1}{3}}}, \tag{13}$$

$$\tilde{t}_1^{\pm} = \frac{(1 \pm i\sqrt{3})\lambda_2(\lambda_3 - \lambda_2) + (-1 \pm i\sqrt{3})\sigma^{\frac{2}{3}}}{2(\lambda_2 - \lambda_3)\sigma^{\frac{1}{3}}}, \tag{14}$$

where

$$\sigma := \lambda_2(\lambda_3 - \lambda_2)\left(\lambda_2 - \lambda_3 + \sqrt{\lambda_3(\lambda_3 - \lambda_2)}\right).$$

For $\lambda_3 > \lambda_2$ it is clear that σ is a real number, so solutions (14) are complex, thus (13) is a unique admissible solution and $\tau(\lambda_2, \lambda_3) := \tilde{t}_1$. For $\lambda_3 < \lambda_2$ function g has three real zeros, but it follows from (12) that only one is in $(0,1)$, namely $\tau(\lambda_2, \lambda_3) := \tilde{t}_1^{+}$.

Let us now analyse the case $\alpha_0 = 0$. It is easy to see from (7) that none of λ_i, $i = 2,3,4$, can be equal to zero. From the first and the third equation in (7) one obtains

$$\alpha_1 = \frac{1}{\lambda_2}\frac{t_1^2}{1 - t_1}, \quad \mu_2 = \frac{3 - 2t_1}{(1 - t_1)\alpha_2}, \tag{15}$$

and from the remaining equations

$$\alpha_1 = \frac{1}{\lambda_3}\frac{(1 - t_1)(2 + t_1)}{t_1}, \quad \alpha_2 = \frac{1}{\lambda_4}\frac{(1 - t_1)(1 + 2t_1)}{t_1^2}. \tag{16}$$

Therefore it follows that the system (15)–(16) has an admissible solution $\alpha_1 > 0$, $\alpha_2 > 0$ and $t_1 \in (0,1)$ iff $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$ and $\mu_2 = \phi_2(\boldsymbol{\lambda})$. The proof for the case $\alpha_2 = 0$ is symmetric to this one and it will be omitted.

Suppose now that $\alpha_1 = 0$. It is clear from (7) that neither λ_1 nor λ_4 can be equal to zero. From the first and the last equation in (7) one obtains

$$\alpha_0 = -\frac{t_1^2}{(t_1 - 1)^2\lambda_1}, \quad \alpha_2 = -\frac{(t_1 - 1)^2}{t_1^2\lambda_4}.$$

The remaining two equations then simplify to

$$-\frac{t_1 + 2}{t_1^3} - \frac{\mu_1}{(t_1 - 1)^2\lambda_1} = 0, \quad \frac{3 - t_1}{(t_1 - 1)^3} - \frac{\mu_2}{t_1^2\lambda_4} = 0. \tag{17}$$

Since the solution must be in $\mathscr{D}$, it is clear that $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$ and $\mu_2 > 0$. Multiplying the first equation in (17) by $\dfrac{1}{\mu_1}t_1^2(1 - t_1)$ it rewrites to $g(t_1; -\lambda_1, \mu_1) =$

0. Since $-\lambda_1 > 0$ and $\mu_1 > 0$ there exists a unique $\tau(-\lambda_1, \mu_1) \in (0,1)$ that solves it. From (17) it then follows that

$$\mu_1 > 0, \quad \mu_2 = \psi_2(\lambda_1, \lambda_4, \mu_1), \quad \text{or equivalently} \quad \mu_2 > 0, \quad \mu_1 = \psi_1(\lambda_1, \lambda_4, \mu_2),$$

where $\psi_1(\lambda_1, \lambda_4, \mu) := \psi_2(\lambda_4, \lambda_1, \mu)$. This completes the proof.

The following properties of functions φ_i, ϕ_i and ψ_i, $i = 1,2$, will be needed.

Lemma 8 *If* $\boldsymbol{\lambda} > 0$ *then*

$$-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < -\frac{\lambda_2\lambda_4}{\lambda_3}, \tag{18}$$

$$-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < -\frac{\lambda_1\lambda_3}{\lambda_2}. \tag{19}$$

Also, if $\boldsymbol{\lambda} < 0$, *then*

$$\mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < -\frac{\lambda_2\lambda_4}{\lambda_3}, \tag{20}$$

$$\mu_2 > -\frac{\lambda_2\lambda_4}{\lambda_3} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < -\frac{\lambda_1\lambda_3}{\lambda_2}. \tag{21}$$

Furthermore, for $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$ *and* $\lambda_4 < 0$ *the following implication holds:*

$$\mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \varphi_2(\boldsymbol{\lambda}, \mu_1) < \frac{\lambda_2\lambda_4}{\lambda_3} + \frac{2\lambda_1\lambda_4}{\mu_1} < \frac{\lambda_2\lambda_4}{\lambda_3}. \tag{22}$$

Lemma 9 *For* $\boldsymbol{\lambda} > 0$ *functions* ϕ_i *are limited from below as*

$$\frac{\lambda_1\lambda_3}{\lambda_2} < \phi_1(\boldsymbol{\lambda}) \quad \textit{and} \quad \frac{\lambda_2\lambda_4}{\lambda_3} < \phi_2(\boldsymbol{\lambda}). \tag{23}$$

Moreover, for $\boldsymbol{\lambda} < 0$

$$\varphi_2(\boldsymbol{\lambda}) < \psi_2(\lambda_1, \lambda_4, \mu_1) \quad \textit{and} \quad \varphi_1(\boldsymbol{\lambda}) < \psi_1(\lambda_1, \lambda_4, \mu_2), \tag{24}$$

$$\frac{2\lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} < \psi_2(\lambda_1, \lambda_4, \mu_1), \tag{25}$$

$$\mu_1 > 0, \quad \mu_2 > \frac{2\lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}) < \mu_1. \tag{26}$$

The proofs of Lemma 8 and Lemma 9 are elementary, but very technical. They can easily be done by using a Computer Algebra system's symbolic facilities, thus they will be omitted.

4 Main theorems

In this section sufficient conditions that imply the existence of a cubic geometric interpolant $\boldsymbol{P}$ that satisfies (2) will be given. There are twelve possibilities for the signs of λ_i as shown in Table 2. Lemma 4 shows that for the last seven options the solution of (7) does not exist. Other possibilities are considered in the following theorems.

Table 2
Twelve possibilities for the signs of λ_i.

sign(λ_1)	+	+	−	+	−	−	−	−	−	+	+	+	+	−	+	−
sign(λ_2)	+	+	+	+	−	+	+	−	−	+	−	−	−	+	−	−
sign(λ_3)	+	+	+	−	+	+	−	+	−	−	+	−	+	−	−	−
sign(λ_4)	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+

Theorem 10 *Suppose that the data $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, satisfy* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$ *and $\boldsymbol{\lambda} > 0$. If one of the listed cases*

(1) $\mu_1 > \phi_1(\boldsymbol{\lambda})$ *and* $\mu_2 > \phi_2(\boldsymbol{\lambda})$,
(2) $\frac{\lambda_1\lambda_3}{\lambda_2} \leq \mu_1 < \phi_1(\boldsymbol{\lambda})$ *and* $\mu_2 < \phi_2(\boldsymbol{\lambda})$,
(3) $\frac{\lambda_2\lambda_4}{\lambda_3} \leq \mu_2 < \phi_2(\boldsymbol{\lambda})$ *and* $\mu_1 < \phi_1(\boldsymbol{\lambda})$,
(4) $-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2}$ *and* $\varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \phi_2(\boldsymbol{\lambda})$,
(5) $-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3}$ *and* $\varphi_1(\boldsymbol{\lambda}, \mu_2) < \mu_1 < \phi_1(\boldsymbol{\lambda})$,

holds, then the interpolating curve $\boldsymbol{P}$ that satisfies (2) *exists.*

Theorem 11 *Suppose that the data $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, satisfy* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$, *$\lambda_i > 0$, $i = 1,2,3$, and $\lambda_4 < 0$. If*

$$\mu_1 > \phi_1(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_2 > \varphi_2(\boldsymbol{\lambda}, \mu_1), \quad \text{or}$$
$$\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \phi_1(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_2 < \varphi_2(\boldsymbol{\lambda}, \mu_1),$$

then the interpolating curve $\boldsymbol{P}$ that satisfies (2) *exists.*

Remark 12 *The case $\lambda_1 < 0$ and $\lambda_i > 0$, $i = 2,3,4$, is symmetric to the one considered in Theorem 11. The result is the following. If $\mu_2 > \phi_2(\boldsymbol{\lambda})$ and $\mu_1 > \varphi_1(\boldsymbol{\lambda}, \mu_2)$, or $\dfrac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \phi_2(\boldsymbol{\lambda})$ and $\mu_1 < \varphi_1(\boldsymbol{\lambda}, \mu_2)$, then the interpolating curve $\boldsymbol{P}$ that satisfies* (2) *exists.*

Theorem 13 *Suppose that the data $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, satisfy* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$

and $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\lambda_4 < 0$. *If*

$$-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 \quad \textit{and} \quad \varphi_2(\boldsymbol{\lambda},\mu_1) < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3} + \frac{2\lambda_1\lambda_4}{\mu_1},$$

then the interpolating curve $\boldsymbol{P}$ *that satisfies* (2) *exists.*

Remark 14 *The case where* $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$, $\lambda_4 > 0$ *is symmetric to the one considered in Theorem 13, and the result is the following. If*

$$-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 \quad \textit{and} \quad \varphi_1(\boldsymbol{\lambda},\mu_2) < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2} + \frac{2\lambda_1\lambda_4}{\mu_2},$$

then the interpolating curve $\boldsymbol{P}$ *that satisfies* (2) *exists.*

Theorem 15 *Suppose that the data* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, *satisfy* $\det(\Delta\boldsymbol{T}_0,\Delta\boldsymbol{T}_1) \neq 0$ *and* $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$. *If one of the following cases holds,*

(1) $\lambda_2 > 0,\ \ \lambda_3 > 0,\ \ \mu_2 > \psi_2(\lambda_1,\lambda_4,\mu_1)$,
(2) $\lambda_2 < 0,\ \ \lambda_3 > 0,\ \ 0 < \mu_1 \leq \frac{\lambda_1\lambda_3}{\lambda_2},\ \ \mu_2 > \psi_2(\lambda_1,\lambda_4,\mu_1)$,
(3) $\lambda_2 > 0,\ \ \lambda_3 < 0,\ \ 0 < \mu_2 \leq \frac{\lambda_2\lambda_4}{\lambda_3},\ \ \mu_1 > \psi_1(\lambda_1,\lambda_4,\mu_2)$,
(4) $\lambda_2 < 0,\ \ \lambda_3 < 0,\ \ \mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2},\ \ \varphi_2(\boldsymbol{\lambda},\mu_1) < \mu_2 < \psi_2(\lambda_1,\lambda_4,\mu_1)$,
(5) $\lambda_2 < 0,\ \ \lambda_3 < 0,\ \ 0 < \mu_1 \leq -\frac{\lambda_1\lambda_3}{\lambda_2},\ \ \frac{2\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} < \mu_2 < \psi_2(\lambda_1,\lambda_4,\mu_1)$,

then the interpolating curve $\boldsymbol{P}$ *that satisfies* (2) *exists.*

Remark 16 *The constants* $\boldsymbol{\lambda}$, μ_1 *and* μ_2 *change if lengths of* $\boldsymbol{d}_i$ *change, but all the relations in Theorems 10–15 stay the same.*

These theorems provide us with sufficient conditions that imply the existence of the interpolating polynomial. If none of these conditions is fulfilled the number of solutions is even, almost always zero. Let us take for example Theorem 10 and choose data so that $\lambda_i = 1$, $i = 1,2,3,4$, and $(\mu_1,\mu_2) = (3,1)$, $(2,3)$, $(0,\pm3)$, $(\pm3,0)$, $(-3,\pm3)$. In all of these examples the data do not satisfy any of the conditions of Theorem 10, and since the solutions can be computed analytically, one can easily check that there is no solution in $\mathscr{D}$.

The proof of these theorems will be made in two steps and will be given as the next section. First the existence of the solution will be proved for the particular data. In the second step conclusions will be carried from the particular case to the general one by a convex homotopy and Brouwer's degree argument.

5 Proofs of main Theorems

5.1 *Step* 1*: Particular cases*

Let the points be chosen as

$$\boldsymbol{T}_0 = (-1,-1)^T, \quad \boldsymbol{T}_1 = (0,0)^T, \quad \boldsymbol{T}_2 = (1,-1)^T, \tag{27}$$

and the tangent directions given in Table 3. Table 4 shows the constants for this

Table 3
The tangent directions for different choices of data.

	$\boldsymbol{d}_0$	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$
data 1	$(3,5)^T$	$(2,0)^T$	$(3,-5)^T$
data 2	$(1,3)^T$	$(2,0)^T$	$(1,-3)^T$
data 3	$(-0.5,1.5)^T$	$(2,0)^T$	$(-0.5,-1.5)^T$
data 4	$(3,5)^T$	$(2,0)^T$	$(5,-3)^T$
data 5	$(1,3)^T$	$(2,0)^T$	$(-7,9)^T$
data 6	$(2,4)^T$	$(0,-2)^T$	$(1,1)^T$
data 7	$(7,5)^T$	$(2,0)^T$	$(7,-5)^T$
data 8	$(1.8,-0.2)^T$	$(0,2)^T$	$(41,-39)^T$
data 9	$(3,1)^T$	$(-2,0)^T$	$(3,-1)^T$
data 10	$(1.5,-0.5)^T$	$(-2,0)^T$	$(11,-9)^T$

data, and Table 5 gives the admissible solutions $(t_1,\boldsymbol{\alpha}) \in \mathscr{D}$. Note that there is a unique admissible solution in all the cases (Fig 4).

5.2 *Step* 2*: Homotopy*

In order to prove Theorems 10, 11, 13 and 15 one needs to show that the system (7) has a solution $(t_1,\boldsymbol{\alpha}) \in \mathscr{D}$. The conclusions for the particular data outlined in Table 4 will be carried to the general case by the use of convex homotopy and Brouwer's degree argument. Let the general data be denoted by $(\boldsymbol{\lambda},\mu_1,\mu_2)$ and particular one by $(\boldsymbol{\lambda}^*,\mu_1^*,\mu_2^*)$. A homotopy is defined as

$$\boldsymbol{H}(t_1,\boldsymbol{\alpha};\xi) := \boldsymbol{F}\left(t_1,\boldsymbol{\alpha};\boldsymbol{\lambda}(\xi),\mu_1(\xi),\mu_2(\xi)\right),$$

where

$$\boldsymbol{\lambda}(\xi) := (1-\xi)\boldsymbol{\lambda}^* + \xi\boldsymbol{\lambda}, \quad \mu_i(\xi) := q_i(\xi;\mu_i^*,\mu_i), \quad i = 1,2,$$

Table 4
The constants for the particular data.

data	λ_1	λ_2	λ_3	λ_4	μ_1	μ_2	$\phi_1(\boldsymbol{\lambda})$	$\phi_2(\boldsymbol{\lambda})$	$\varphi_1(\boldsymbol{\lambda},\mu_2)$	$\varphi_2(\boldsymbol{\lambda},\mu_1)$	$\psi_2(\boldsymbol{\lambda},\mu_1)$
1	1	1	1	1	4	4	2.8571	2.8571	/	/	/
2	1	1	1	1	2	2	2.8571	2.8571	/	/	/
3	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	2.8571	2.8571	-4	-4	/
4	1	1	1	-1	4	4	2.8571	/	/	$-\frac{2}{3}$	/
5	1	1	1	-1	2	-8	2.8571	/	/	-2	/
6	1	1	-1	-1	3	0	/	/	2	$-\frac{1}{2}$	/
7	-1	1	1	-1	6	6	/	/	/	/	4.1724
8	-1	-1	1	-1	0.8	40	/	/	/	/	36.2152
9	-1	-1	-1	-1	2	2	/	/	$\frac{2}{3}$	$\frac{2}{3}$	12.96
10	-1	-1	-1	-1	0.5	10	/	/	$\frac{2}{11}$	/	63.1769

Table 5
The admissible solutions for the particular data.

	t_1	α_0	α_1	α_2
data 1	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{9}{10}$	$\frac{4}{5}$
data 2	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{7}{6}$	$\frac{4}{3}$
data 3	$\frac{1}{2}$	$\frac{8}{3}$	$\frac{11}{6}$	$\frac{8}{3}$
data 4	0.3624	1.2392	0.9961	0.3470
data 5	0.1902	3.8203	3.1384	1.6290
data 6	0.4462	3.0658	2.0574	6.1521
data 7	0.5	0.8	0.1	0.8
data 8	0.6860	4.8270	0.0169	0.1849
data 9	0.5	4	1.5	4
data 10	0.7620	13.2434	0.7134	1.0338

and $q_i := q_i(\cdot;\mu_i^*,\mu_i) : [0,1] \to \mathbb{R}$ will be chosen later on as continuous piecewise linear functions that satisfy $q_i(0;\mu_i^*,\mu_i) = \mu_i^*$, $q_i(1;\mu_i^*,\mu_i) = \mu_i$. The idea of the proof is to connect particular and general data so that a set of solutions

$$\mathscr{S} := \{(t_1(\xi),\boldsymbol{\alpha}(\xi)) \in \mathscr{D}; \quad \boldsymbol{H}(t_1(\xi),\boldsymbol{\alpha}(\xi);\xi) = \mathbf{0}, \quad \xi \in [0,1]\}$$

stays away from the boundary $\partial\mathscr{D}$. If this can be done, one can find a compact set $K \subset \mathscr{D}$, such that

$$\mathscr{S} \subset K \subset \mathscr{D}, \quad \mathscr{S} \cap \partial K = \emptyset. \tag{28}$$

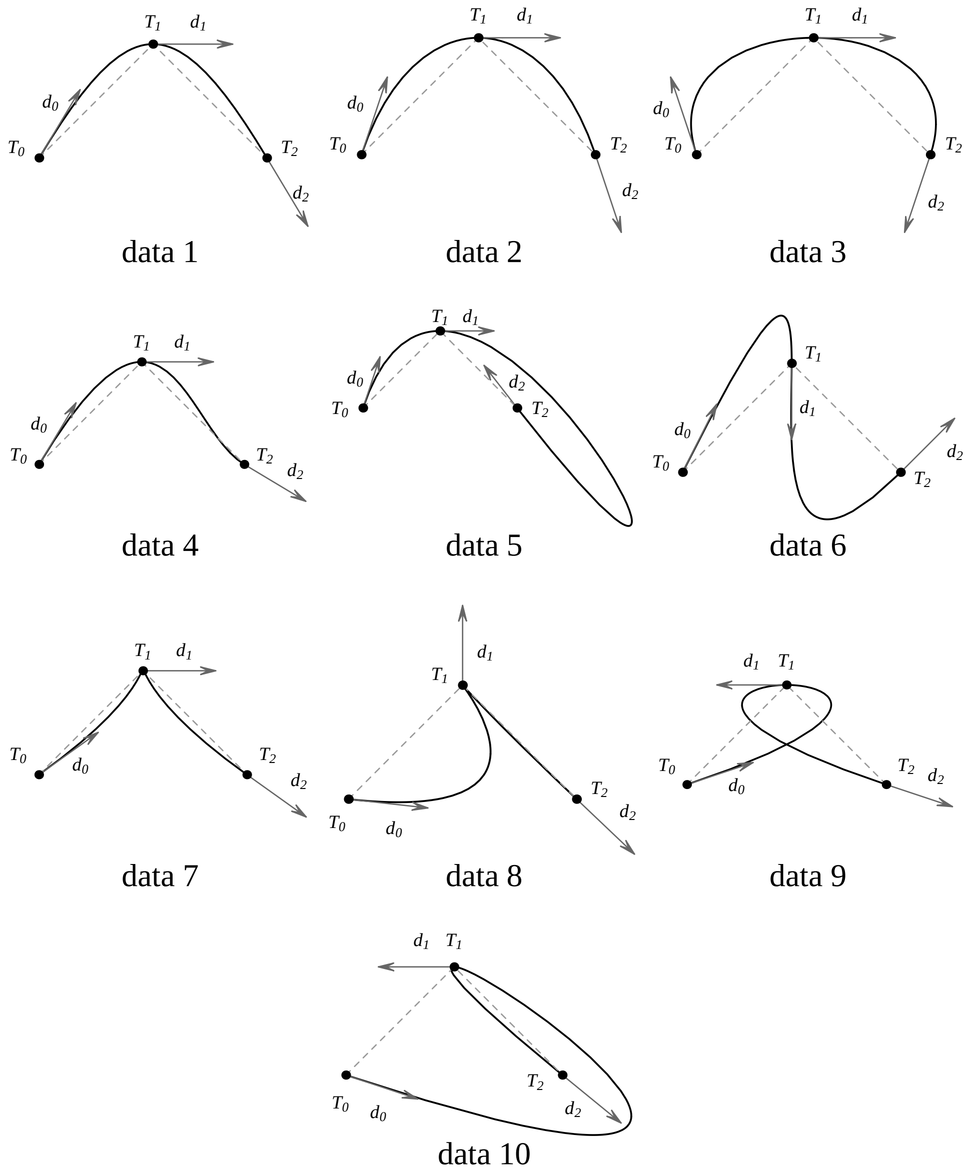

Fig. 4. Cubic Hermite geometric interpolants for particular data points defined by (27) and tangent directions given in Table 3.

Therefore the map $\boldsymbol{H}$ does not vanish at the boundary ∂K, and a Brouwer's degree ([14]) of $\boldsymbol{H}$ on K is invariant for all $\xi \in [0,1]$. But since it is odd for the particular map $\boldsymbol{F}(\cdot,\cdot;\boldsymbol{\lambda}^*,\mu_1^*,\mu_2^*)$, equations $\boldsymbol{F}(t_1,\boldsymbol{\alpha};\boldsymbol{\lambda},\mu_1,\mu_2) = \boldsymbol{0}$ must have at least one admissible solution.

One is now left to show how to choose q_1 and q_2, and to prove that (28) holds. Namely, it must be shown that solutions of $\boldsymbol{H}(t_1(\xi),\boldsymbol{\alpha}(\xi);\xi) = \boldsymbol{0}$ satisfy

$$t_1(\xi) \geq \text{const} > 0, \quad 1 - t_1(\xi) \geq \text{const} > 0, \quad \alpha_i(\xi) \geq \text{const} > 0, \quad i = 0,1,2, \tag{29}$$

for all $\xi \in [0,1]$. Here and throughout the rest of the paper term 'const' will stand for an arbitrary constant. The inequality

$$|\lambda_i(\xi)| \geq \min_{\xi\in[0,1]} \{|(1-\xi)\lambda_i^* + \xi\lambda_i|\} \geq \min\{|\lambda_i^*|, |\lambda_i|\} \geq \text{const} > 0$$

will be fulfilled, and the results of Lemma 5, Lemma 6 and Lemma 7 will be the main tool in the proof of all four theorems. Each of them will be analysed separately.

Theorem 10: There are five cases to be considered. Choosing data 1, 2 or 3 as particular data yield $\boldsymbol{\lambda}(\xi) > 0$, and thus $\alpha_1(\xi)$ cannot approach zero, i.e., $\alpha_1(\xi) \geq \text{const} > 0$ for all $\xi \in [0,1]$. In the first case where $\mu_i > \phi_i(\boldsymbol{\lambda})$, $i = 1,2$, choose data 1 as the particular data. Since $\mu_i^* > \phi_i(\boldsymbol{\lambda}^*)$ and ϕ_i does not depend on μ_i, there obviously exists q_i, such that

$$\mu_i(\xi) > \phi_i(\boldsymbol{\lambda}(\xi)), \quad \xi \in [0,1], \quad i = 1,2.$$

Therefore $\alpha_0(\xi)$ and $\alpha_2(\xi)$ cannot approach zero for any $\xi \in [0,1]$, and by (23) parameter $t_1(\xi)$ cannot approach zero or one either.

Suppose now that $\dfrac{\lambda_1\lambda_3}{\lambda_2} \leq \mu_1 < \phi_1(\boldsymbol{\lambda})$ and $\mu_2 < \phi_2(\boldsymbol{\lambda})$ (case 2) and choose data 2. There clearly exists q_2 that satisfies $\mu_2(\xi) < \phi_2(\boldsymbol{\lambda}(\xi))$ for every $\xi \in [0,1]$. Moreover, from (23) it follows that $\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \phi_1(\boldsymbol{\lambda}(\xi))$, so one can find q_1 such that

$$\frac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \mu_1(\xi) < \phi_1(\boldsymbol{\lambda}(\xi)), \quad \xi \in [0,1].$$

Now, it is clear that $\alpha_0(\xi), \alpha_2(\xi), t_1(\xi) \geq \text{const} > 0$ for all $\xi \in [0,1]$. Moreover, by (19) parameter $t_1(\xi)$ cannot approach one either which completes the proof. Case 3 is symmetric to the second one and will be omitted.

Consider now case 4 and choose data 3 that satisfy

$$-\frac{\lambda_1^*\lambda_3^*}{\lambda_2^*} < \mu_1^* < \frac{\lambda_1^*\lambda_3^*}{\lambda_2^*} \quad \text{and} \quad \varphi_2(\boldsymbol{\lambda}^*, \mu_1^*) < \mu_2^* < \phi_2(\boldsymbol{\lambda}^*).$$

There obviously exists q_1 such that

$$-\frac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \mu_1(\xi) < \frac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)}, \quad \xi \in [0,1].$$

Since by (18) and (23)

$$\varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < -\frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)} < \frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)} < \phi_2(\boldsymbol{\lambda}(\xi)), \quad \xi \in [0,1],$$

there exists q_2 that satisfies $\varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < \mu_2(\xi) < \phi_2(\boldsymbol{\lambda}(\xi))$. Thus $\alpha_0(\xi)$, $t_1(\xi) \geq \text{const} > 0$. Further, by (19) and (23), $\alpha_2(\xi), 1 - t_1(\xi) \geq \text{const} > 0$, and (29) holds.

Case 5 is symmetric to case 4 and will be omitted. This completes the proof of Theorem 10.

Theorem 11: Particular data must be chosen so that $\lambda_i(\xi) > 0$, $i = 1,2,3$, and $\lambda_4(\xi) < 0$. It is then clear that there exists a constant 'const' such that $\alpha_i(\xi) \geq$ const > 0, $i = 0,1$, and $1 - t_1(\xi) \geq$ const > 0, no matter how we define q_1 and q_2. By (23), $\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \phi_1(\boldsymbol{\lambda}(\xi))$. Now, for the first case choose data 4, define q_1 so that $\mu_1(\xi) > \phi_1(\boldsymbol{\lambda}(\xi))$, and then choose such q_2 that $\mu_2(\xi) > \varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi))$, $\xi \in [0,1]$. In the second case choose data 5, define q_1 so that

$$\frac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \mu_1(\xi) < \phi_1(\boldsymbol{\lambda}(\xi)), \quad \xi \in [0,1],$$

and choose such q_2 that $\mu_2(\xi) < \varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi))$. Now, (29) obviously holds and Theorem 11 is proved.

Theorem 13: Let us choose data 6 as the particular data. Since $\lambda_1(\xi) > 0$, $\lambda_2(\xi) > 0$, $\lambda_3(\xi) < 0$ and $\lambda_4(\xi) < 0$ it is clear from Lemma 7 that $\alpha_i(\xi) \geq$ const > 0, $i = 0,1,2$, for $\xi \in [0,1]$. Let q_1 be chosen in such a way that $-\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)} < \mu_1(\xi)$ for $\xi \in [0,1]$. By using (22) one can find q_2 that satisfies

$$-\frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)} < \varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < \mu_2(\xi) < \frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)} + \frac{2\lambda_1(\xi)\lambda_4(\xi)}{\mu_1(\xi)} < \frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)}.$$

Therefore $t_1(\xi)$ cannot approach zero for any $\xi \in [0,1]$. Now, it can easily be proved that $\mu_1(\xi) > \varphi_1(\boldsymbol{\lambda}(\xi), \mu_1(\xi))$, thus parameter $t_1(\xi)$ cannot approach one either and the proof is completed.

Theorem 15: Particular data must be chosen so that $\lambda_1(\xi) < 0$ and $\lambda_4(\xi) < 0$. Then it is clear that $\alpha_i(\xi) \geq$ const > 0, $i = 0,2$, for all $\xi \in [0,1]$. In case 1 choose data 7. It is clear that $t_1(\xi)$ cannot approach zero or one. The only problem could be if $\alpha_1(\xi)$ would go to zero. But for $q_1(\xi; \mu_1^*, \mu_1) := (1-\xi)\mu_1^* + \xi\mu_1$, and q_2 chosen so that $\mu_2(\xi) > \psi_2(\lambda_1(\xi), \lambda_4(\xi), \mu_1(\xi))$, this cannot happen.
In the second case choose data 8 and define q_1 so that $0 < \mu_1(\xi) \leq \dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)}$. Now, $1 - t_1(\xi), t_1(\xi) \geq$ const > 0 for all $\xi \in [0,1]$. Choosing q_2 as in the previous case completes the proof. Next case is symmetric to this one and will be omitted.
In case 4 choose data 9 and q_1 so that $\mu_1(\xi) > -\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)}$. Now, by (20) and (24),

$$\varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < -\frac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)} \text{ and } \varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < \psi_2(\lambda_1(\xi)\lambda_4(\xi), \mu_1(\xi)).$$

So there exists q_2 such that $\varphi_2(\boldsymbol{\lambda}(\xi), \mu_1(\xi)) < \mu_2(\xi) < \psi_2(\lambda_1(\xi), \lambda_4(\xi), \mu_1(\xi))$, and thus $t_1(\xi), \alpha_1(\xi) \geq$ const > 0 for $\xi \in [0,1]$. Parameter $t_1(\xi)$ can approach

1 only for $\mu_2(\xi) > -\dfrac{\lambda_2(\xi)\lambda_4(\xi)}{\lambda_3(\xi)}$, but in this case, by (21), $\varphi_1(\boldsymbol{\lambda}(\xi),\mu_2(\xi)) < -\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)}$, so $1-t_1(\xi) \geq \text{const} > 0$ for $\xi \in [0,1]$.

For the last case choose data 10 and q_1 so that $0 < \mu_1(\xi) \leq -\dfrac{\lambda_1(\xi)\lambda_3(\xi)}{\lambda_2(\xi)}$. Moreover, by (25), q_2 can be chosen so that

$$\frac{2\lambda_1(\xi)\lambda_3(\xi)\lambda_4(\xi)+\lambda_2(\xi)\lambda_4(\xi)\mu_1(\xi)}{\lambda_3(\xi)\mu_1(\xi)} < \mu_2(\xi) < \psi_2(\lambda_1(\xi),\lambda_4(\xi),\mu_1(\xi)).$$

It is clear that $t_1(\xi) \geq \text{const} > 0$ and $\alpha_1(\xi) \geq \text{const} > 0$ for $\xi \in [0,1]$. But, by (26), $\mu_1(\xi) > \varphi_1(\boldsymbol{\lambda}(\xi),\mu_2(\xi))$, so $1-t_1(\xi) \geq \text{const} > 0$ too and the proof is completed.

6 Approximation order

In this section Theorem 2 will be proved. Recall the notation declared there. One needs to show that there exists $h_0 > 0$ small enough and a constant $C > 0$, so that for every $h = \max_{\ell} \Delta s_\ell$, $0 < h \leq h_0$, the G^1 spline exists as well as

$$\operatorname{dist}(\boldsymbol{f},\boldsymbol{P}) = \operatorname{dist}(\boldsymbol{f},\boldsymbol{P})_{[a,b]} = \inf_{\theta} \|\boldsymbol{f} - \boldsymbol{P}\circ\theta\| \leq C\,h^6, \tag{30}$$

where $\theta : [a,b] \to [0,1]$ is a regular reparameterization. Since

$$\operatorname{dist}(\boldsymbol{f},\boldsymbol{P}) \leq \max\left\{\operatorname{dist}(\boldsymbol{f},\boldsymbol{P})_{[s_{2\ell-2},s_{2\ell}]};\ \ell = 1,2,\ldots,m\right\},$$

it is again enough to analyse the polynomial case $m = 1$ only. Without losing generality, one can assume $a = 0$, $\boldsymbol{f}(0) = (0,0)^T$, $\boldsymbol{f}'(0) = (1,0)^T$. Further, let h be redefined as $h := s_2 - s_0$. For h small enough, $\boldsymbol{f}$ can be parameterized by the first component,

$$\boldsymbol{f}(s) = \begin{pmatrix} s \\ y(s) \end{pmatrix}, \quad y(s) = \frac{1}{2}y''(0)s^2 + \frac{1}{3!}y^{(3)}(0)s^3 + \mathcal{O}\left(s^4\right), \quad y''(0) \neq 0.$$

Moreover, let $\eta_i := s_i/h$, $i = 0,1,2$. By Remark 3 the tangent directions can be redefined as $\boldsymbol{d}_i = h\boldsymbol{f}'(h\eta_i)$, $i = 0,1,2$. Since $\boldsymbol{f}$ is convex, $\det(\boldsymbol{T}_0,\boldsymbol{T}_1) \neq 0$ and constants (6) are well defined. It is straightforward to compute

$$\begin{aligned}
\lambda_1 &= \frac{\eta_1}{1-\eta_1} + \mathcal{O}(h), \quad \lambda_2 = \frac{\eta_1}{1-\eta_1} + \mathcal{O}(h), \quad \lambda_3 = \frac{1-\eta_1}{\eta_1} + \mathcal{O}(h),\\
\lambda_4 &= \frac{1-\eta_1}{\eta_1} + \mathcal{O}(h), \quad \mu_1 = 1 + \frac{1}{\eta_1} + \mathcal{O}(h), \quad \mu_2 = 1 + \frac{1}{1-\eta_1} + \mathcal{O}(h),
\end{aligned}$$

and the nonlinear system (7) becomes

$$\boldsymbol{F}\left(t_1, \boldsymbol{\alpha}; \tilde{\boldsymbol{\lambda}}, 1+\frac{1}{\eta_1}, 1+\frac{1}{1-\eta_1}\right) + \mathscr{O}(h) = 0, \tag{31}$$

where $\tilde{\boldsymbol{\lambda}} := \left(\frac{\eta_1}{1-\eta_1}, \frac{\eta_1}{1-\eta_1}, \frac{1-\eta_1}{\eta_1}, \frac{1-\eta_1}{\eta_1}\right)$. It is easy to check that the solution at the limit $h = 0$ is $t_1 = \eta_1$, $\alpha_i = 1$, $i = 0,1,2$. But unfortunately, the Jacobian at the limit solution is singular, and one can not make use of the Implicit Function Theorem. To show that the solution exists also for all h small enough Theorem 1 will be used. There obviously exists such h_0 that $\boldsymbol{\lambda} > 0$ for all $0 < h \leq h_0$. With some elementary mathematics one can prove that the inequalities

$$1+\frac{1}{\eta_1} > \phi_1(\tilde{\boldsymbol{\lambda}}), \quad 1+\frac{1}{1-\eta_1} > \phi_2(\tilde{\boldsymbol{\lambda}})$$

hold. Furthermore,

$$\phi_i(\boldsymbol{\lambda}) = \phi_i(\tilde{\boldsymbol{\lambda}}) + \mathscr{O}(h),$$

thus h_0 can be chosen so small that $\mu_i > \phi_i(\boldsymbol{\lambda})$ for all $0 < h \leq h_0$. The existence of the solution is now guaranteed by Theorem 10 (case 1), which also shows that this is probably the most important existence result. Now that the existence is provided it is well known (see [7], [4]) that approximation order is optimal which completes the proof.

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