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GEOMETRIC INTERPOLATION BY PLANAR PARAMETRIC POLYNOMIAL CURVES

Doctoral thesis

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GEOMETRIJSKA INTERPOLACIJA Z RAVNINSKIMI PARAMETRIČNIMI POLINOMSKIMI KRIVULJAMI

Doktorska disertacija

MENTOR: Prof. Dr. Jernej Kozak

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Zahvala

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Abstract

In the thesis the geometric interpolation by planar parametric polynomial curves is considered. In the introduction general geometric interpolation schemes, their properties and advantages are outlined, and main results about these schemes are presented. The Lagrange problem of interpolating $2n$ points in the plane by a polynomial curve of degree n is considered in detail. Since the problem is nonlinear, the question of the existence of the solution is very difficult. In Chapter 2, sufficient geometric conditions that ensure the existence of the cubic curve that interpolates six points in the plane are given. The conditions turn out to be quite simple and depend only on certain determinants derived from data points. The results cover convex and nonconvex data. In the next chapter the geometric interpolation by cubic G^1 spline curves is considered. Sufficient geometric conditions are derived that admit the existence of a G^1 spline curve, where on each polynomial segment four points and two tangent directions are interpolated. An algorithm that determines the areas for tangent directions, such that the existence of the spline is guaranteed, is presented. In Chapter 4 the Hermite geometric interpolation by cubic polynomial curves is studied. Again, sufficient geometric conditions are derived that cover most of the cases. From the analysis of geometric interpolation by cubic curves it is clear that it would be impossible to consider the problem for a general degree without some further assumptions. In Chapter 5 the asymptotic approach is applied, which means that the data are sampled from a smooth convex curve $\boldsymbol{f} : [0, h] \to \mathbb{R}^2$ with h small enough. For a general degree n a special nonlinear system of equations is derived and it is proven that in the case that it has at least one real solution the approximation order is optimal, i.e., $2n$. The existence of the solution of this system is proven for $n \leq 5$ for general curves. For a general degree n the existence of the solution is established for a special class of functions, so called *circle-like curves*. Tools used in the analysis are resultants, Gröbner basis and Brouwer's mapping degree. Their definitions and main properties are given in the last chapter.

Key-words: geometric interpolation, polynomial curve, spline curve, geometric continuity, existence of the solution, parametric distance, asymptotic analysis, approximation order, CAGD.

Math. Subj. Class. (2000): 65D05, 65D07, 65D10, 65D17.

Povzetek

V disertaciji je obravnavana geometrijska interpolacija z ravninskimi parametričnimi polinomskimi krivuljami. V uvodu so predstavljene splošne geometrijske interpolacijske sheme, njihove glavne lastnosti in prednosti. Podani so najpomembnejši rezultati s tega področja. Podrobno je predstavljen Lagrangeev problem interpolacije $2n$ ravninskih točk s polinomsko krivuljo stopnje n. Ker je problem nelinearen, je vprašanje o obstoju rešitve precej težko. V drugem poglavju so izpeljani geometrijski pogoji, ki zagotavljajo obstoj kubične interpolacijske krivulje, ki interpolira šest točk v ravnini. Pogoji so preprosto preverljivi in odvisni le od geometrije danih točk. Rezultati pokrijejo tako konveksne kot nekonveksne podatke. V naslednjem poglavju je obravnavan problem geometrijske interpolacije s kubičnimi G^1 zlepki. Izpeljani so zadostni pogoji za obstoj G^1 zlepka, kjer so na vsakem odseku interpolirane štiri točke in dve smeri tangent. Dodan je algoritem, s katerim določimo območja za smeri tangent, da je obstoj zlepka zagotovljen. V četrtem poglavju je obravnavana Hermitova interpolacija s kubičnimi polinomskimi krivuljami in G^1 zlepki. Izpeljani so geometrijski pogoji, ki zagotavljajo obstoj interpolanta, ki pokrijejo večino primerov. Iz analize problema interpolacije s kubičnimi polinomi se vidi, da je v splošnem problem nemogoče obravnavati brez kakšnih dodatnih predpostavk. V petem poglavju je uporabljen asimptotični pristop, kar pomeni, da so podatki vzeti iz gladke konveksne krivulje $\boldsymbol{f} : [0, h] \to \mathbb{R}^2$, kjer je h dovolj majhen. Za poljubno stopnjo n je izpeljan poseben sistem nelinearnih enačb in dokazano je, da je v primeru, ko ima ta sistem vsaj eno realno rešitev, red aproksimacije optimalen, to je $2n$. Obstoj rešitve tega sistema je za poljubne krivulje dokazan za stopnje $n \leq 5$. Za poljubne stopnje polinomov pa je obstoj rešitve dokazan za poseben razred krivulj, tako imenovane *krivulje blizu krožnice*. V dokazih so uporabljene rezultante, Gröbnerjeve baze in Brouwerjeva stopnja. Njihove definicije in glavne lastnosti so podane v zadnjem poglavju.

Ključne besede: geometrijska interpolacija, polinomska krivulja, krivulja zlepkov, geometrijska zveznost, obstoj rešitve, parametrična razdalja, asimptotična analiza, red aproksimacije, CAGD.

Math. Subj. Class. (2000): 65D05, 65D07, 65D10, 65D17.

Contents

Chapter 1

Introduction

The geometric interpolation by parametric polynomial curves has received considerable attention since it was introduced in [4], where a Hermite cubic interpolation of two points, tangent directions and curvatures is studied. It was shown that under some natural restrictions a planar convex curve can be approximated up to the sixth order accuracy. High approximation order is one of the reasons for a further work on the subject. The other is the fact that geometric interpolants in many cases please the human eye more than their usual standard counterparts. The underlying basic concept of geometric interpolation is to consider the curve itself independent of its actual parameterization. The interpolating curve depends only on geometric quantities such as data points, tangent directions, curvatures, etc. This makes the geometric interpolant a valuable tool in the computer aided geometric design. No additional artificial conditions are imposed on the curve such as at which parameter values the interpolation conditions should be met. Since no free parameters are used ineffectively, geometric interpolation often results in higher approximation order than one would expect from the functional case.

But, unfortunately, geometric interpolating schemes involve a nonlinear part. This drawback makes it hard to analyse the existence of the interpolating curve and to establish the approximation order. Numerical computations have to be done with some care too, usually by the continuation method ([2]). Thus it is quite clear why the analysis of geometric interpolation schemes is usually based upon the assumption that data are sampled densely enough from a smooth curve, and the asymptotic analysis is applied.

Commonly, a curve is not approximated by a single polynomial or rational curve, but by a geometric spline, i.e., a finite number of consecutive segments, joining at the breakpoints with an appropriate order of geometric continuity. Two curves join with G^0 continuity if they meet at a common join point, with G^1 continuity if they share a common tangent direction at the join point, and with G^2 continuity if additionally the signed curvature at the join point is continuous. In general, geometric continuity can be defined in the following way.

DEFINITION 1.1. *Two parametric curves*

$$\boldsymbol{f}_1 : [t_0, t_1] \to \mathbb{R}^d, \quad \boldsymbol{f}_2 : [s_0, s_1] \to \mathbb{R}^d,$$

join with a geometric continuity of order k or shortly with G^k-continuity if the end point of $\boldsymbol{f}_1$ coincides with the starting point of $\boldsymbol{f}_2$, $\boldsymbol{f}_1(t_1) = \boldsymbol{f}_2(s_0)$, and there exists a regular reparameterization $\phi : [t_0, t_1] \to [s_0, s_1]$ such that

$$\left.\frac{d^j \boldsymbol{f}_1}{dt^j}(t)\right|_{t=t_1} = \left.\frac{d^j (\boldsymbol{f}_2 \circ \phi)}{dt^j}(t)\right|_{t=t_1}, \quad j = 0, 1, \ldots, k.$$

In order to study the approximation order in parametric case we need to know how to measure the distance between parametric objects. Since objects are usually considered as sets of points the well known *Hausdorff distance* d_H is one of the possible metrics. It is defined in the following way.

DEFINITION 1.2. *Let X and Y be two subsets of a metric space M. The Hausdorff distance $d_H(X, Y)$ is defined by*

$$d_H(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\},$$

where $d(x, y)$ is a metric on M.

But unfortunately, it is difficult to compute the Hausdorff distance in practice. As its upper bound the so called *parametric distance* d_P has been proposed by Lyche and Mørken ([34]).

DEFINITION 1.3. *Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be two parametric curves defined on intervals I_1 and I_2, respectively. The parametric distance between $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is defined by*

$$\operatorname{dist}(\boldsymbol{f}_1, \boldsymbol{f}_2) := \inf_{\phi} \|\boldsymbol{f}_1 \circ \phi - \boldsymbol{f}_2\| = \inf_{\phi} \max_{t \in I_2} \|\boldsymbol{f}_1(\phi(t)) - \boldsymbol{f}_2(t)\|,$$

where $\phi : I_2 \to I_1$ is a regular reparameterization, i.e., $\phi' \neq 0$ on I_2.

Let us now summarize some of the well known results on geometric interpolation. In [4], C. de Boor, K. Höllig, and M. Sabin introduce the concept of geometric continuity. They study the approximation of a curve $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^2$ by planar piecewise cubic polynomials. Data points, tangent directions and curvatures of $\boldsymbol{f}$ at given nodes t_i are interpolated. The authors show that for $h := \sup_i \|\boldsymbol{f}(t_{i+1}) - \boldsymbol{f}(t_i)\|$ sufficiently small an interpolant which approximates $\boldsymbol{f}$ at the surprisingly high order h^6 exists, except when the curvature of $\boldsymbol{f}$ vanishes. Then the approximation order is locally reduced to h^4. They also show that the interpolant need not be unique.

In [22], K. Höllig and J. Koch make the following conjecture.

CONJECTURE 1.4. *Under appropriate generic assumptions, a polynomial curve of degree n can interpolate*

$$m = n + 1 + \left\lfloor \frac{n-1}{d-1} \right\rfloor$$

points on a smooth curve $\boldsymbol{f} \in \mathbb{R}^d$. This interpolant approximates $\boldsymbol{f}$ with order m as the distance of interpolation points tends to zero.

They give a proof of this conjecture for planar quadratic polynomial and spline curves and describe a simple construction of curvature continuous quadratic splines from control polygons. In [21], the same two authors consider the approximation of space curves by cubic splines and prove that a smooth space curve can be approximated by a cubic polynomial with fifth-order accuracy at any point where the curvature and torsion do not vanish. They describe a natural generalization of the standard Hermite interpolation which achieves the optimal approximation order 5 for degree 3. In addition to the position and the tangent direction at the endpoints, a third point within the parameter interval is interpolated.

In [37], K. Mørken and K. Scherer establish a general framework for geometric interpolation methods for parametric curves that includes interpolation methods in all dimensions. Questions of solvability and stability are considered. As a special case of the general result, they prove that four points on a planar curve can be interpolated by a quadratic with fourth-order accuracy, if the points are sufficiently close to a point with a nonvanishing curvature. They also prove that six points on a planar curve can be interpolated by a cubic with sixth-order accuracy, provided the points are sufficiently close to a point where the curvature does not have a double zero. In space it turns out that five points sufficiently close to a point with nonvanishing torsion can be interpolated by a cubic, with fifth-order accuracy. The case of quartic curves in the plane is studied by K. Scherer in [45] and [46].

In [34], T. Lyche and K. Mørken define a metric on the set of parametric curves (see Definition 1.3). They give a definition of approximation rate for parametric approximation schemes in terms of this metric, and present a simple family of odd degree parametric polynomial approximations to circle segments with the approximation order twice the degree of the polynomial.

Degen ([7]) extends the work of C. de Boor, K. Höllig, and M. Sabin ([4]). He shows, by using rational cubics, that it is possible to have third-order contact at the endpoints, which raises the approximation order to eight. Analysing the solvability conditions for third-order contact by rational cubics he establishes that purely geometric properties of the given curve determine whether the desired approximant exists. In [8] Degen introduces the notion of geometric contact elements and designs a unified theory of geometric Hermite interpolation for parametric curves. He predicts the order of approximation to be $2n$ for a polynomial curve and $3n-1$ for a rational curve of degree n. In his recent work [9] Degen presents an excellent overview of the developments of geometric Hermite approximation theory for planar curves. A general method to solve these problems is presented. Geometric interpretations, examples and a detailed discussion of the case of degree $n = 4$ with one contact point is given.

In [38], Rababah describes a Taylor polynomial interpolation for space curves which yields the order $(n+1) + \left[\frac{n+1}{2d-1}\right]$ for a curve in $\mathbb{R}^d$, where n is the degree of the approximating polynomial curve. The cubic case is studied with examples. He extends these results to the piecewise polynomial curve interpolation in [39].

An important contribution to the subject is given by J. Kozak, Y. Y. Feng and E. Žagar ([16], [15], [14], [26], [27], [28], [47]). In [16] the interpolation by G^2 continuous Bézier spline curves in $\mathbb{R}^d$ is outlined. Each segment of the spline curve interpolates r interior and two boundary points. A general approach is followed in detail for the case $d = 3$, $n = 3$, $r = 1$. For the single component case, the optimal approximation order is proved, and asymptotic existence established. In [15] the interpolation by G^2 continuous planar cubic Bézier spline curves is studied. The interpolation is based only on the underlying curve points and the end tangent directions. On each polynomial segment four points and two tangent directions are interpolated. The authors show the existence of the interpolant in asymptotic sense and prove that the approximation order is optimal. In [47], E. Žagar considers an interpolation by G^2 spline curves of degree d in $\mathbb{R}^d$ that interpolate d points on each segment. He shows that if the data points are sampled regularly and are sufficiently dense, then the interpolant exists. The optimal approximation order $d+2$ is achieved.

Results on rational and polynomial geometric interpolation by R. Schaback can be found in [40], [41], [44], [42], and [43].

Most of the mentioned results are obtained by the asymptotic approach. However, results offered by the asymptotic analysis are not always adequate in practical applications. If one is merely looking for an interpolant of a nice shape, suppositions like *"if data points are sampled dense enough"* are not very encouraging. Therefore robust algorithms should be based upon conditions that ensure the existence in advance if possible at all. But in geometric interpolation, this can be rarely achieved. Beside some special cases, like the interpolation of a circle ([34] [10], [19] [35], [17], [18], [11], [12]), there are only few results concerning geometric conditions for the existence of the interpolant. The interpolation by a parametric parabola at four distinct planar points is studied in [33], where the conditions are established through geometric arguments. In [36] the algebraic approach is applied, and results are extended to all possible cases (Taylor, Hermite, Lagrange). The interpolation of an arbitrary number of points in a plane with composite G^2 quadratic curves is studied in [41], [14], [26], where sufficient conditions for solvability and the uniqueness of the solution are derived. Perhaps the most general results are given by J. Kozak and E. Žagar in [27], where necessary and sufficient geometric conditions for the simplest nontrivial geometric interpolation schemes in all dimensions, i.e., the interpolation of $d+2$ distinct points in $\mathbb{R}^d$ by a polynomial curve of degree $\leq d$, are outlined.

In this thesis the study of geometric interpolation is restricted to the planar case $d = 2$. Planar schemes are probably the most important in practice, and the gap between the parametric and the functional case is the largest. Namely, Conjecture 1.4 states that $2n$ planar points can be interpolated by a polynomial curve of degree n, and the approximation order $2n$ can be achieved, while in the functional case a polynomial of degree n can interpolate only $n+1$ points with the approximation order $n+1$.

The Lagrange interpolation problem considered is the following. For $2n$ given data points

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2n-1} \in \mathbb{R}^2, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \tag{1.1}$$

find a parametric polynomial curve

$$\boldsymbol{P}_n : [t_0, t_{2n-1}] \to \mathbb{R}^2$$

of degree $\leq n$ that interpolates these points at some values t_i in an increasing order,

$$t_0 < t_1 < \cdots < t_{2n-2} < t_{2n-1}. \tag{1.2}$$

Since a linear transformation of the parameter preserves the degree of a parametric polynomial curve, one can assume $t_0 := 0$ and $t_{2n-1} := 1$, but the remaining parameters

$$\boldsymbol{t} := (t_i)_{i=1}^{2n-2}$$

are unknown. The admissible parameters t_i can be viewed as components of a point in the open simplex

$$\mathcal{D}_n := \left\{(t_i)_{i=1}^{2n-2}; \quad 0 = t_0 < t_1 < \cdots < t_{2n-2} < t_{2n-1} = 1\right\},$$

with the boundary $\partial\mathcal{D}_n$ where at least two different t_i coincide. The system of equations

$$\boldsymbol{P}_n(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 2n-1, \tag{1.3}$$

should determine the unknown $\boldsymbol{P}_n$ as well as the parameters $\boldsymbol{t}$. Once the parameters are determined it is straightforward to obtain the coefficients of the polynomial curve $\boldsymbol{P}_n$. One only has to take any $n+1$ distinct interpolating conditions in (1.3), and apply any standard interpolation scheme like Newton or Lagrange componentwise.

Since the system (1.3) is nonlinear, it is usually difficult to prove the existence of a solution. Also, a solution does not necessary exist. As a simple example take four points ($n = 2$) with a nonconvex data polygon. Since each component of the interpolating $\boldsymbol{P}_2$ is a parabola and since a parabola cannot have more than two zeros it is clear that the interpolant can not exist. But for this case very nice necessary and sufficient conditions for the existence are given in [27]. Namely, the interpolant $\boldsymbol{P}_2$ exists if and only if determinants

$$\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1), \quad \det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_2), \quad \det(\Delta\boldsymbol{T}_1, \Delta\boldsymbol{T}_2),$$

where $\Delta\boldsymbol{T}_i := \boldsymbol{T}_{i+1} - \boldsymbol{T}_i$, are of the same sign.

As a motivation to geometric interpolation schemes let us consider some numerical examples. First, let us compare the cubic geometric scheme with a componentwise quintic interpolation, where a parameterization is chosen in advance as the uniform,

$$t_0 = 0, \quad t_{i+1} = t_i + 1,$$

and the chord length parameterization,

$$t_0 = 0, \quad t_{i+1} = t_i + \|\boldsymbol{T}_{i+1} - \boldsymbol{T}_i\|.$$

The cubic curve (black) clearly does the job much better than its quintic counterparts as one can observe in Figure 1.1. The shape of the geometric interpolation curve is as one would require for the given data points, without any visible extraneous inflections. Also, the computational effort to compute these six cubic interpolants turns out to be negligible. The Newton method with equidistant starting values $t_i = \frac{i}{5}$ converges within a machine precision accuracy in eight iterations on average. There is perhaps a simple explanation of the fact that the cubic geometric interpolation curves are superior. An approximate curvature, with denominator neglected, is a parabola

$$\det\left(\dot{\boldsymbol{P}}_3, \quad \ddot{\boldsymbol{P}}_3\right),$$

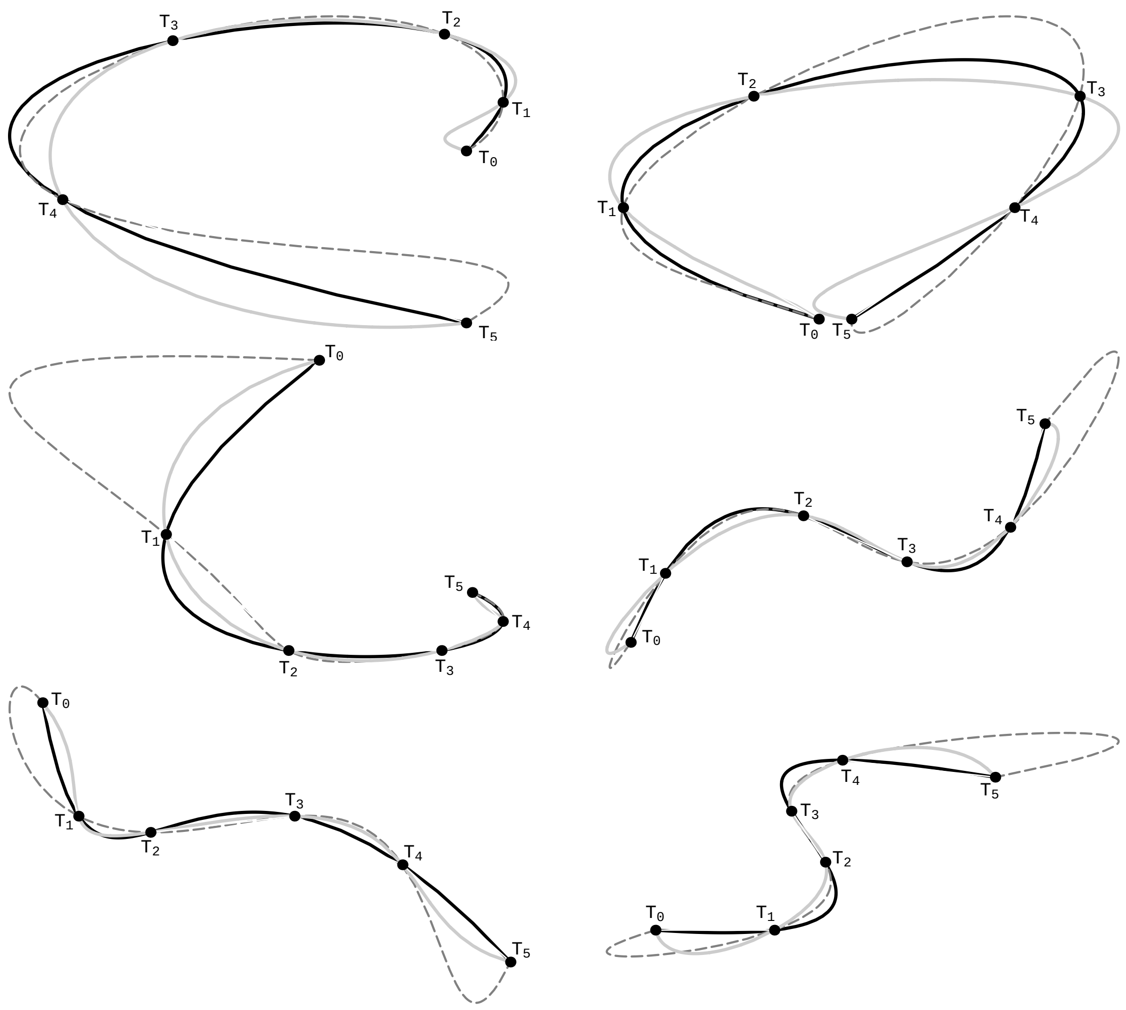


Figure 1.1: A geometric cubic interpolant (black) and quintic polynomial interpolating curves with uniform (grey) and chord length parameterization (dashed).

so the rate of change of the curvature is approximately linear what pleases most the human eye.

As a next example let us approximate a particular logaritmic spiral

$$\boldsymbol{f}(t) = \log(t+\pi)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{1.4}$$

by geometric interpolants of degrees $n = 3, 4, 5, 6$. The data points are obtained from (1.4) by the equidistant splitting of the parameter domain. Table 1.1 numerically suggests that the approximation order is $2n$, where the error is measured as the parametric distance between the curve and its interpolants.

The outline of the thesis is the following. In Chapter 2 the geometric conditions that imply the existence of a cubic geometric interpolant that interpolates six points in the plane are presented. The conditions depend only on certain determinants of data points and are very simple to verify. The results cover both the convex and nonconvex

Interval	Approximation error		Decay exponent	
	$n=3$	$n=4$	$n=3$	$n=4$
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	2.2662×10^{-2}	1.6485×10^{-3}	—	—
$\left[-\frac{5\pi}{12}, \frac{5\pi}{12}\right]$	8.0154×10^{-3}	3.6922×10^{-4}	5.70	8.21
$\left[-\frac{4\pi}{12}, \frac{4\pi}{12}\right]$	2.1793×10^{-3}	5.8837×10^{-5}	5.84	8.23
$\left[-\frac{3\pi}{12}, \frac{3\pi}{12}\right]$	3.9573×10^{-4}	5.5716×10^{-6}	5.93	8.19
$\left[-\frac{2\pi}{12}, \frac{2\pi}{12}\right]$	3.4941×10^{-5}	2.0533×10^{-7}	5.99	8.14
$\left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$	5.4085×10^{-7}	7.5616×10^{-10}	6.01	8.09
	$n=5$	$n=6$	$n=5$	$n=6$
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	5.0251×10^{-5}	1.7288×10^{-7}	—	—
$\left[-\frac{5\pi}{12}, \frac{5\pi}{12}\right]$	7.7733×10^{-6}	2.3347×10^{-8}	10.24	10.98
$\left[-\frac{4\pi}{12}, \frac{4\pi}{12}\right]$	7.7842×10^{-7}	2.1547×10^{-9}	10.31	10.68
$\left[-\frac{3\pi}{12}, \frac{3\pi}{12}\right]$	4.0522×10^{-8}	8.1861×10^{-11}	10.27	11.37
$\left[-\frac{2\pi}{12}, \frac{2\pi}{12}\right]$	6.4814×10^{-10}	6.6848×10^{-13}	10.20	11.86
$\left[-\frac{\pi}{12}, \frac{\pi}{12}\right]$	5.8338×10^{-13}	1.6306×10^{-16}	10.12	12.00

Table 1.1: The error and approximation order in interpolation of the logaritmic spiral (1.4) by geometric interpolants.

data. Chapter 3 extends the results to geometric interpolation by cubic G^1 splines. Geometric conditions that imply the existence of a spline are derived, where on each segment four points and two tangent directions are interpolated. The conditions again depend only on positions of data. The algorithm that carries out the verification is added. In the next chapter Hermite interpolation of three points and three tangent directions is considered. Geometric conditions for the existence of the interpolant are given and the optimal approximation order is confirmed. In Chapter 5 the asymptotic analysis is done for all degrees n. This means that the data are sampled from a smooth convex curve $\boldsymbol{f} : [0, h] \to \mathbb{R}^2$, with h small enough. The Conjecture 1.4 is proven for degree $n \leq 5$. For a general n, a special nonlinear system of equations is derived and it is proven that in the case when this system has one real solution, the approximation order is optimal, i.e., $2n$. The proof of the existence of a real solution is very difficult. In the proof for degree $n = 5$ the theory of resultants, Gröbner basis, varieties and ideals is used. Chapter 6 is a continuation of the previous one, where the conjecture is proven for a special class of functions, so called *circle-like* curves, for general degrees of polynomials.

The results of this thesis are published in the following papers: [29], [30], [31], [23], [24], and [25].

Chapter 2

Geometric interpolation by cubic polynomials

In this chapter the Lagrange interpolation of six points in $\mathbb{R}^2$ by a cubic polynomial curve is studied. Namely, the interpolation problem introduced in the first chapter is here considered for $n = 3$ and simple sufficient geometric conditions that ensure the existence of the interpolant are given. The conditions turn out to be quite simple and depend only on certain determinants derived from the data points.

2.1. The main results

The conditions that imply the nonlinear system

$$\boldsymbol{P}_3(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 5, \tag{2.1}$$

to have at least one admissible solution, i.e.,

$$0 =: t_0 < t_1 < \cdots < t_5 := 1, \tag{2.2}$$

will be determined here. The key role is played by the matrix of data differences,

$$\left(\Delta \boldsymbol{T}_i\right)_{i=0}^{4} \in \mathbb{R}^{2\times 5},$$

and by the signs and ratios of its minors

$$D_{i,j} := \det\left(\Delta \boldsymbol{T}_i, \Delta \boldsymbol{T}_j\right). \tag{2.3}$$

These are the volumes of parallelograms spanned by the vectors $\Delta \boldsymbol{T}_i, \Delta \boldsymbol{T}_j$. Let us define

$$\lambda_1 := \frac{D_{0,1}}{D_{1,2}},\quad \lambda_2 := \frac{D_{0,2}}{D_{1,2}},\quad \lambda_3 := \frac{D_{2,4}}{D_{2,3}},\quad \lambda_4 := \frac{D_{3,4}}{D_{2,3}},\quad \delta := \frac{D_{1,3}}{D_{1,2}},\quad \mu := \frac{D_{2,3}}{D_{1,2}},$$
$$\gamma_1 := \frac{\lambda_2(1+\lambda_2)}{\lambda_1(1+\lambda_2)+\sqrt{\lambda_1(1+\lambda_2)(\lambda_1+\lambda_2)}},$$
$$\gamma_2 := \frac{\lambda_3(1+\lambda_3)}{\lambda_4(1+\lambda_3)+\sqrt{\lambda_4(1+\lambda_3)(\lambda_3+\lambda_4)}}.$$

Note that the data points with a convex control polygon, as in the first three figures of Figure 1.1, have $\mu > 0$ and $\lambda_i > 0$, $i = 1, 2, 3, 4$. The control polygons of data points in the last three figures of Figure 1.1 change from convexity to concavity at $\Delta \boldsymbol{T}_2$. Such data have $\lambda_i > 0$ and $\mu < 0$. We will restrict our study to these two types of data. Geometric interpretation of λ_i, δ and μ is shown in Figure 2.1.

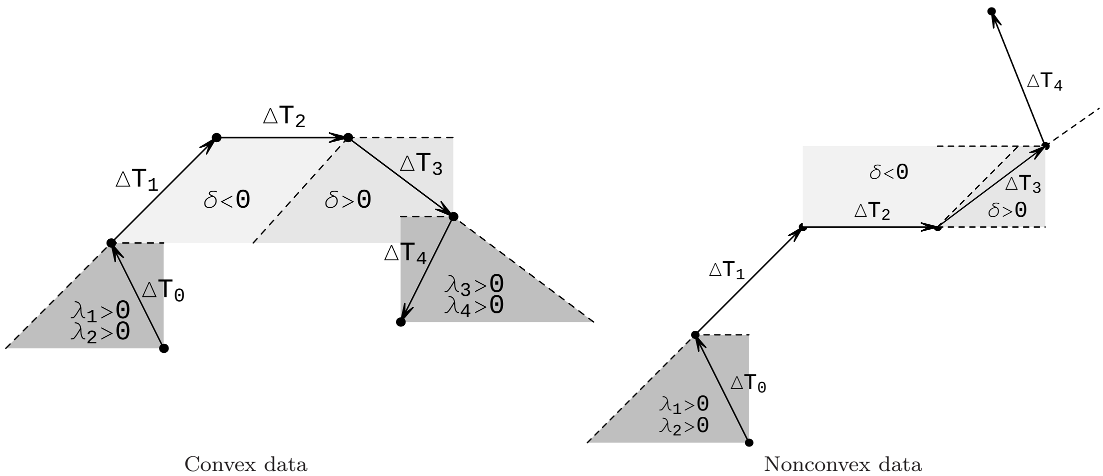


Figure 2.1: Geometric interpretation of λ_i, δ and μ for convex and nonconvex data.

Further let us define $\boldsymbol{\lambda} := (\lambda_i)_{i=1}^4$, and the functions

$$\vartheta_1(\boldsymbol{\lambda},\mu) := \frac{2\mu-\gamma_1+\sqrt{\gamma_1^2+4\mu(1+\gamma_1)}}{2\gamma_1},$$
$$\vartheta_2(\boldsymbol{\lambda},\mu) := \frac{2-\mu\gamma_2+\sqrt{\mu^2\gamma_2^2+4\mu(1+\gamma_2)}}{2\gamma_2},$$
$$\vartheta_3(\boldsymbol{\lambda},\mu) := \frac{\lambda_1\mu}{\lambda_2}+\frac{\lambda_4}{\lambda_3}+\frac{\mu}{\lambda_2}\sqrt{\frac{\lambda_1(\lambda_1+\lambda_2)}{1+\lambda_2}},$$
$$\vartheta_4(\boldsymbol{\lambda},\mu) := \frac{\lambda_1\mu}{\lambda_2}+\frac{\lambda_4}{\lambda_3}+\frac{1}{\lambda_3}\sqrt{\frac{\lambda_4(\lambda_3+\lambda_4)}{1+\lambda_3}},$$

that will be used in boundary relations between the constants, that ensure the existence of the solution. The main results are the following.

THEOREM 2.1. *Suppose that $D_{1,2}D_{2,3} \neq 0$ and the data are convex, i.e., $\mu > 0$ and $\lambda_i > 0$, $i = 1,2,3,4$. If either $\vartheta_1(\boldsymbol{\lambda}, \mu) = \vartheta_2(\boldsymbol{\lambda}, \mu)$, or one of the following conditions is met,*

$$\delta < \min_{\ell=1,2} \{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\} \quad \text{or} \quad \delta > \max_{\ell=1,2} \{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\},$$

then the cubic interpolating curve $\boldsymbol{P}_3$ that satisfies (2.1) *exists.*

THEOREM 2.2. *Suppose that $D_{1,2}D_{2,3} \neq 0$, and the data imply an inflection point, i.e., $\mu < 0$ and $\lambda_i > 0$, $i = 1,2,3,4$. If*

$$\delta \in (\vartheta_3(\boldsymbol{\lambda}, \mu), \vartheta_4(\boldsymbol{\lambda}, \mu)),$$

then the cubic interpolating curve $\boldsymbol{P}_3$ that satisfies (2.1) *exists.*

Theorem 2.1 and Theorem 2.2 provide only sufficient conditions for the existence of a cubic geometric interpolant. But the next conclusion excludes most of the data that do not satisfy these two theorems.

THEOREM 2.3. *The cases where the solution of the interpolation problem* (2.1) *does not exist are summarized in Table 2.1.*

$D_{1,2}D_{2,3} \neq 0$		$D_{1,2}D_{2,3} = 0$
$\mu > 0$	$\mu < 0$	
$\lambda_2 \leq 0,\ \lambda_3 \leq 0$	$\lambda_2 \leq 0$	$D_{1,2} = 0,\ D_{2,3} = 0$
$\delta \leq 0,\ \lambda_1 \leq 0$	$\lambda_3 \leq 0$	$D_{1,2} = 0,\ \lambda_3 \leq 0$
$\delta \leq 0,\ \lambda_4 \leq 0$	$\lambda_1 \leq 0,\ \delta \leq 0$	$D_{2,3} = 0,\ \lambda_2 \leq 0$
$\lambda_1 \leq 0,\ \lambda_3 \leq 0,\ \lambda_4 \geq 0$	$\lambda_4 \leq 0,\ \delta \geq 0$	$D_{1,2} = 0,\ D_{0,1}D_{2,3} \geq 0$
$\lambda_2 \leq 0,\ \lambda_4 \leq 0,\ \lambda_1 \geq 0$		$D_{2,3} = 0,\ D_{1,2}D_{3,4} \geq 0$

Table 2.1: The cases where the solution of the interpolation problem (2.1) does not exist.

Some possibilities are not covered by Theorem 2.1, Theorem 2.2 or Theorem 2.3. As an example, consider the points

$$\boldsymbol{T}_0 = \begin{pmatrix} -20-\xi \\ 3 \end{pmatrix}, \quad \boldsymbol{T}_1 = \begin{pmatrix} -10 \\ 1 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} -5 \\ 0 \end{pmatrix},$$
$$\boldsymbol{T}_3 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_4 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}, \quad \boldsymbol{T}_5 = \begin{pmatrix} 20+\xi \\ 3 \end{pmatrix}, \quad \xi > 0, \tag{2.4}$$

with $\lambda_1 = \lambda_4 = -\frac{\xi}{10}$, $\lambda_2 = \lambda_3 = 2$, $\delta = \mu = 1$. Note that neither the requirements of Theorem 2.1, Theorem 2.2 nor of Theorem 2.3 are met. Now, the data (2.4) admit two solutions for $\xi \in (0, \xi_0]$, where $\xi_0 := 2.95373852$ (Figure 2.2). For $\xi = \xi_0$ both of the solutions coincide with a cusp, but for $\xi > \xi_0$ no solution can be found.

The examples in Figure 1.1 all satisfy the conditions of Theorem 2.1 or of Theorem 2.2. Let us look at two of them more precisely. In the first one $\delta < \vartheta_1(\boldsymbol{\lambda}, \mu)$. Figure 2.3 (left) shows how the positions of points change as δ approaches $\vartheta_1(\boldsymbol{\lambda}, \mu)$. For $\delta \in$

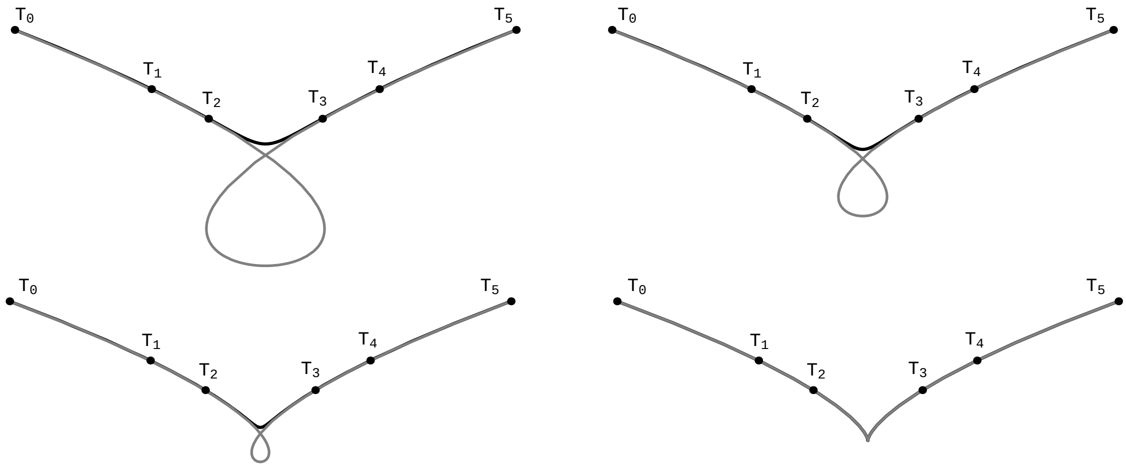


Figure 2.2: Four cubic geometric interpolants at points (2.4), with $\xi = 2,\ 2.5,\ 2.8,\ \xi_0$.

$[\vartheta_1(\boldsymbol{\lambda}, \mu), \vartheta_2(\boldsymbol{\lambda}, \mu)]$ no proper solution exists. Similarly, Figure 2.3 (right) shows the displacement of points as δ changes from $\vartheta_3(\boldsymbol{\lambda}, \mu)$ to $\vartheta_4(\boldsymbol{\lambda}, \mu)$ for the last example of Figure 1.1. For $\delta < \vartheta_3(\boldsymbol{\lambda}, \mu)$ or $\delta > \vartheta_4(\boldsymbol{\lambda}, \mu)$ two solutions $\boldsymbol{t} \in \mathcal{D}_3$ were found and the problem similar as in the example above has happened.

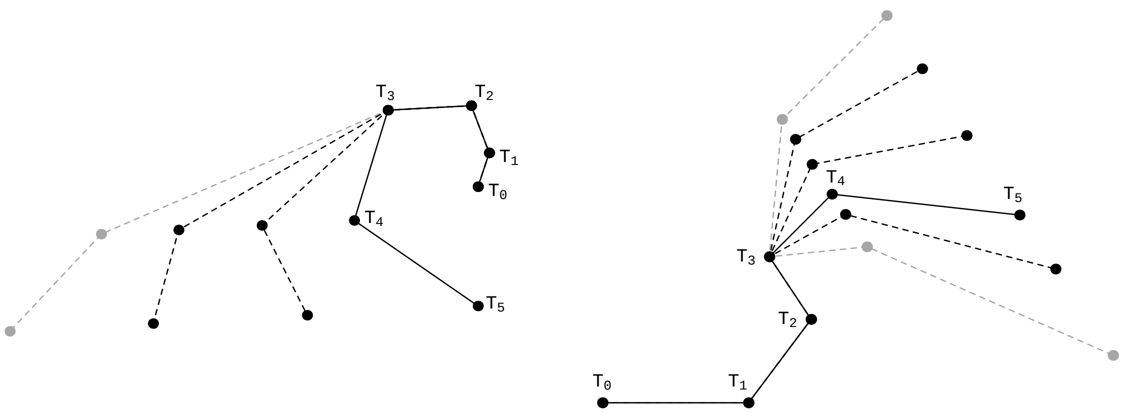


Figure 2.3: The change of point positions as δ approaches $\vartheta_1(\boldsymbol{\lambda}, \mu)$ (left), and as δ changes from $\vartheta_3(\boldsymbol{\lambda}, \mu)$ to $\vartheta_4(\boldsymbol{\lambda}, \mu)$ (right).

The requirements of Theorem 2.1 and Theorem 2.2 are quite simple, but the proof takes several steps. First, the system (2.1) is transformed into a form more suitable for further analysis. Then it is proved that any solution of (2.1) satisfying (2.2) cannot have the parameters t_i arbitrary close to the boundary $\partial\mathcal{D}_3$. For particular data it is proved that the nonlinear system has an odd number of solutions and then this fact is extended to a general case by a convex homotopy and Brouwer's degree argument.

2.2. The equations

First let us split (2.1) into the nonlinear system for unknown parameters only. The divided difference $[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]$, applied to the system (2.1), maps any $\boldsymbol{P}_3$ to zero. Let

$$\omega_{i,j}(t) := (t - t_i)(t - t_{i+1}) \cdots (t - t_{j-1})(t - t_j), \quad \dot{\omega}_{i,j}(t) = \frac{d}{dt}\omega_{i,j}(t), \quad i < j. \tag{2.5}$$

Since t_i are assumed to be distinct, one can express the divided difference in terms of $\dot{\omega}_{\ell,\ell+4}(t_i)$. The nonlinear part of the system (2.1), that should determine the unknowns t_1, t_2, t_3, t_4 thus becomes

$$\sum_{i=\ell}^{\ell+4} \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_i)} \boldsymbol{T}_i = \boldsymbol{0}, \quad \ell = 0, 1. \tag{2.6}$$

The equations (2.6) were derived as necessary conditions for the existence of the solution of the interpolation problem (2.1), but they are sufficient too. A quintic polynomial curve $\boldsymbol{p}_5$ that solves the interpolation problem

$$\boldsymbol{p}_5(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 5, \tag{2.7}$$

at distinct t_i is determined uniquely. But if $\boldsymbol{t} \in \mathcal{D}_3$ satisfies (2.6), one may apply $[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]$, $\ell = 0, 1$, to both sides of (2.7). The right hand side vanishes, so should the left hand one. This reveals that the quintic polynomial curve $\boldsymbol{p}_5$ in this case is actually a cubic one, the unique solution of (2.1). But

$$[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]1 = \sum_{i=\ell}^{\ell+4} \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_i)} = 0, \quad \ell = 0, 1, \tag{2.8}$$

and the system (2.6) can be rewritten as

$$\left(\boldsymbol{T}_i - \boldsymbol{T}_0\right)_{i=1}^{4} \left(\frac{1}{\dot{\omega}_{0,4}(t_j)}\right)_{j=1}^{4} = \boldsymbol{0}, \quad \left(\boldsymbol{T}_5 - \boldsymbol{T}_{5-i}\right)_{i=1}^{4} \left(\frac{1}{\dot{\omega}_{1,5}(t_{5-j})}\right)_{j=1}^{4} = \boldsymbol{0},$$

or, after inserting

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{Id}$$

between the two factors, and using (2.8), as

$$\left(\Delta \boldsymbol{T}_i\right)_{i=\ell}^{\ell+3} \boldsymbol{\sigma}_\ell = \boldsymbol{0}, \quad \boldsymbol{\sigma}_\ell := \left(\sum_{i=\ell}^{\ell+j} \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_i)}\right)_{j=0}^{3}, \quad \ell = 0, 1.$$

From now on let us assume that $D_{\ell+1,\ell+2} \neq 0$, $\ell = 0, 1$, as required in Theorem 2.1 and Theorem 2.2. The kernel of the matrix $\left(\Delta \boldsymbol{T}_i\right)_{i=\ell}^{\ell+3}$ is therefore two-dimensional, spanned by

$$\left(1, \quad -\frac{D_{\ell,\ell+2}}{D_{\ell+1,\ell+2}}, \quad \frac{D_{\ell,\ell+1}}{D_{\ell+1,\ell+2}}, \quad 0\right)^T, \quad \left(0, \quad -\frac{D_{\ell+2,\ell+3}}{D_{\ell+1,\ell+2}}, \quad \frac{D_{\ell+1,\ell+3}}{D_{\ell+1,\ell+2}}, \quad -1\right)^T.$$

Since $\boldsymbol{\sigma}_\ell$ must be in the kernel,

$$\boldsymbol{\sigma}_\ell = a_\ell \begin{pmatrix} 1 \\ -\frac{D_{\ell,\ell+2}}{D_{\ell+1,\ell+2}} \\ \frac{D_{\ell,\ell+1}}{D_{\ell+1,\ell+2}} \\ 0 \end{pmatrix} + b_\ell \begin{pmatrix} 0 \\ -\frac{D_{\ell+2,\ell+3}}{D_{\ell+1,\ell+2}} \\ \frac{D_{\ell+1,\ell+3}}{D_{\ell+1,\ell+2}} \\ -1 \end{pmatrix}, \quad \ell = 0, 1, \tag{2.9}$$

for some a_ℓ and b_ℓ. After the elimination of a_ℓ and b_ℓ,

$$a_\ell = \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_\ell)}, \quad b_\ell = \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_{\ell+4})}, \quad \ell = 0, 1,$$

and the use of (2.8), the equations (2.9) become

$$\frac{1}{\dot{\omega}_{0,4}(t_0)}(1 + \lambda_2) + \frac{1}{\dot{\omega}_{0,4}(t_1)} + \frac{1}{\dot{\omega}_{0,4}(t_4)}\mu = 0, \tag{2.10}$$

$$\frac{1}{\dot{\omega}_{0,4}(t_0)}\lambda_1 + \frac{1}{\dot{\omega}_{0,4}(t_3)} + \frac{1}{\dot{\omega}_{0,4}(t_4)}(1 + \delta) = 0, \tag{2.11}$$

$$\frac{1}{\dot{\omega}_{1,5}(t_1)}\left(1 + \frac{\delta}{\mu}\right) + \frac{1}{\dot{\omega}_{1,5}(t_2)} + \frac{1}{\dot{\omega}_{1,5}(t_5)}\lambda_4 = 0, \tag{2.12}$$

$$\frac{1}{\dot{\omega}_{1,5}(t_1)}\frac{1}{\mu} + \frac{1}{\dot{\omega}_{1,5}(t_4)} + \frac{1}{\dot{\omega}_{1,5}(t_5)}(1 + \lambda_3) = 0. \tag{2.13}$$

The system (2.10)–(2.13) is clearly equivalent to (2.6) since only nonsingular linear transformations were applied.

It will now be shown, that under certain restrictions the solutions $\boldsymbol{t} \in \mathcal{D}_3$ must stay aside from the boundary $\partial\mathcal{D}_3$. The theorem is stated as follows.

THEOREM 2.4. *Suppose that the requirements of Theorem 2.1 or of Theorem 2.2 are met. Then the system* (2.10) - (2.13) *cannot have a solution arbitrary close to the boundary* $\partial\mathcal{D}_3$.

The proof of Theorem 2.4 is quite technical, and will be given as the next subsection.

2.3. Proof of Theorem 2.4

In order to prove Theorem 2.4 one has to show that

$$\Delta t_i := t_{i+1} - t_i \geq \text{const} > 0, \quad i = 0, 1, \ldots, 4.$$

Here and throughout the rest of the thesis, the term 'const' will stand for an arbitrary positive constant. Suppose that at least two parameters approach, i.e., $\Delta t_i \to 0$ for some i. It is enough to consider the following four possibilities:

Case 1: $\Delta t_0 \geq \text{const} > 0,\ \Delta t_4 \geq \text{const} > 0,$

Case 2: $\Delta t_0 \geq \text{const} > 0,\ \Delta t_4 \to 0,$

Case 3: $\Delta t_0 \to 0,\ \ \Delta t_4 \geq \text{const} > 0,$

Case 4: $\Delta t_0 \to 0,\ \ \Delta t_4 \to 0.$

In order to proceed, the following lemmas are needed.

LEMMA 2.5. *Suppose that* $\Delta t_i \to 0,\ i = 0, 3,$ *and* $\Delta t_2 \geq \text{const} > 0$. *Then*

$$\frac{\dot{\omega}_{0,4}(t_0)}{\dot{\omega}_{0,4}(t_4)} \to -\frac{\lambda_1}{\delta}.$$

Similarly, $\Delta t_i \to 0,\ i = 1, 4,$ *and* $\Delta t_2 \geq \text{const} > 0$ *imply*

$$\frac{\dot{\omega}_{1,5}(t_5)}{\dot{\omega}_{1,5}(t_1)} \to -\frac{\mu\lambda_4}{\delta}.$$

Proof. Consider the first assertion. From

$$\frac{1}{t_3 - t_i} = \frac{1}{t_4 - t_i}\left(1 + \frac{\Delta t_3}{\Delta t_2 + (t_2 - t_i)}\right),$$

one obtains

$$\frac{1}{\dot{\omega}_{0,4}(t_3)} = -\frac{1}{\dot{\omega}_{0,4}(t_4)}\left(1 + \Delta t_3 \sum_{i=0}^{2} \frac{1}{\Delta t_2 + (t_2 - t_i)} + \mathcal{O}\left(\Delta t_3^2\right)\right).$$

Thus the expression

$$\frac{1}{\dot{\omega}_{0,4}(t_3)} + \frac{1}{\dot{\omega}_{0,4}(t_4)} = \prod_{i=0}^{2} \frac{1}{\Delta t_2 + \Delta t_3 + (t_2 - t_i)} \left(-\sum_{i=0}^{2} \frac{1}{\Delta t_2 + (t_2 - t_i)} + \mathcal{O}\left(\Delta t_3\right)\right)$$

stays bounded. Since $\dot{\omega}_{0,4}(t_0) \to 0$, the equation (2.11) gives

$$\frac{\dot{\omega}_{0,4}(t_0)}{\dot{\omega}_{0,4}(t_4)} = -\frac{1}{\delta}\left(\lambda_1 + \dot{\omega}_{0,4}(t_0)\left(\frac{1}{\dot{\omega}_{0,4}(t_3)} + \frac{1}{\dot{\omega}_{0,4}(t_4)}\right)\right) \to -\frac{\lambda_1}{\delta}.$$

The second assertion follows similarly. □

LEMMA 2.6. *Suppose that* $\mu > 0$. *Then* $\Delta t_i \to 0,\ i = 0, 1, 2, 3,$ *implies* $\delta > 0$ *and* $\delta \to \vartheta_1(\boldsymbol{\lambda}, \mu)$. *Similarly, from* $\Delta t_i \to 0,\ i = 1, 2, 3, 4,$ *it follows* $\delta > 0$ *and* $\delta \to \vartheta_2(\boldsymbol{\lambda}, \mu)$.

Proof. Let us prove the first assertion only. The proof of the second one is similar. After rewriting the equations (2.10) - (2.13) in a polynomial form, the last two equations simplify to

$$\delta\Delta t_2\left(\Delta t_2 + \Delta t_3\right) - \mu\Delta t_1\left(\Delta t_1 + 2\Delta t_2 + \Delta t_3\right) + \text{h.o.t.} = 0,$$
$$-\mu\Delta t_1\left(\Delta t_1 + \Delta t_2\right) + \Delta t_3\left(\Delta t_2 + \Delta t_3\right) + \text{h.o.t.} = 0,$$

where 'h. o. t.' stands for higher order terms that are small compared to the terms left in the expressions. Since $\Delta t_i > 0$, it is clear that $\delta > 0$. Moreover, by solving the first part of these two equations on Δt_2, Δt_3, the only admissible relation is

$$\Delta t_2 = \frac{\mu}{\delta}\left(1 + \sqrt{\frac{\delta+\mu}{\mu(1+\delta)}}\right)\Delta t_1 =: c_2 \Delta t_1, \quad \Delta t_3 = \mu\sqrt{\frac{\delta+\mu}{\mu(1+\delta)}}\,\Delta t_1.$$

After substituting this into the remaining equations, we obtain

$$\begin{aligned}\Delta t_0^2 + (2+c_2)\Delta t_0\Delta t_1 - \lambda_2(1+c_2)\Delta t_1^2 + \text{h.o.t.} &= 0,\\ -\Delta t_0(\Delta t_0+\Delta t_1) + \lambda_1 c_2(1+c_2)\Delta t_1^2 + \text{h.o.t.} &= 0.\end{aligned}$$

Then, by the Gröbner basis one obtains an equivalent system

$$\begin{aligned}(1+c_2)\Delta t_1^2\left(c_2^2\lambda_1(1-\lambda_1) + 2c_2\lambda_1(1+\lambda_2) - \lambda_2(1+\lambda_2)\right) + \text{h.o.t.} &= 0,\\ -(1+c_2)\Delta t_1\left(\Delta t_0 + (c_2\lambda_1-\lambda_2)\Delta t_1\right) + \text{h.o.t.} &= 0,\\ -\Delta t_0(\Delta t_0+\Delta t_1) + c_2(1+c_2)\lambda_1\Delta t_1^2 + \text{h.o.t.} &= 0.\end{aligned}$$

Only particular constants will admit the solution of this system for small positive Δt_i. Since $c_2 > 0$, a straightforward computation shows that the solution exists only if $c_2 \to \gamma_1$. Since $\lambda_1, \lambda_2 > 0$, it is easy to verify that $\gamma_1 > 0$. Therefrom by solving $c_2 = \gamma_1$ for δ, one obtains $\delta \to \vartheta_1(\boldsymbol{\lambda}, \mu)$, where $\vartheta_1(\boldsymbol{\lambda}, \mu) > 0$ as can easily be checked. □

REMARK 2.7. *Note that if* $\vartheta_1(\boldsymbol{\lambda}, \mu) = \vartheta_2(\boldsymbol{\lambda}, \mu)$, *the parameters* t_i, $i = 1,2,3,4$, *cannot approach* t_0 *and* t_5 *at the same time.*

LEMMA 2.8. *Suppose that* $\mu < 0$. *Then* $\Delta t_i \to 0$, $i = 0,1,2,4$, *implies* $\delta \to \vartheta_3(\boldsymbol{\lambda}, \mu)$, *and similarly,* $\Delta t_i \to 0$, $i = 0,2,3,4$, *implies* $\delta \to \vartheta_4(\boldsymbol{\lambda}, \mu)$.

Proof. Let us prove the second statement. After rewriting the equations (2.10) - (2.13) in a polynomial form, the last two equations simplify to

$$\begin{aligned}\lambda_4\Delta t_2(\Delta t_2+\Delta t_3) - \Delta t_4(\Delta t_3+\Delta t_4) + \text{h.o.t.} &= 0,\\ \lambda_3\Delta t_3(\Delta t_2+\Delta t_3) - \Delta t_4(\Delta t_2+2\Delta t_3+\Delta t_4) + \text{h.o.t.} &= 0.\end{aligned}$$

By solving the main part of these two equations for Δt_2, Δt_3, the only admissible relation is given as

$$\begin{aligned}\Delta t_2 &= \sqrt{\frac{1+\lambda_3}{\lambda_4(\lambda_3+\lambda_4)}}\,\Delta t_4 =: c_2\Delta t_4,\\ \Delta t_3 &= \frac{1}{\lambda_3}\left(1+\sqrt{\frac{\lambda_4(1+\lambda_3)}{\lambda_3+\lambda_4}}\right)\Delta t_4 =: c_3\Delta t_4.\end{aligned}$$

After substituting these expressions into the remaining equations, we obtain

$$\begin{aligned}\mu\Delta t_0 + c_3(c_2+c_3)\lambda_2\Delta t_4^2 + \text{h.o.t.} &= 0,\\ (c_2\delta - c_3)\Delta t_0 + c_2c_3(c_2+c_3)\lambda_1\Delta t_4^2 + \text{h.o.t.} &= 0.\end{aligned}$$

Again, with the help of the Gröbner basis, the equivalent system reads as

$$\begin{aligned}
c_3(c_2+c_3)\Delta t_4^2\left(c_2\left(\lambda_2\delta-\lambda_1\mu\right)-c_3\lambda_2\right)+\text{h.o.t.}&=0,\\
\mu\Delta t_0+c_3(c_2+c_3)\lambda_2\Delta t_4^2+\text{h.o.t.}&=0,\\
(c_2\delta-c_3)\Delta t_0+c_2c_3(c_2+c_3)\lambda_1\Delta t_4^2+\text{h.o.t.}&=0.
\end{aligned}$$

Since $c_2>0$, $c_3>0$, it is easy to verify that this system will have a solution for small Δt_i only if $\delta\to\vartheta_4\left(\boldsymbol{\lambda},\mu\right)$. The first statement is proved in a similar way. □

Now to the proof of Theorem 2.4. Note that

$$\operatorname{sign}\left(\dot\omega_{\ell,\ell+4}(t_i)\right)=(-1)^{\ell+i},\quad i=\ell,\ell+1,\ldots,\ell+4.$$

Also, from the equations (2.10) - (2.13) it is straightforward to derive a useful relation

$$\frac{\dot\omega_{0,4}(t_2)}{\dot\omega_{0,4}(t_3)}\frac{\dot\omega_{1,5}(t_3)}{\dot\omega_{1,5}(t_2)}=\frac{1+\delta+\lambda_1\dfrac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_0)}}{\delta+\mu+(\lambda_1+\lambda_2)\dfrac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_0)}}\,\frac{1+\frac{\delta}{\mu}+\lambda_4\dfrac{\dot\omega_{1,5}(t_1)}{\dot\omega_{1,5}(t_5)}}{\dfrac{1+\delta}{\mu}+(\lambda_3+\lambda_4)\dfrac{\dot\omega_{1,5}(t_1)}{\dot\omega_{1,5}(t_5)}}. \tag{2.14}$$

Case 1: In this case $\dot\omega_{0,4}(t_0)\geq\text{const}>0$, $\dot\omega_{1,5}(t_5)\geq\text{const}>0$. From the equations (2.10) - (2.13) it is straightforward to see that $\Delta t_1\to0$ or $\Delta t_2\to0$ implies $\Delta t_3\to0$. Consequently

$$\frac{\dot\omega_{0,4}(t_4)}{t_4-t_1}=\left(1-\Delta t_4\right)\left(\Delta t_2+\Delta t_3\right)\Delta t_3\to0.$$

From (2.10) and (2.13) it is easy to derive

$$\frac{t_4-t_1}{\dot\omega_{0,4}(t_4)}=\frac{1+\lambda_2}{\mu}\frac{\Delta t_0\Delta t_4}{\dot\omega_{0,4}(t_0)}+(1+\lambda_3)\frac{(1-\Delta t_0)\Delta t_4}{\dot\omega_{1,5}(t_5)}.$$

Since the right hand side is bounded, but the left hand one is not, we have a contradiction that excludes the case 1.

Case 2: In this case $\dot\omega_{0,4}(t_0)\geq\text{const}>0$, and $\dot\omega_{1,5}(t_5)\to0$. Suppose first that $\Delta t_2\geq\text{const}>0$. The equation (2.12) then implies $\Delta t_1\to0$. But then, (2.10) implies $\Delta t_3\to0$, and further

$$\dot\omega_{0,4}(t_4)=\left(1-\Delta t_4\right)\left(\Delta t_1+\Delta t_2+\Delta t_3\right)\left(\Delta t_2+\Delta t_3\right)\Delta t_3\to0.$$

Moreover, the equation (2.11) yields

$$-\delta=1+\lambda_1\frac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_0)}+\frac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_3)}=1+\frac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_0)}-\prod_{i=0}^{2}\left(1+\frac{\Delta t_3}{\Delta t_2+(t_2-t_i)}\right)\to0.$$

Now, by Lemma 2.5 and the use of relation (2.14) one obtains

$$\frac{\dot\omega_{0,4}(t_4)}{\dot\omega_{0,4}(t_0)}\to0,\qquad\frac{\dot\omega_{1,5}(t_1)}{\dot\omega_{1,5}(t_5)}\to0,\qquad\frac{\dot\omega_{0,4}(t_2)}{\dot\omega_{0,4}(t_3)}\frac{\dot\omega_{1,5}(t_3)}{\dot\omega_{1,5}(t_2)}\to1.$$

However, on the other hand,

$$\frac{\dot\omega_{0,4}(t_2)}{\dot\omega_{0,4}(t_3)}\frac{\dot\omega_{1,5}(t_3)}{\dot\omega_{1,5}(t_2)} = \frac{\Delta t_0+\Delta t_1}{\Delta t_0+\Delta t_1+\Delta t_2}\,\frac{\Delta t_3+\Delta t_4}{\Delta t_2+\Delta t_3+\Delta t_4} \to 0,$$

which is a contradiction. Therefore $\Delta t_2 \to 0$. But then the equations (2.10) and (2.11) imply $\Delta t_1 \to 0$, $\Delta t_3 \to 0$, $\mu > 0$, and Lemma 2.6 excludes the second case. The third case is a mirror view of the second one, and needs not to be proved.
Case 4: Here $\dot\omega_{0,4}(t_0)$, $\dot\omega_{1,5}(t_5) \to 0$. Suppose again for a moment that $\Delta t_2 \geq \text{const} > 0$. The equations (2.11) and (2.12) then imply $\Delta t_1 \to 0$ and $\Delta t_3 \to 0$. So, by Lemma 2.5,

$$\frac{\dot\omega_{0,4}(t_0)}{\dot\omega_{0,4}(t_4)} \to -\frac{\lambda_1}{\delta},\quad \frac{\dot\omega_{1,5}(t_5)}{\dot\omega_{1,5}(t_1)} \to -\frac{\mu\lambda_4}{\delta}.$$

Therefrom by using the relation (2.14) we obtain

$$\frac{\dot\omega_{0,4}(t_2)\dot\omega_{1,5}(t_3)}{\dot\omega_{1,5}(t_2)\dot\omega_{0,4}(t_3)} \to \frac{\lambda_1}{\lambda_1\mu-\lambda_2\delta}\,\frac{\lambda_4\mu}{\lambda_4-\lambda_3\delta} \neq 0,$$

but on the other hand

$$\frac{\dot\omega_{0,4}(t_2)}{\dot\omega_{0,4}(t_3)}\frac{\dot\omega_{1,5}(t_3)}{\dot\omega_{1,5}(t_2)} = \frac{\Delta t_0+\Delta t_1}{\Delta t_0+\Delta t_1+\Delta t_2}\,\frac{\Delta t_3+\Delta t_4}{\Delta t_2+\Delta t_3+\Delta t_4} \to 0.$$

Therefore $\Delta t_2 \to 0$. Suppose now that $\Delta t_1 \geq \text{const} > 0$. The equation (2.10) gives

$$\dot\omega_{0,4}(t_4) = -\frac{\mu\,\dot\omega_{0,4}(t_0)}{1+\lambda_2+\dfrac{\dot\omega_{0,4}(t_0)}{\dot\omega_{0,4}(t_1)}} = -\frac{\mu\,\dot\omega_{0,4}(t_0)}{1+\lambda_2-\prod\limits_{i=2}^{4}\left(1+\dfrac{\Delta t_0}{t_i-t_2+\Delta t_1}\right)} \to 0,$$

so $\Delta t_3 \to 0$, and $\mu < 0$. But by Lemma 2.8 this cannot happen. Similarly one can prove that $\Delta t_3 \geq \text{const} > 0$ implies $\Delta t_1 \to 0$, and $\mu < 0$. But, again by Lemma 2.8, this cannot happen either, which excludes the case 4, and therefore completes the proof of Theorem 2.4.

2.4. A particular case

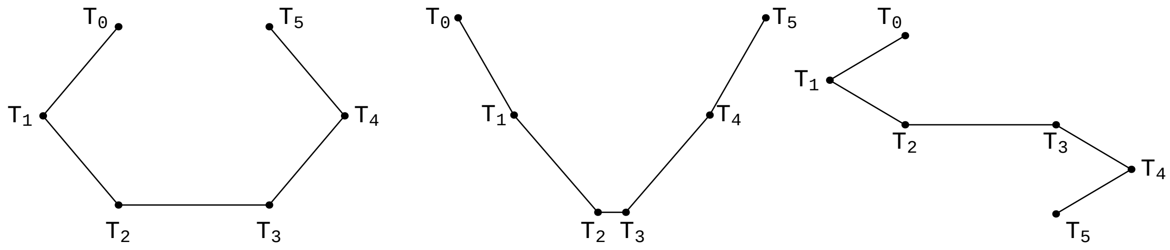
Let us now consider the system (2.10) - (2.13) for particular data points

$$\begin{aligned} &\boldsymbol{T}_0^* = \begin{pmatrix}1-2c\\2\end{pmatrix}, &&\boldsymbol{T}_1^* = \begin{pmatrix}-1-c\\1\end{pmatrix}, &&\boldsymbol{T}_2^* = \begin{pmatrix}-1\\0\end{pmatrix}, && \\ &\boldsymbol{T}_3^* = \begin{pmatrix}1\\0\end{pmatrix}, &&\boldsymbol{T}_4^* = \begin{pmatrix}1+c\\(-1)^s\end{pmatrix}, &&\boldsymbol{T}_5^* = \begin{pmatrix}-1+2c\\2(-1)^s\end{pmatrix}, &&s=0,1, \end{aligned} \tag{2.15}$$

where s determines whether μ^* is positive or negative, and $c = 0$ or $c = 6$ as will be needed in the proof of Theorem 2.1 (see Figure 2.4). It is straightforward to compute

$$\boldsymbol{\lambda}^* = \mathbf{1},\quad \delta^* = \frac{1}{2}\left(1+(-1)^s\right)c,\quad \mu^* = (-1)^s.$$

The number of all admissible solutions, i.e., solutions $\boldsymbol{t} \in \mathcal{D}_3$, is given in the next theorem.



$s = 0,\ c = 1$ $\quad$ $s = 0,\ c = 6$ $\quad$ $s = 1,\ c = 1$

Figure 2.4: Particular data points (2.15).

THEOREM 2.9. *Suppose the data points* $\boldsymbol{T}_\ell$ *are given by* (2.15) *with* $c = 0$ *or* $c = 6$. *The number of admissible solutions* $\boldsymbol{t} \in \mathcal{D}_3$, *counted with multiplicity, is odd. More precisely, the symmetric solution that satisfies* $\Delta t_0 = \Delta t_4$ *and* $\Delta t_1 = \Delta t_3$, *is unique. The number of the other solutions is even.*

Proof. System (2.10) - (2.13) for data points (2.15) simplifies to

$$\begin{aligned}
\frac{2}{\dot\omega_{0,4}(t_0)} + \frac{1}{\dot\omega_{0,4}(t_1)} + (-1)^s \frac{1}{\dot\omega_{0,4}(t_4)} &= 0,\\
\frac{1}{\dot\omega_{0,4}(t_0)} + \frac{1}{\dot\omega_{0,4}(t_3)} + \left(1 + \frac{c}{2} + (-1)^s \frac{c}{2}\right) \frac{1}{\dot\omega_{0,4}(t_4)} &= 0,\\
\frac{2}{\dot\omega_{1,5}(t_5)} + \frac{1}{\dot\omega_{1,5}(t_4)} + (-1)^s \frac{1}{\dot\omega_{1,5}(t_1)} &= 0,\\
\frac{1}{\dot\omega_{1,5}(t_5)} + \frac{1}{\dot\omega_{1,5}(t_2)} + \left(1 + \frac{c}{2} + (-1)^s \frac{c}{2}\right) \frac{1}{\dot\omega_{1,5}(t_1)} &= 0.
\end{aligned} \tag{2.16}$$

If there exists an admissible nonsymmetric solution $(t_i)_{i=0}^5$, then $(1 - t_{5-i})_{i=0}^5$ is also an admissible solution, since

$$\dot\omega_{0,4}(1 - t_{5-i}) = \dot\omega_{1,5}(t_{5-i}), \quad i = 0, 1, \ldots, 4, \quad \dot\omega_{1,5}(1 - t_{5-i}) = \dot\omega_{0,4}(t_{5-i}), \quad i = 1, 2, \ldots, 5.$$

Therefore the number of solutions, that are not symmetric, must be even. Let us examine the symmetric solutions now. It is easy to see that the first and the last two equations in (2.16) are then identical, and one is left with two equations

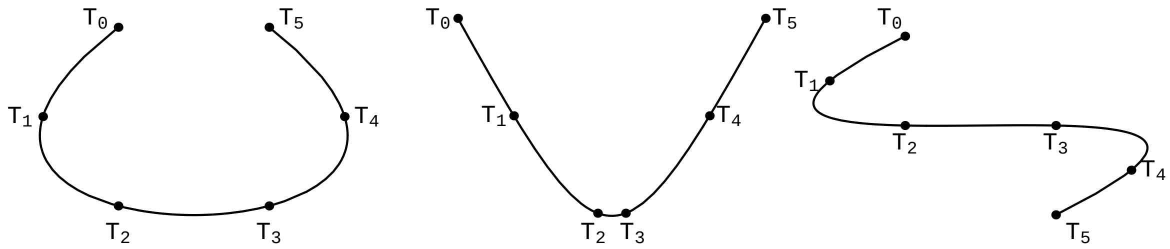
$$\frac{2t_3(t_3-1)\left(3t_4 - 2 + (-1)^s(1-t_4)\right) - 4t_4(2t_4-1)(t_4-1)}{t_3 t_4 (t_3-1)(t_4-1)(t_3-t_4)(t_3+t_4-1)(2t_4-1)} = 0,$$

$$\frac{2t_4(t_3-t_4)(2t_3+2t_4-3) - c(t_3-1)(t_4-1)(2t_3-1)\left(1+(-1)^s\right)}{t_4(2t_3-1)(t_3-1)(t_4-1)(2t_4-1)(t_3-t_4)(t_3+t_4-1)} = 0,$$

for two unknowns ordered as $\frac{1}{2} < t_3 < t_4 < 1$. This yields a polynomial system that can be solved analytically. The admissible solution is unique (Table 2.2) and it is shown in Figure 2.5. The proof of Theorem 2.9 is completed. □

	$s=0,\ c=0$	$s=0,\ c=6$	$s=1,\ c\in\mathbb{R}$
t_3	$\frac{1}{2}\left(3-\sqrt{3}\right)$	$\frac{1}{44}\left(21-9\sqrt{3}+\sqrt{300+38\sqrt{3}}\right)$	$\frac{3}{5}$
t_4	$\frac{\sqrt{3}}{2}$	$\frac{1}{44}\left(36+5\sqrt{3}-\sqrt{243-112\sqrt{3}}\right)$	$\frac{9}{10}$

Table 2.2: The admissible symmetric solutions of the system (2.15).



$s=0,\ c=1$ $\qquad$ $s=0,\ c=6$ $\qquad$ $s=1,\ c=1$

Figure 2.5: Interpolating cubic curves for particular data points (2.15).

2.5. Proofs of main theorems

In order to prove Theorem 2.1 and Theorem 2.2 one must show that the nonlinear system (2.10) - (2.13) has at least one solution $\boldsymbol{t}\in\mathcal{D}_3$. The convex homotopy and Brouwer's degree argument (see Chapter 7) will help us carry the conclusions from a particular to the general case.

Let us multiply (2.12) and (2.13) by μ and denote the obtained system (2.10) - (2.13) by $\boldsymbol{F}(\boldsymbol{t};\boldsymbol{\lambda},\delta,\mu)=\boldsymbol{0}$. Now, $\boldsymbol{F}$ can be split as $\boldsymbol{F}(\cdot;\boldsymbol{\lambda},\delta,\mu)=\boldsymbol{F}_1(\cdot;\boldsymbol{\lambda},\mu)+\delta\boldsymbol{F}_2$, where

$$\boldsymbol{F}_1(\cdot;\boldsymbol{\lambda},\mu):=\boldsymbol{F}(\cdot;\boldsymbol{\lambda},0,\mu),\quad \boldsymbol{F}_2:=\boldsymbol{F}(\cdot;\boldsymbol{\lambda},1,\mu)-\boldsymbol{F}_1(\cdot;\boldsymbol{\lambda},\mu).$$

A general data will be denoted by $(\boldsymbol{\lambda},\delta,\mu)$, and the particular data (2.15), where s is chosen so that $\operatorname{sign}(\mu^*)=\operatorname{sign}(\mu)$, by $(\boldsymbol{\lambda}^*,\delta^*,\mu^*)$. The homotopy is now defined as

$$\boldsymbol{H}(\boldsymbol{t};\zeta):=(1-\zeta)\boldsymbol{F}_1(\boldsymbol{t};\boldsymbol{\lambda}^*,\mu^*)+\zeta\boldsymbol{F}_1(\boldsymbol{t};\boldsymbol{\lambda},\mu)+q(\zeta,\delta^*,\delta)\boldsymbol{F}_2(\boldsymbol{t}),$$

where $q(\cdot;\delta^*,\delta):[0,1]\to\mathbb{R}$, satisfies $q(0;\delta^*,\delta)=\delta^*$, $q(1;\delta^*,\delta)=\delta$. Moreover, let

$$\boldsymbol{\lambda}(\zeta):=(1-\zeta)\boldsymbol{\lambda}^*+\zeta\boldsymbol{\lambda},\quad \delta(\zeta):=q(\zeta,\delta^*,\delta),\quad \mu(\zeta):=(1-\zeta)\mu^*+\zeta\mu.$$

Then

$$\lambda_i(\zeta)\geq\min_{\zeta\in[0,1]}\left((1-\zeta)\lambda_i^*+\zeta\lambda_i\right)\geq\min\left\{\lambda_i^*,\lambda_i\right\}\geq\text{const}>0,$$
$$|\mu(\zeta)|\geq\min_{\zeta\in[0,1]}|(1-\zeta)\mu^*+\zeta\mu|\geq\min\left\{|\mu^*|,|\mu|\right\}\geq\text{const}>0.$$

Consider the case $\mu>0$ as in Theorem 2.1 first. Note that $\vartheta_1(\boldsymbol{\lambda}^*,\mu^*)=\vartheta_2(\boldsymbol{\lambda}^*,\mu^*)=4$. If $\delta<\min_{\ell=1,2}\left\{\vartheta_\ell\left(\boldsymbol{\lambda},\mu\right)\right\}$ let us choose $c=\delta^*=0$. It is then clear, that there exists a piecewise linear function $q(\zeta,\delta^*,\delta)$, such that

$$q(\zeta,\delta^*,\delta)<\min_{\ell=1,2}\left\{\vartheta_\ell\left(\boldsymbol{\lambda}(\zeta),\mu(\zeta)\right)\right\},\quad \zeta\in[0,1].$$

Similarly we can do for $\delta > \max_{\ell=1,2} \{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\}$ by choosing $c = \delta^* = 6$. In the case when $\mu < 0$, as in Theorem 2.2, we have

$$\vartheta_3(\boldsymbol{\lambda}^*, \mu^*) = -1 < \delta^* = 0 < \vartheta_4(\boldsymbol{\lambda}^*, \mu^*) = 1.$$

Since $\mu(\zeta) < 0$, it is straightforward to see that $\vartheta_3(\boldsymbol{\lambda}(\zeta), \mu(\zeta))$ and $\vartheta_4(\boldsymbol{\lambda}(\zeta), \mu(\zeta))$ can not intersect for $\zeta \in [0, 1]$. Thus there obviously exists a piecewise linear function $q(\zeta, \delta^*, \delta)$, such that

$$\vartheta_3(\boldsymbol{\lambda}(\zeta), \mu(\zeta)) < q(\zeta, \delta^*, \delta) < \vartheta_4(\boldsymbol{\lambda}(\zeta), \mu(\zeta)), \quad \zeta \in [0, 1].$$

Therefore $\boldsymbol{H}(\boldsymbol{t}, \zeta) = \boldsymbol{0}$ meets the requirements of Theorem 2.4 for any $\zeta \in [0, 1]$. As a consequence, a set of solutions

$$V := \{\boldsymbol{t} \in \mathcal{D}_3; \quad \boldsymbol{H}(\boldsymbol{t}, \zeta) = \boldsymbol{0}\}$$

lies aside from the boundary $\partial\mathcal{D}_3$. More precisely, one can find a compact set $K \subset \mathcal{D}_3$, such that

$$V \subset K \subset \mathcal{D}_3, \quad V \cap \partial K = \emptyset.$$

Thus the map $\boldsymbol{H}$ does not vanish at the boundary ∂K, and Brouwer's degree of $\boldsymbol{H}$ on K is invariant for all $\zeta \in [0, 1]$. But by Theorem 2.9, it is odd for the particular map $\boldsymbol{F}(\cdot; \boldsymbol{\lambda}^*, \delta^*, \mu^*)$. Therefore $\boldsymbol{F}(\boldsymbol{t}; \boldsymbol{\lambda}, \delta, \mu) = \boldsymbol{0}$ must have at least one admissible solution and Theorem 2.1 and Theorem 2.2 are proved.

Let us now prove Theorem 2.3. Since the geometric interpolation is independent of affine transformations of data points, one can choose the coordinate system so that one axis is in the direction of $\Delta\boldsymbol{T}_1$, $\Delta\boldsymbol{T}_2$ or $\Delta\boldsymbol{T}_3$. It is then straightforward to verify that the conditions of Theorem 2.3 imply that the other component of the interpolating curve as a cubic polynomial should have four zeros, which is a contradiction. The proof for the

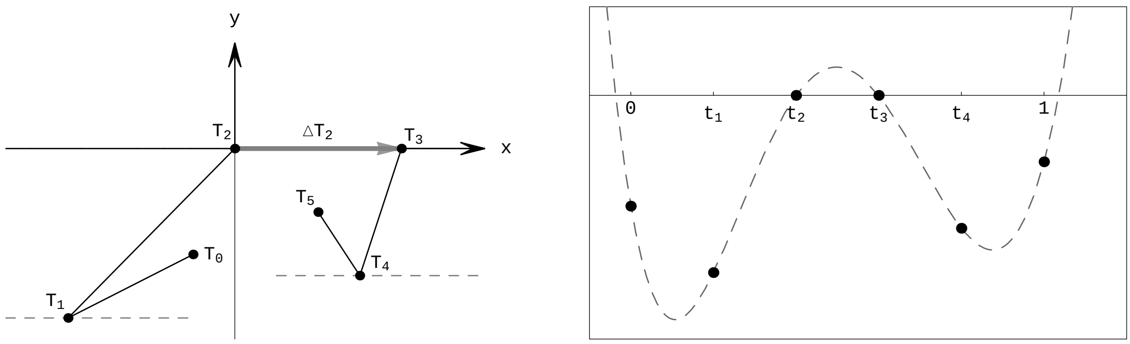


Figure 2.6: The data points with $\mu > 0$, $\lambda_2 \leq 0$, $\lambda_4 \leq 0$ (left), and the y - component of $\boldsymbol{P}_3$ (right).

case $\mu > 0$, $\lambda_2 \leq 0$, and $\lambda_4 \leq 0$, is sketched in Figure 2.6. The other cases follow by the same approach. This completes the proof of Theorem 2.3.

2.6. Examples

Let us conclude this chapter with a few numerical examples. The easiest way to compute the solution of the system (2.10)–(2.13) is by using the Newton method. But its convergence depends on the chosen starting value. By choosing a different starting value a different solution may be computed, and some of the solutions may not be obtained this way. Even numerically, it is not an easy task to get all the admissible solutions. One way to do it is by computing the Gröbner basis. But for our system this was impossible to do in a real time even on very efficient computers. All the admissible solutions can be computed by the continuation method ([2]). Another tool that can be used are resultants (see Chapter 7). Let us describe how one can use them for finding all the admissible solutions. Denote the equations (2.10)–(2.13) rewritten in a polynomial form by $\boldsymbol{q}(\boldsymbol{t}) = (q_i(\boldsymbol{t}))_{i=1}^4$. The idea is to eliminate the variables one after another until we are left with only one polynomial equation for one unknown. Namely, to eliminate the variable t_4 compute the resultants

$$\begin{aligned} r_1(t_1,t_2,t_3) &:= \mathrm{Res}(q_1,q_2;t_4),\\ r_2(t_1,t_2,t_3) &:= \mathrm{Res}(q_3,q_4;t_4),\\ r_3(t_1,t_2,t_3) &:= \mathrm{Res}(q_1,q_4;t_4). \end{aligned}$$

Polynomials r_1, r_2, r_3 may contain extraneous factors like $(t_i - t_j)$, that do not produce the admissible solution. Exclude these factors and denote the remaining polynomials by s_1, s_2 and s_3. Further, eliminate the variable t_3,

$$\begin{aligned} r_4(t_1,t_2) &:= \mathrm{Res}(s_1,s_3;t_3),\\ r_5(t_1,t_2) &:= \mathrm{Res}(s_2,s_3;t_3). \end{aligned}$$

Again exclude the extraneous factors and denote the polynomials by s_4 and s_5. Finally compute

$$r_6(t_1) := \mathrm{Res}(s_4,s_5;t_2)$$

to eliminate the variable t_2. Solve the equation $r_6(t_1) = 0$ and select only the solutions that satisfy $0 < t_1 < 1$. Now, insert each of the solution t_1 into r_4 and r_5 and solve $r_4(t_1,t_2) = 0$, $r_5(t_1,t_2) = 0$ on t_2. Again select only those solutions that satisfy $0 < t_1 < t_2 < 1$. Further, insert each pair of the solutions (t_1,t_2) into r_i and solve $r_i(t_1,t_2,t_3) = 0$, $i = 1,2,3$, on t_3. Keep only the solutions that satisfy $0 < t_1 < t_2 < t_3 < 1$. Finally, insert each such pair (t_1,t_2,t_3) into q_i, solve $q_i(t_1,t_2,t_3,t_4) = 0$, $i = 1,2,3,4$, and keep the admissible solutions. Finally check if the computed $(t_i)_{i=1}^4$ really is the solution of the system (2.10)–(2.13), since some redundant solutions may be produced.

Let us use the described method to compute all the admissible solutions of the next two interpolation problems. First the data points are given as

$$\begin{aligned} &\boldsymbol{T}_0 = \begin{pmatrix}-2\\-3\end{pmatrix}, && \boldsymbol{T}_1 = \begin{pmatrix}-6\\0\end{pmatrix}, && \boldsymbol{T}_2 = \begin{pmatrix}-4\\3\end{pmatrix},\\ &\boldsymbol{T}_3 = \begin{pmatrix}0\\4\end{pmatrix}, && \boldsymbol{T}_4 = \begin{pmatrix}2\\1\end{pmatrix}, && \boldsymbol{T}_5 = \begin{pmatrix}1\\0\end{pmatrix}. \end{aligned} \tag{2.17}$$

It is straightforward to compute

$$\lambda_1 = \frac{9}{5}, \quad \lambda_2 = \frac{8}{5}, \quad \lambda_3 = \frac{3}{14}, \quad \lambda_4 = \frac{5}{14}, \quad \delta = \frac{6}{5}, \quad \mu = \frac{7}{5},$$

and

$$\vartheta_1(\boldsymbol{\lambda}, \mu) = 5.45835, \quad \vartheta_2(\boldsymbol{\lambda}, \mu) = 7.72159.$$

Since $\delta < \vartheta_1$ and $\delta < \vartheta_2$, the conditions of Theorem 2.1 are fulfilled. By computing resultants r_i, $i = 1, 2, \ldots, 6$, one can see that the final polynomial r_6 is of degree 111. Furthermore, there is only one admissible solution

$$t_1 = 0.179372, \quad t_2 = 0.418313, \quad t_3 = 0.611056, \quad t_4 = 0.953539.$$

As the second example the points are chosen as

$$\boldsymbol{T}_0 = \begin{pmatrix} -9 \\ -8 \end{pmatrix}, \quad \boldsymbol{T}_1 = \begin{pmatrix} -7 \\ -1 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \boldsymbol{T}_3 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \quad \boldsymbol{T}_4 = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \quad \boldsymbol{T}_5 = \begin{pmatrix} 10 \\ 9 \end{pmatrix}. \tag{2.18}$$

Now,

$$\lambda_1 = \frac{12}{11}, \quad \lambda_2 = \frac{41}{33}, \quad \lambda_3 = \frac{41}{27}, \quad \lambda_4 = \frac{22}{27}, \quad \delta = -\frac{2}{11}, \quad \mu = -\frac{9}{11},$$

and

$$\vartheta_3(\boldsymbol{\lambda}, \mu) = -0.883441, \quad \vartheta_4(\boldsymbol{\lambda}, \mu) = 0.390352,$$

so that the conditions of Theorem 2.2 are met. The polynomial r_6 is of degree 111 and the admissible solution is again unique, i.e.,

$$t_1 = 0.113433, \quad t_2 = 0.418394, \quad t_3 = 0.665168, \quad t_4 = 0.878749.$$

Interpolating curves are shown in Figure 2.7.

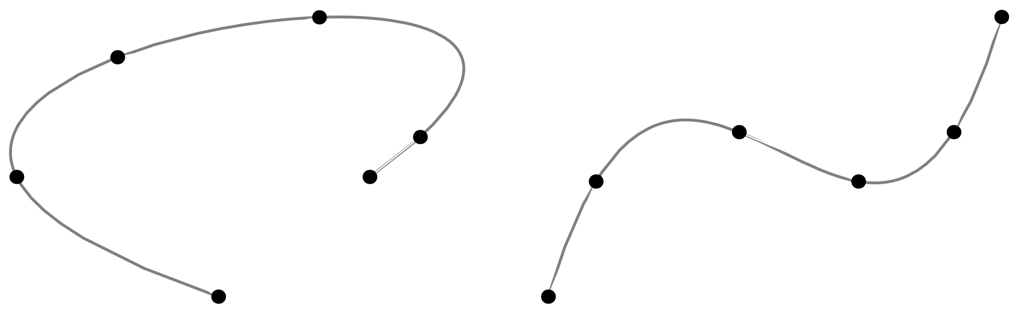


Figure 2.7: Geometric interpolants for data (2.17) (left) and (2.18) (right).

Chapter 3

Geometric interpolation by cubic G^1 splines

In the previous chapter the Lagrange geometric interpolation of six points by a cubic polynomial curve is studied. Here the results are extended to an interpolation of four points and two tangent directions. Moreover, the geometric interpolation by a cubic G^1 spline is considered. A wide class of sufficient conditions that admit a cubic G^1 spline interpolant is determined. In particular, convex data as well as data with inflection point are included. The existence requirements are based upon geometric properties of data entirely, and can easily be verified in advance. An algorithm that carries out the verification is added.

3.1. Interpolation problem

The interpolation problem concerned is the following. Let

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad i = 0, 1, 2, \ldots, 3m, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \tag{3.1}$$

be a given sequence of data points. Find a cubic G^1 spline curve $\boldsymbol{S} : [a, b] \to \mathbb{R}^2$ with breakpoints

$$a := u_0 < u_1 < \cdots < u_m := b$$

that interpolates the data $\boldsymbol{T}_i$ in the prescribed order so that $\boldsymbol{S}(u_\ell) = \boldsymbol{T}_{3\ell}$. Let $\boldsymbol{d}_{3\ell}$, $\|\boldsymbol{d}_{3\ell}\|_2 = 1$, denote the tangent directions of the spline curve $\boldsymbol{S}$ at u_ℓ. A piecewise representation

$$\boldsymbol{P}^\ell\left(t^\ell\right) := \boldsymbol{S}(u)\big|_{[u_{\ell-1}, u_\ell]}, \quad t^\ell := \frac{u - u_{\ell-1}}{\Delta u_{\ell-1}} \in [0, 1], \quad \ell = 1, 2, \ldots, m,$$

rewrites the interpolation problem as follows: find cubic polynomials $\boldsymbol{P}^\ell$ such that

$$\begin{aligned} &\boldsymbol{P}^\ell\left(t_i^\ell\right) = \boldsymbol{T}_{3(\ell-1)+i}, \quad i = 0, 1, \ldots, 3, \\ &\frac{d}{dt^\ell}\boldsymbol{P}^\ell(0) = \alpha_0^\ell \boldsymbol{d}_{3(\ell-1)}, \quad \frac{d}{dt^\ell}\boldsymbol{P}^\ell(1) = \alpha_3^\ell \boldsymbol{d}_{3\ell}, \end{aligned} \qquad \ell = 1, 2, \ldots, m, \tag{3.2}$$

where the unknown parameters t_1^ℓ, t_2^ℓ, $\alpha_0^\ell, \alpha_3^\ell$ must satisfy

$$0 =: t_0^\ell < t_1^\ell < t_2^\ell < t_3^\ell := 1, \quad \alpha_0^\ell > 0, \quad \alpha_3^\ell > 0, \quad \ell = 1, 2, \ldots, m. \tag{3.3}$$

Note that α_i^ℓ are chosen as local derivative lengths rather than global in order to simplify the notation for further discussion.

The tangent directions $\boldsymbol{d}_{3\ell}$, $\ell = 1, 2, \ldots, m-1$, have clearly not been prescribed by the data (3.1) yet. However, they may be known as data or given as an approximation, perhaps as interactive shape parameters, or implicitly prescribed by the requirement that $\boldsymbol{S}$ is G^2 too. In the latter case, $\boldsymbol{d}_0$ and $\boldsymbol{d}_{3m}$ would be known, and the following $m-1$ equations

$$\begin{aligned} &\frac{1}{\left(\alpha_3^\ell\right)^2} \det\left(3\left(\boldsymbol{T}_{3\ell} - \boldsymbol{T}_{3\ell-3}\right) - \alpha_0^\ell \boldsymbol{d}_{3\ell-3},\, \boldsymbol{d}_{3\ell}\right) = \\ &\frac{1}{\left(\alpha_0^{\ell+1}\right)^2} \det\left(\boldsymbol{d}_{3\ell},\, 3\left(\boldsymbol{T}_{3\ell+3} - \boldsymbol{T}_{3\ell}\right) - \alpha_3^{\ell+1} \boldsymbol{d}_{3\ell+3}\right), \end{aligned} \qquad \ell = 1, 2, \ldots, m-1, \tag{3.4}$$

added (see [13]). But, in general, the problem (3.2) and (3.3) need not have a solution. So it is quite possible that the curve $\boldsymbol{S}$ could not interpolate all the prescribed data. For this reason the interpolation problem (3.2) and (3.3) is split into two steps. At the first and the main step, the region for $(\boldsymbol{d}_\ell)_{\ell=1}^m$ that admits a solution of (3.2) is determined. The second step is left to the user, but with clear bounds on $\boldsymbol{d}_{3\ell}$. Some suggestions on how to choose the tangent directions are given in Subsection 3.4.

An example of a cubic G^1 spline that interpolates the prescribed data points and tangent directions is given in Figure 3.1.

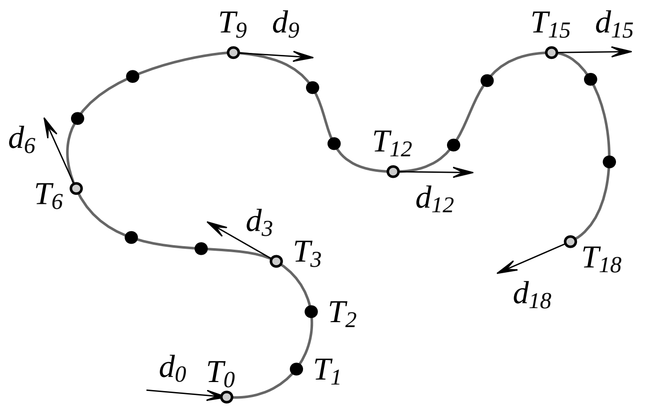


Figure 3.1: Interpolating G^1 spline for given data points and given tangent directions.

As expected, it is not possible to break apart sufficient conditions that admit a solution to a local level. However, if the data are convex, we are able to determine the allowed angles for $\boldsymbol{d}_{3\ell}$ by the local data only. To be precise, at a point $\boldsymbol{T}_{3\ell}$ the angle between $\Delta\boldsymbol{T}_{3\ell-1}$ and $\Delta\boldsymbol{T}_{3\ell}$ gives a range for $\boldsymbol{d}_{3\ell}$ that is further split into at most three subangles. This partition depends only on data $\boldsymbol{T}_{3\ell-3}, \boldsymbol{T}_{3\ell-2}, \ldots, \boldsymbol{T}_{3\ell+3}$. All that is left is to connect particular subangles in an allowable global choice by taking into account certain simple additional relations between subangles at different breakpoints. This is carried out by a straightforward backtracking algorithm. Figure 3.2 shows three such possible choices (gray). But if the data imply an inflexion point, the answer is not so obvious, and is left to Subsection 3.4, as well as a precise explanation of the convex case.

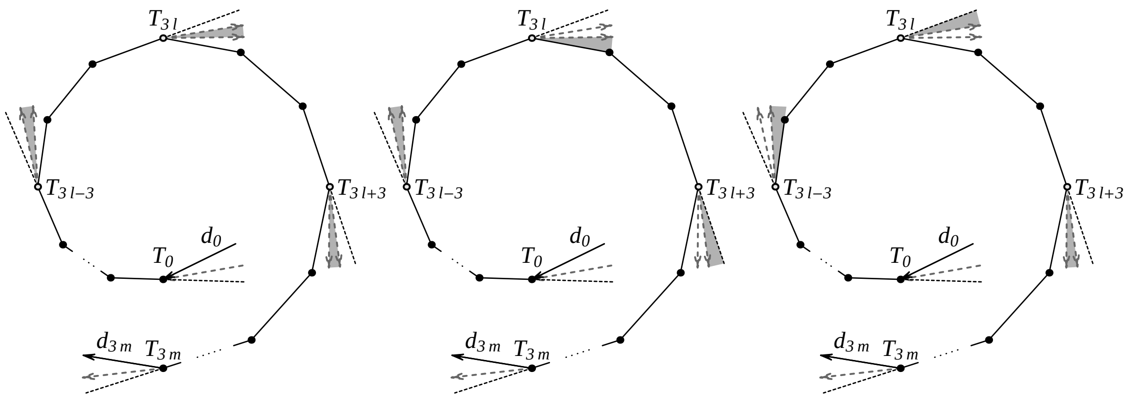


Figure 3.2: The directions for tangents (gray area) that imply the existence of a G^1 spline $\boldsymbol{S}$ for convex data.

The outline of this chapter is the following. In Subsection 3.2 a polynomial case is considered and geometric conditions that imply the existence of the interpolant are derived. Subsection 3.3 is devoted to a proof of two main theorems of Subsection 3.2. In Subsection 3.4 the results are carried over to G^1 cubic spline curves, and the conclusions are presented as an algorithm.

3.2. Polynomial case

The first step to the G^1 spline construction is a single polynomial case. So, $m = 1$ and let $\boldsymbol{P} := \boldsymbol{P}^1$. Further, let us shorten the notation by

$$\boldsymbol{d}_0 := \boldsymbol{d}_0^1, \quad \boldsymbol{d}_3 := \boldsymbol{d}_3^1, \quad t_1 := t_1^1, \quad t_2 := t_2^1, \quad \alpha_0 := \alpha_0^1, \quad \alpha_3 := \alpha_3^1.$$

The nonlinear part of the interpolation problem (3.2) is to compute the admissible parameters $(t_1, t_2, \alpha_0, \alpha_3) \in \mathcal{U}$, where by (3.3)

$$\mathcal{U} := \{(t_1, t_2);\ 0 =: t_0 < t_1 < t_2 < t_3 := 1\} \times \{(\alpha_0, \alpha_3);\ \alpha_0 > 0,\, \alpha_3 > 0\},$$

is an open set with the boundary $\partial\mathcal{U}$, determined by $t_i = t_{i+1}$ for at least one $i \in \{0, 1, 2\}$, $\alpha_0 = 0$ or $\alpha_3 = 0$. Once these parameters are determined, the coefficients of $\boldsymbol{P}$ are obtained by using any standard interpolation scheme componentwise.

To reduce the interpolation problem (3.2) to the nonlinear system for unknowns $(t_1, t_2, \alpha_0, \alpha_3)$ only, divided differences that map polynomials of degree ≤ 3 to zero are applied to (3.2). Therefore,

$$[t_0,t_0,t_1,t_2,t_3]\boldsymbol{P}=\boldsymbol{0}=\frac{\alpha_0}{\dot\omega_{0,3}(t_0)}\boldsymbol{d}_0+\sum_{j=1}^{3}\left(\sum_{i=j}^{3}\frac{1}{\dot\omega_{0,3}(t_i)}\frac{1}{t_i-t_0}\right)\Delta\boldsymbol{T}_{j-1}, \tag{3.5}$$

$$[t_0,t_1,t_2,t_3,t_3]\boldsymbol{P}=\boldsymbol{0}=\frac{\alpha_3}{\dot\omega_{0,3}(t_3)}\boldsymbol{d}_3+\sum_{j=0}^{2}\left(\sum_{i=0}^{j}\frac{1}{\dot\omega_{0,3}(t_i)}\frac{1}{t_3-t_i}\right)\Delta\boldsymbol{T}_{j}, \tag{3.6}$$

where $\omega_{0,3}$ is defined by (2.5). Further, with linear functionals $\det(\cdot,\Delta\boldsymbol{T}_0)$, $\det(\cdot,\Delta\boldsymbol{T}_1)$ applied to (3.5), and $\det(\cdot,\Delta\boldsymbol{T}_1)$, $\det(\cdot,\Delta\boldsymbol{T}_2)$ applied to (3.6) one obtains

$$\left.\begin{aligned}&\frac{\alpha_0}{\dot\omega_{0,3}(t_0)}\det(\boldsymbol{d}_0,\Delta\boldsymbol{T}_k)+\sum_{j=1}^{3}\left(\sum_{i=j}^{3}\frac{1}{\dot\omega_{0,3}(t_i)}\frac{1}{t_i-t_0}\right)\det(\Delta\boldsymbol{T}_{j-1},\Delta\boldsymbol{T}_k)=0,\\ &\frac{\alpha_3}{\dot\omega_{0,3}(t_3)}\det(\boldsymbol{d}_3,\Delta\boldsymbol{T}_{k+1})+\sum_{j=0}^{2}\left(\sum_{i=0}^{j}\frac{1}{\dot\omega_{0,3}(t_i)}\frac{1}{t_3-t_i}\right)\det(\Delta\boldsymbol{T}_{j},\Delta\boldsymbol{T}_{k+1})=0,\end{aligned}\right\}\quad k=0,1. \tag{3.7}$$

Let us recall that $t_0=0$ and $t_3=1$. After eliminating α_0 from the first and α_3 from the last equation, the system transforms to

$$\begin{aligned}\frac{1}{t_1^2(1-t_1)}-\frac{1}{t_2^2(1-t_2)}(1+\mu_1)+\frac{t_2-t_1}{(1-t_1)(1-t_2)}\left(1+\mu_1(1+\lambda_1)-\frac{\lambda_1}{\lambda_2}\right)&=0,\\ \frac{1}{t_2(1-t_2)^2}-\frac{1}{t_1(1-t_1)^2}(1+\mu_2)+\frac{t_2-t_1}{t_1t_2}\left(1+\mu_2(1+\lambda_2)-\frac{\lambda_2}{\lambda_1}\right)&=0,\end{aligned} \tag{3.8}$$

and

$$\alpha_0=\delta_1\frac{t_1t_2}{t_2-t_1}\left(\frac{1}{t_2^2(1-t_2)}-\frac{t_2-t_1}{(1-t_1)(1-t_2)}(1+\lambda_1)\right), \tag{3.9}$$

$$\alpha_3=\delta_2\frac{(1-t_1)(1-t_2)}{t_2-t_1}\left(\frac{1}{t_1(1-t_1)^2}-\frac{t_2-t_1}{t_1t_2}(1+\lambda_2)\right), \tag{3.10}$$

where the new constants are defined more generally as

$$\begin{aligned}\lambda_{2\ell-1}&:=\frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-3,3\ell-2}}, &\lambda_{2\ell}&:=\frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-2,3\ell-1}},\\ \mu_{2\ell-1}&:=\frac{\det(\boldsymbol{d}_{3\ell-3},\Delta\boldsymbol{T}_{3\ell-2})}{\det(\boldsymbol{d}_{3\ell-3},\Delta\boldsymbol{T}_{3\ell-3})}, &\mu_{2\ell}&:=\frac{\det(\Delta\boldsymbol{T}_{3\ell-2},\boldsymbol{d}_{3\ell})}{\det(\Delta\boldsymbol{T}_{3\ell-1},\boldsymbol{d}_{3\ell})},\\ \delta_{2\ell-1}&:=\frac{D_{3\ell-3,3\ell-2}}{\det(\boldsymbol{d}_{3\ell-3},\Delta\boldsymbol{T}_{3\ell-3})}, &\delta_{2\ell}&:=\frac{D_{3\ell-2,3\ell-1}}{\det(\Delta\boldsymbol{T}_{3\ell-1},\boldsymbol{d}_{3\ell})},\end{aligned} \tag{3.11}$$

and $D_{i,j}$ is defined by (2.3). Note that the definition of the constants λ_i, μ_i, and δ_i in this chapter is different from the one in Chapter 2. The constants again have a clear geometric meaning, for example, $D_{i,j}$ is the volume of a parallelogram spanned by vectors $\Delta\boldsymbol{T}_i$, $\Delta\boldsymbol{T}_j$ and the other constants are the ratios of such volumes. Figure 3.3 illustrates the sign change in λ_1 and λ_2 (μ_1 and δ_1) as $\boldsymbol{T}_3$ (tangent direction $\boldsymbol{d}_0$) changes. Note also that $\lambda_{2\ell-1},\lambda_{2\ell}$ depend on data points $\boldsymbol{T}_i$ only. Further, for the future use, we add the following observation.

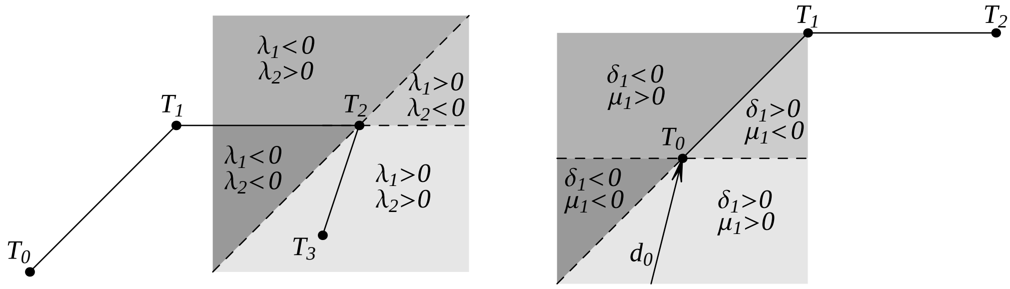


Figure 3.3: The signs of λ_1, λ_2 in dependence of the position of $\boldsymbol{T}_3$ (left), and the signs of μ_1, δ_1 in dependence of the tangent direction $\boldsymbol{d}_0$ (right).

REMARK 3.1. *The constants $\mu_{2\ell-1}, \mu_{2\ell}$, and* sign $\delta_{2\ell-1}$, sign $\delta_{2\ell}$ *do not depend on the length of tangents involved.*

REMARK 3.2. *The system of equations* (3.7) *could also be derived from Chapter 2 (eq.* (2.6)*) by replacing $\boldsymbol{T}_1$, $\boldsymbol{T}_4$ with $\boldsymbol{T}_0 + (t_1 - t_0)\alpha_0\boldsymbol{d}_0$, $\boldsymbol{T}_5 - (t_5 - t_4)\alpha_3\boldsymbol{d}_3$ accordingly, and passing to the limits $t_1 \to t_0$ and $t_4 \to t_5$ as well as renumbering the remaining points $\boldsymbol{T}_i$ and parameters t_i, $i = 0, 2, 3, 5$, by $0, 1, 2, 3$. Some of the properties of the nonlinear system* (3.7) *are thus inherited from Chapter 2 (eq.* (2.10)– (2.13)*), but not all. In particular, the requirement $\alpha_i > 0$ has to be considered thoroughly.*

In order to make the analysis bearable some restrictions on the data must be made. Namely, $\lambda_k > 0$, $\mu_k > 0$ and $\delta_k > 0$, $k = 1, 2$, will be assumed for the convex data and $\lambda_1 \cdot \lambda_2 < 0$, $\delta_k > 0$ for the data that imply an inflection point. Since the individual pieces will be composed in a spline curve, these assumptions are very natural as one can see from Figure 3.3.

It is straightforward to compute the solution of the system (3.8) in a closed form by using Gröbner basis or resultants. But not all of the solutions will satisfy $0 < t_1 < t_2 < 1$. Even if this is true, the solution may not produce positive α_0 and α_3. This means that we are dealing with a problem that is only partially algebraic. The following lemmas reveal the possibility that $\boldsymbol{P}'$ vanishes at $t = 0$ or $t = 1$, i.e., $\alpha_0 = 0$ or $\alpha_3 = 0$.

LEMMA 3.3. *Suppose that $\lambda_1 > 0$. There exists a unique solution of the system* (3.8) *and* (3.9) *such that $0 < t_1 < t_2 < 1$ and $\alpha_0 = 0$ if and only if $\lambda_2 > 0$ and $\mu_2 = \phi_2(\lambda_1, \lambda_2)$, where*

$$\phi_2(\lambda_1, \lambda_2) := \frac{\lambda_2 \dfrac{1 - \tilde{t}_1}{\tilde{t}_2^2} - \lambda_1 \dfrac{\tilde{t}_1}{(1 - \tilde{t}_2)^2}}{\lambda_2 \dfrac{1 - \tilde{t}_2}{\tilde{t}_1^2} - \lambda_1 \dfrac{\tilde{t}_2}{(1 - \tilde{t}_1)^2}} - 1,$$

and $\left(\tilde{t}_1, \tilde{t}_2\right)$ is the unique solution of the system

$$\frac{1 - t_1}{t_2^2 (t_2 - t_1)} = 1 + \lambda_1, \quad \frac{1 - t_2}{t_1^2 (t_2 - t_1)} = \frac{\lambda_1}{\lambda_2} (1 + \lambda_2), \quad 0 < t_1 < t_2 < 1. \qquad (3.12)$$

LEMMA 3.4. *Suppose that $\lambda_1 > 0$. There exists a unique solution of the system (3.8) and (3.10) such that $0 < t_1 < t_2 < 1$ and $\alpha_3 = 0$ if and only if $\lambda_2 > 0$ and $\mu_1 = \phi_1(\lambda_1, \lambda_2) := \phi_2(\lambda_2, \lambda_1)$.*

Proof. Let us prove Lemma 3.3. The proof of Lemma 3.4 is similar and will be omitted. When $\alpha_0 = 0$ the equations (3.8) and (3.9) simplify to (3.12) and

$$\mu_2 = \frac{\lambda_2 \dfrac{1-t_1}{t_2^2} - \lambda_1 \dfrac{t_1}{(1-t_2)^2}}{\lambda_2 \dfrac{1-t_2}{t_1^2} - \lambda_1 \dfrac{t_2}{(1-t_1)^2}} - 1.$$

From the first equation in (3.12), one obtains

$$t_1(t_2) = \frac{(1+\lambda_1)t_2^3 - 1}{(1+\lambda_1)t_2^2 - 1}.$$

Since $\lambda_1 > 0$, the function $t_1(t_2)$ has only one real zero $t_2 = \dfrac{1}{\sqrt[3]{1+\lambda_1}}$ and one positive real pole $t_2 = \dfrac{1}{\sqrt{1+\lambda_1}}$, where

$$0 < \frac{1}{\sqrt{1+\lambda_1}} < \frac{1}{\sqrt[3]{1+\lambda_1}} < 1.$$

Moreover, $t_1(0) = t_1(1) = 1$, $t_1(t_2) = t_2$ iff $t_2 = 1$, and $t_1(t_2)$ is monotonically increasing. Namely,

$$\frac{d}{dt_2} t_1(t_2) = \frac{(1+\lambda_1)t_2(2 - 3t_2 + (1+\lambda_1)t_2^3)}{((1+\lambda_1)t_2^2 - 1)^2} > 0, \quad t_2 \in (0, 1].$$

The condition $0 < t_1(t_2) < t_2 < 1$ is thus fulfilled iff $t_2 \in \left(\dfrac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$. By substituting $t_1(t_2)$ into the second equation in (3.12) it simplifies to

$$(t_2 - 1)g(t_2) = 0, \quad g(t_2) := \lambda_2 - \frac{\lambda_1(1+\lambda_2)((1+\lambda_1)t_2^3 - 1)^2}{((1+\lambda_1)t_2^2 - 1)^3}.$$

Now,

$$g\left(\frac{1}{\sqrt[3]{1+\lambda_1}}\right) = \lambda_2, \quad g(1) = -1,$$

and the sign of the derivative

$$\frac{d}{dt_2} g(t_2) = \frac{6\lambda_1(1+\lambda_1)(1+\lambda_2)t_2(t_2-1)((1+\lambda_1)t_2^3 - 1)}{((1+\lambda_1)t_2^2 - 1)^4}$$

is equal to the sign of $1+\lambda_2$ for $t_2 \in \left(\dfrac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$. Therefore a unique $\tilde{t}_2 \in \left(\dfrac{1}{\sqrt[3]{1+\lambda_1}}, 1\right)$ that solves $g(\tilde{t}_2) = 0$ exists iff $\lambda_2 > 0$. Then $\left(\tilde{t}_1, \tilde{t}_2\right) := \left(t_1(\tilde{t}_2), \tilde{t}_2\right)$ is the unique solution of the system (3.12), which concludes the proof. □

Let us now define two additional functions that will play a major role in the formulation of main results, namely

$$\phi_3(\lambda_1, \lambda_2, \mu_1) := \frac{\lambda_2\mu_1}{\lambda_1(\lambda_2\mu_1 - 1 - \sqrt{1+\mu_1})},$$
$$\phi_4(\lambda_1, \lambda_2, \mu_1) := \frac{\lambda_2\mu_1(\lambda_2\mu_1(1+2\lambda_1) - 2\lambda_1)}{\lambda_1^2(\lambda_2\mu_1 - 1)^2}.$$

The next lemma collects some of their properties that can easily be verified.

LEMMA 3.5. *Suppose that $\lambda_1 > 0$, $\lambda_2 < 0$ and $\mu_1 > 0$. Then $\phi_3(\lambda_1, \lambda_2, \cdot)$ and $\phi_4(\lambda_1, \lambda_2, \cdot)$ are monotonically increasing functions of μ_1,*

$$\lim_{\mu_1\to\infty} \phi_3(\lambda_1, \lambda_2, \mu_1) = \frac{1}{\lambda_1}, \quad \lim_{\mu_1\to\infty} \phi_4(\lambda_1, \lambda_2, \mu_1) = \frac{1+2\lambda_1}{\lambda_1^2},$$

and $\phi_3(\lambda_1, \lambda_2, \cdot) < \phi_4(\lambda_1, \lambda_2, \cdot)$. Moreover $\phi_3(\lambda_1, \lambda_2, \mu_1) = \mu_2$ if and only if $\phi_4(\lambda_2, \lambda_1, \mu_2) = \mu_1$, and $\phi_4(\lambda_1, \lambda_2, \mu_1) = \mu_2$ if and only if $\phi_3(\lambda_2, \lambda_1, \mu_2) = \mu_1$.

The following results now give sufficient conditions on data points and tangent directions that imply the existence of the interpolant $\boldsymbol{P}$. The first assertion covers convex data, and the second one covers data with an inflection point.

THEOREM 3.6. *Suppose that the data $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ satisfy*

$$\lambda_k > 0, \quad \delta_k > 0, \quad \mu_k > 0, \quad k = 1, 2.$$

If

$$0 < \mu_1 < \phi_1(\lambda_1, \lambda_2) \quad \text{and} \quad 0 < \mu_2 < \phi_2(\lambda_1, \lambda_2),$$

or

$$\mu_1 > \phi_1(\lambda_1, \lambda_2) \quad \text{and} \quad \mu_2 > \phi_2(\lambda_1, \lambda_2),$$

then a cubic interpolating curve $\boldsymbol{P}$ that satisfies (3.2) *exists.*

THEOREM 3.7. *Suppose that the data $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ satisfy*

$$\lambda_1 > 0, \quad \lambda_2 < 0, \quad \delta_1 > 0 \quad \text{and} \quad \delta_2 > 0.$$

If $\mu_1 > 0$ and

$$\phi_3(\lambda_1, \lambda_2, \mu_1) < \mu_2 < \phi_4(\lambda_1, \lambda_2, \mu_1),$$

then a cubic interpolating curve $\boldsymbol{P}$ that satisfies (3.2) *exists.*

REMARK 3.8. *The symmetry of equations* (3.8)-(3.10) *implies that Theorem 3.7 holds also if the role of λ_1, λ_2, and μ_1, μ_2 is reversed.*

A geometric interpretation of Theorem 3.6 is shown in Figure 3.4. For any chosen tangent direction $\boldsymbol{d}_0$ in light (dark) gray area and any chosen tangent direction $\boldsymbol{d}_3$ in light (dark) gray area the interpolating polynomial curve exists. For geometric interpretation of Theorem 3.7 see Figure 3.5. Choosing the direction $\boldsymbol{d}_0$ defines the gray area where the direction $\boldsymbol{d}_3$ must lie so that the existence of the interpolating polynomial curve is guaranteed by Theorem 3.7.

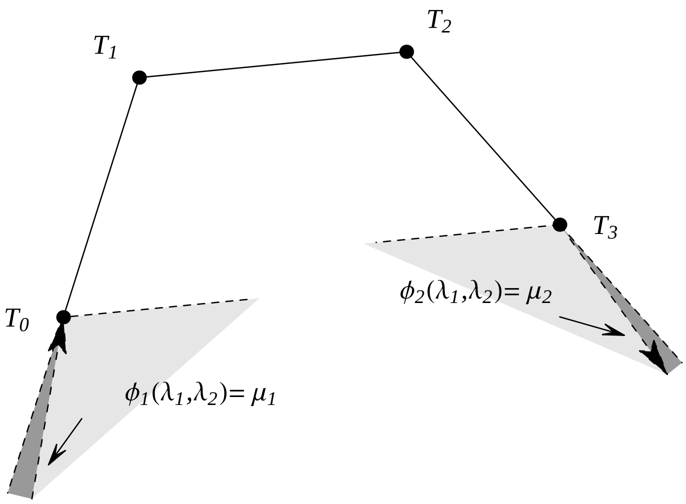


Figure 3.4: Geometric interpretation of Theorem 3.6.

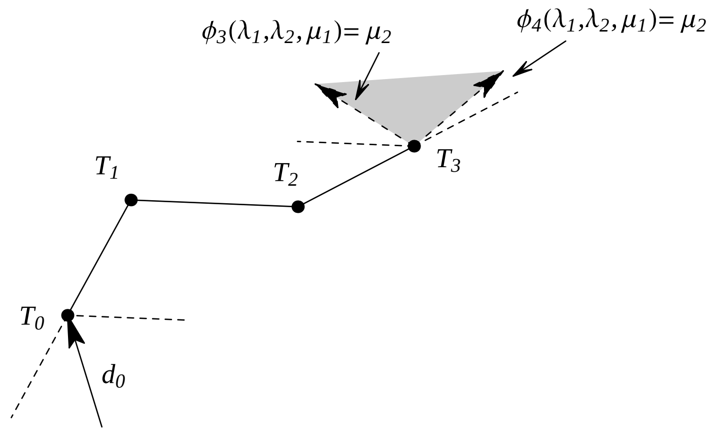


Figure 3.5: Geometric interpretation of Theorem 3.7.

3.3. Proof of Theorem 3.6 and Theorem 3.7

The key part of the proof is the following observation.

LEMMA 3.9. *Suppose that the assumptions of Theorem 3.6 or Theorem 3.7 are met. Then the system* (3.8)–(3.10) *cannot have a solution arbitrary close to the boundary* $\partial\mathcal{U}$.

Proof. Since by Lemma 3.3 and Lemma 3.4 no solution can have α_0 or α_3 arbitrary close to zero, it remains to show that

$$\Delta t_i := t_{i+1} - t_i \geq \text{const} > 0, \quad i = 0, 1, 2.$$

Note that $\Delta t_i > 0$, $\sum_{i=0}^{2} \Delta t_i = 1$. To show that $\Delta t_i \to 0$ cannot happen, the following possibilities need to be disproved:

1. $\Delta t_0, \Delta t_2 \geq \text{const} > 0$, and $\Delta t_1 \to 0$,
2. $\Delta t_0 \to 0$, $\Delta t_2 \to 0$,
3. $\Delta t_0 \geq \text{const} > 0$, $\Delta t_2 \to 0$:
   (a) $\Delta t_1 \geq \text{const} > 0$, (b) $\Delta t_1 \to 0$,
4. $\Delta t_0 \to 0$, $\Delta t_2 \geq \text{const} > 0$:
   (a) $\Delta t_1 \geq \text{const} > 0$, (b) $\Delta t_1 \to 0$.

Let us examine each possibility more precisely. Further, let us assume that the equations (3.8) are rewritten in a polynomial form in all four cases.
Case 1. The equations (3.8) simplify to

$$-\lambda_2\mu_1\Delta t_0^2\Delta t_2(\Delta t_0+\Delta t_2)^2+\mathcal{O}(\Delta t_1)=0,$$
$$\lambda_1\mu_2\Delta t_0\Delta t_2^2(\Delta t_0+\Delta t_2)^2+\mathcal{O}(\Delta t_1)=0,$$

and clearly can not have a solution if $\Delta t_1 \to 0$.
Case 2. In this case, the equations (3.8) simplify to

$$\lambda_2\Delta t_2+\lambda_1\Delta t_0^2(\lambda_2\mu_1-1)+\text{h.o.t.}=0,$$
$$-\lambda_1\Delta t_0+\lambda_2\Delta t_2^2(1-\lambda_1\mu_2)+\text{h.o.t.}=0,$$

where 'h.o.t.' again stands for higher order terms that are small in comparison to the terms left in expressions. If one determines Δt_2 from the first equation, and substitutes it in the second one, the equation reads

$$-\Delta t_0\lambda_1+\text{h.o.t.}=0.$$

This implies $\lambda_1=0$, and this case is not possible either.
Case 3. From the second equation (3.8) it follows immediately that the case (a) cannot happen. Therefore only (b), $\Delta t_1 \to 0$, $\Delta t_2 \to 0$, has to be considered. The equations (3.8) in this case simplify to

$$\lambda_1\Delta t_1(\lambda_2\mu_1-1)-\lambda_2\mu_1\Delta t_2+\text{h.o.t.}=0,$$
$$-\lambda_1(\Delta t_1^2+2\Delta t_1\Delta t_2-\mu_2\Delta t_2^2)+\text{h.o.t.}=0.$$

The solution of the dominant part reads

$$\Delta t_1=\frac{\lambda_2\mu_1}{\lambda_1(\lambda_2\mu_1-1)}\Delta t_2, \quad \mu_2=\phi_4(\lambda_1,\lambda_2,\mu_1).$$

But $\Delta t_1>0$ and $\Delta t_2>0$, which implies

$$\lambda_1>0,\ \lambda_2>0 \Longrightarrow \mu_1<0 \ \text{ or } \ \mu_1>\frac{1}{\lambda_2},$$
$$\lambda_1>0,\ \lambda_2<0 \Longrightarrow \mu_1>0 \ \text{ or } \ \mu_1<\frac{1}{\lambda_2}.$$

Further, the equations (3.9)–(3.10) reduce to

$$\alpha_0 = -\delta_1 \frac{\lambda_1^2(\lambda_2\mu_1 - 1)}{\lambda_2\mu_1(\lambda_2\mu_1 + \lambda_1(\lambda_2\mu_1 - 1))} \frac{1}{\Delta t_2^2} + \mathcal{O}\left(\frac{1}{\Delta t_2}\right),$$
$$\alpha_3 = \delta_2 \frac{\lambda_1^2(\lambda_2\mu_1 - 1)^2}{\lambda_2\mu_1(\lambda_2\mu_1 + \lambda_1(\lambda_2\mu_1 - 1))} \frac{1}{\Delta t_2} + \mathcal{O}(1).$$

Therefore the parameters α_0 and α_3 are strictly positive if

$$\lambda_2\mu_1 < 0 \quad \text{or} \quad \text{sign}(\lambda_2)\frac{\lambda_1}{\lambda_2 + \lambda_1\lambda_2} < \text{sign}(\lambda_2)\,\mu_1 < \text{sign}(\lambda_2)\frac{1}{\lambda_2},$$

and the case $\Delta t_1 \to 0$, $\Delta t_2 \to 0$ can happen only if

$$\lambda_2\mu_1 < 0 \quad \text{and} \quad \mu_2 = \phi_4(\lambda_1, \lambda_2, \mu_1), \tag{3.13}$$

which disproves case 3. Note that by Lemma 3.5 the condition (3.13) is equivalent to

$$0 < \mu_2 < \frac{1 + 2\lambda_1}{\lambda_1^2} \quad \text{and} \quad \mu_1 = \phi_3(\lambda_2, \lambda_1, \mu_2).$$

Case 4. From the first equation in (3.8) it is clear that case (a) is not possible. Therefore only $\Delta t_0 \to 0$, $\Delta t_1 \to 0$ need to be considered, and the equations (3.8) simplify to

$$\lambda_2(-\mu_1\Delta t_0^2 + 2\Delta t_0\Delta t_1 + \Delta t_1^2) + \text{h.o.t.} = 0,$$
$$\lambda_1\mu_2\Delta t_0 - \lambda_2\Delta t_1(\lambda_1\mu_2 - 1) + \text{h.o.t.} = 0.$$

The solution of the main part is

$$\Delta t_1 = \frac{\lambda_1\mu_2}{\lambda_2(\lambda_1\mu_2 - 1)}\Delta t_0, \quad \mu_1 = \phi_4(\lambda_2, \lambda_1, \mu_2).$$

Since $\Delta t_0 > 0$ and $\Delta t_1 > 0$, one concludes

$$\lambda_1 > 0,\ \lambda_2 > 0 \implies \mu_2 < 0 \ \text{ or } \ \mu_2 > \frac{1}{\lambda_1},$$
$$\lambda_1 > 0,\ \lambda_2 < 0 \implies 0 < \mu_2 < \frac{1}{\lambda_1}.$$

Moreover, the equations (3.9)–(3.10) simplify to

$$\alpha_0 = \delta_1 \frac{\lambda_2^2(\lambda_1\mu_2 - 1)^2}{\lambda_1\mu_2(\lambda_1\mu_2 + \lambda_2(\lambda_1\mu_2 - 1))} \frac{1}{\Delta t_0} + \mathcal{O}(1),$$
$$\alpha_3 = -\delta_2 \frac{\lambda_2^2(\lambda_1\mu_2 - 1)}{\lambda_1\mu_2(\lambda_1\mu_2 + \lambda_2(\lambda_1\mu_2 - 1))} \frac{1}{\Delta t_0^2} + \mathcal{O}\left(\frac{1}{\Delta t_0}\right).$$

For $\lambda_1 > 0$ and $\lambda_2 > 0$ the parameters α_0 and α_3 are strictly positive if

$$\mu_2 < 0 \quad \text{or} \quad \frac{\lambda_2}{\lambda_1 + \lambda_1\lambda_2} < \mu_2 < \frac{1}{\lambda_1}.$$

Similarly for $\lambda_1 > 0$ and $\lambda_2 < 0$ the parameters α_0 and α_3 are strictly positive if

$$0 < \mu_2 < \frac{1}{\lambda_1} \quad \text{or} \quad \left(\mu_2 < \frac{\lambda_2}{\lambda_1 + \lambda_1\lambda_2} \quad \text{and} \quad -1 < \lambda_2 < 0\right).$$

Now by using Lemma 3.5 one concludes that $\Delta t_0 \to 0$, $\Delta t_1 \to 0$ can happen for the convex data if

$$\begin{aligned} &\mu_2 < 0 \quad \text{and} \quad \mu_1 = \phi_4(\lambda_2, \lambda_1, \mu_2), \quad \text{or equivalently} \\ &0 < \mu_1 < \frac{1+2\lambda_2}{\lambda_2^2} \quad \text{and} \quad \mu_2 = \phi_3(\lambda_1, \lambda_2, \mu_1), \end{aligned}$$

and similarly for $\lambda_1 > 0$ and $\lambda_2 < 0$ if

$$\begin{aligned} &0 < \mu_2 < \frac{1}{\lambda_1} \quad \text{and} \quad \mu_1 = \phi_4(\lambda_2, \lambda_1, \mu_2), \quad \text{or equivalently} \\ &\mu_1 > 0 \quad \text{and} \quad \mu_2 = \phi_3(\lambda_1, \lambda_2, \mu_1). \end{aligned}$$

That excludes the case 4 and therefore concludes the proof of the theorem.

□

REMARK 3.10. *Lemma 3.9 can also be proved by passing to the limit mentioned by Remark 3.2 in the discussion in Chapter 2 (Subsection 2.3). One can check that only two possible cases $\Delta t_i \to 0$ are to be considered. The first one, $\Delta t_0 \to 0$ and $\Delta t_1 \to 0$, implies $\lambda_1\lambda_2 < 0$ and $\lambda_1 \to \frac{1}{\mu_2} + \frac{\lambda_1(1+\sqrt{1+\mu_1})}{\lambda_2\mu_1}$ or equivalently $\mu_2 \to \phi_3(\lambda_1, \lambda_2, \mu_1)$. Similarly, the second one, $\Delta t_1 \to 0$ and $\Delta t_2 \to 0$, implies $\lambda_1\lambda_2 < 0$ too, and $\lambda_1 \to \frac{\lambda_1}{\lambda_2\mu_1} + \frac{1+\sqrt{1+\mu_2}}{\mu_2}$ or equivalently $\mu_2 \to \phi_4(\lambda_1, \lambda_2, \mu_1)$. The assertion follows now from Lemma 3.3 and Lemma 3.4.*

A standard degree type argument will now conclude the proofs. Let us first show that the number of admissible solutions for the particular data that satisfy the conditions of the theorems is odd. The data points are chosen as

$$\boldsymbol{T}_0 = \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \quad \boldsymbol{T}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \boldsymbol{T}_3 = \begin{pmatrix} 9+(-1)^s \\ (-1)^s 4 \end{pmatrix}, \quad s \in \{0,1\},$$

where $s = 1$ corresponds to the convex case and $s = 0$ to the other one. Further, the tangent directions are chosen as

data 1: $\boldsymbol{d}_0 = (2,3)^T, \quad \boldsymbol{d}_3 = (2,-3)^T, \quad s = 1,$

data 2: $\boldsymbol{d}_0 = (-2,2)^T, \quad \boldsymbol{d}_3 = (-2,-2)^T, \quad s = 1,$

data 3: $\boldsymbol{d}_0 = (-2,2)^T, \quad \boldsymbol{d}_3 = (-1,2)^T, \quad s = 0.$

	λ_1	λ_2	μ_1	μ_2	δ_1	δ_2
data 1	2	2	3	3	4	4
data 2	2	2	$\frac{1}{2}$	$\frac{1}{2}$	1	1
data 3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1

Table 3.1: The constants for the particular data.

	t_1	t_2	α_0	α_3	multiplicity
data 1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{2}$	3
data 2	$\frac{1}{3}(3-\sqrt{3})$	$\frac{\sqrt{3}}{3}$	$3(1+\sqrt{3})$	$3(1+\sqrt{3})$	1
data 3	0.450047	0.583425	12.1642	12.1828	1

Table 3.2: The admissible solutions for the particular data.

Table 3.1 shows the values of the constants (3.11) and Table 3.2 gives the corresponding admissible solutions in $\mathcal{U}$. The interpolating polynomial curves are shown in Figure 3.6.

Since $\phi_1(\lambda_1,\lambda_2) = \phi_2(\lambda_1,\lambda_2) = 2.80828$ for data 1 and data 2, and

$$\phi_3(\lambda_1,\lambda_2,\mu_1) = 10 - 4\sqrt{6} < \mu_2 = \frac{1}{2} < \phi_4(\lambda_1,\lambda_2,\mu_1) = \frac{24}{25}$$

for data 3, the assumptions of theorems are fulfilled.

A homotopy will now help us carry the conclusions from the particular case outlined in Table 3.1 to a general one. Let us rewrite the system (3.8)–(3.10) as

$$\boldsymbol{F}(\boldsymbol{t},\boldsymbol{\alpha};\boldsymbol{\lambda},\boldsymbol{\delta},\mu_1,\mu_2) = \boldsymbol{0}, \tag{3.14}$$

where

$$\boldsymbol{t} = (t_1,t_2), \quad \boldsymbol{\alpha} = (\alpha_1,\alpha_2), \quad \boldsymbol{\lambda} = (\lambda_1,\lambda_2), \quad \boldsymbol{\delta} = (\delta_1,\delta_2).$$

Further, let $(\boldsymbol{\lambda},\boldsymbol{\delta},\mu_1,\mu_2)$ stand for general data, and $(\boldsymbol{\lambda}^*,\boldsymbol{\delta}^*,\mu_1^*,\mu_2^*)$ for the particular case. A homotopy is chosen as

$$\boldsymbol{H}(\boldsymbol{t},\boldsymbol{\alpha};\zeta) := \boldsymbol{F}\left(\boldsymbol{t},\boldsymbol{\alpha};\boldsymbol{\lambda}(\zeta),\boldsymbol{\delta}(\zeta),q_1(\zeta;\mu_1^*,\mu_1),q_2(\zeta;\mu_2^*,\mu_2)\right),$$

where

$$\boldsymbol{\lambda}(\zeta) := (1-\zeta)\boldsymbol{\lambda}^* + \zeta\boldsymbol{\lambda}, \quad \boldsymbol{\delta}(\zeta) := (1-\zeta)\boldsymbol{\delta}^* + \zeta\boldsymbol{\delta}, \quad \mu_k(\zeta) := q_k(\zeta;\mu_k^*,\mu_k),$$

and $q_k(\cdot;\mu_k^*,\mu_k) : [0,1] \to \mathbb{R}$ is a continuous function that satisfies

$$q_k(0;\mu_k^*,\mu_k) = \mu_k^*, \quad q_k(1;\mu_k^*,\mu_k) = \mu_k, \quad k = 1,2.$$

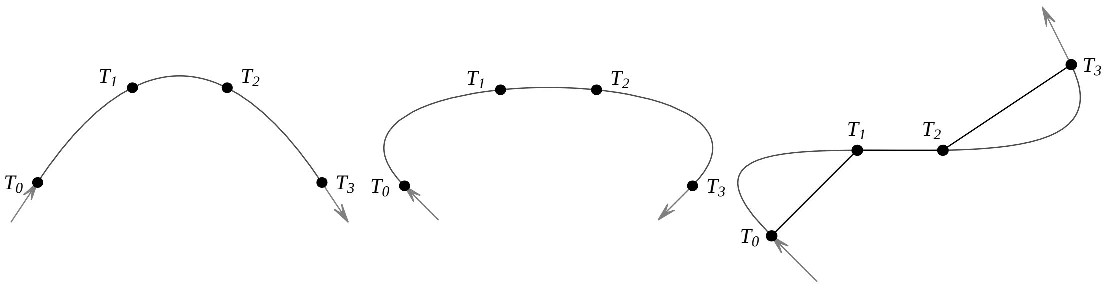


Figure 3.6: Interpolating polynomial curves for the particular data.

It is clear that

$$|\lambda_k(\zeta)| \geq \min_{\zeta\in[0,1]} \{|(1-\zeta)\lambda_k^* + \zeta\lambda_k|\} \geq \min\{|\lambda_k^*|, |\lambda_k|\} \geq \text{const} > 0,$$
$$|\delta_k(\zeta)| \geq \min_{\zeta\in[0,1]} \{|(1-\zeta)\delta_k^* + \zeta\delta_k|\} \geq \min\{|\delta_k^*|, |\delta_k|\} \geq \text{const} > 0.$$

Consider Theorem 3.6 first. For the first possibility $0 < \mu_k < \phi_k(\lambda_1, \lambda_2)$, data 2 are appropriate since they satisfy $0 < \mu_k^* < \phi_k(\lambda_1^*, \lambda_2^*)$ too. It is then clear, that there exists a continuous piecewise linear function $q_k(\zeta, \mu_k^*, \mu_k)$, such that

$$0 < q_k(\zeta, \mu_k^*, \mu_k) < \phi_k(\lambda_1(\zeta), \lambda_2(\zeta)), \quad \zeta \in [0,1], \quad k = 1, 2.$$

A similar conclusion follows for $\mu_k > \phi_k(\lambda_1, \lambda_2)$, with the use of data 1.

In case of Theorem 3.7 where $\lambda_1 > 0$, $\lambda_2 < 0$, we choose data 3 that satisfy the required suppositions. By Lemma 3.5 functions

$$\phi_3(\lambda_1(\zeta), \lambda_2(\zeta), \mu_1(\zeta)) < \phi_4(\lambda_1(\zeta), \lambda_2(\zeta), \mu_1(\zeta))$$

cannot intersect for any $\zeta \in [0,1]$. If one defines

$$\mu_1(\zeta) = q_1(\zeta, \mu_1^*, \mu_1) = (1-\zeta)\mu_1^* + \zeta\mu_1,$$

then there obviously exists a continuous piecewise linear function $q_2(\zeta, \mu_2^*, \mu_2)$, such that

$$\phi_3(\lambda_1(\zeta), \lambda_2(\zeta), \mu_1(\zeta)) < q_2(\zeta; \mu_2^*, \mu_2) < \phi_4(\lambda_1(\zeta), \lambda_2(\zeta), \mu_1(\zeta)), \quad \zeta \in [0,1].$$

Therefore $\boldsymbol{H}(\boldsymbol{t}, \boldsymbol{\alpha}; \zeta) = \boldsymbol{0}$ meets the requirements of Theorem 2.4 for any $\zeta \in [0,1]$. As a consequence, a set of solutions

$$V := \{(\boldsymbol{t}, \boldsymbol{\alpha}) \in \mathcal{U}; \quad \boldsymbol{H}(\boldsymbol{t}, \boldsymbol{\alpha}; \zeta) = \boldsymbol{0}, \quad \zeta \in [0,1]\}$$

lies aside from the boundary $\partial\mathcal{U}$. More precisely, one can find a compact set $K \subset \mathcal{U}$, such that

$$V \subset K \subset \mathcal{U}, \quad V \cap \partial K = \emptyset.$$

Therefore the map $\boldsymbol{H}$ does not vanish at the boundary ∂K, and Brouwer's degree of $\boldsymbol{H}$ on K is invariant for all $\zeta \in [0,1]$. But since it is odd for the particular map $\boldsymbol{F}(\cdot,\cdot; \boldsymbol{\lambda}^*, \boldsymbol{\delta}^*, \mu_1^*, \mu_2^*)$, equations $\boldsymbol{F}(\boldsymbol{t}, \boldsymbol{\alpha}; \boldsymbol{\lambda}, \boldsymbol{\delta}, \mu_1, \mu_2) = \boldsymbol{0}$ must have at least one admissible solution and Theorem 3.6 and Theorem 3.7 are proved.

Theorem 3.6 and Theorem 3.7 give only sufficient conditions that cover most often met practical cases. Additional ones can be found in Table 3.3.

$\lambda_1 > 0$, $\lambda_2 > 0$	$\delta_1 > 0$, $\delta_2 > 0$	$\mu_1 \leq 0$	$\phi_4(\lambda_1, \lambda_2, \mu_1) < \mu_2 < \phi_2(\lambda_1, \lambda_2)$
		$0 < \mu_1 \leq \frac{1+2\lambda_2}{\lambda_2^2}$	$\phi_3(\lambda_1, \lambda_2, \mu_1) < \mu_2 \leq 0$
	$\delta_1 < 0$, $\delta_2 > 0$	$\mu_1 > \frac{1}{\lambda_2}$	$\phi_4(\lambda_1, \lambda_2, \mu_1) < \mu_2 < \phi_2(\lambda_1, \lambda_2)$
		$\mu_1 \leq \frac{1}{\lambda_2}$	$\phi_2(\lambda_1, \lambda_2) < \mu_2$
	$\delta_1 > 0$, $\delta_2 < 0$	$\mu_2 > \frac{1}{\lambda_1}$	$\phi_4(\lambda_2, \lambda_1, \mu_2) < \mu_1 < \phi_1(\lambda_1, \lambda_2)$
		$\mu_2 \leq \frac{1}{\lambda_1}$	$\phi_1(\lambda_1, \lambda_2) < \mu_1$
$\lambda_1 < 0$, $\lambda_2 < 0$	$\delta_1 > 0$, $\delta_2 > 0$	$\mu_1 \geq 0$	$\phi_3(\lambda_1, \lambda_2, \mu_1) < \mu_2$
		$\frac{1}{\lambda_2} < \mu_1 \leq 0$	$\phi_4(\lambda_1, \lambda_2, \mu_1) < \mu_2$

Table 3.3: Conditions that imply the existence of an interpolating curve $\boldsymbol{P}$ too.

3.4. The G^1 spline curve

We tackle now the G^1 cubic spline interpolation as introduced at the beginning of this chapter, with tangent directions in (3.2) considered to be unknown. Each tangent direction $\boldsymbol{d}_{3\ell}$ depends on one parameter only. If vectors $\Delta\boldsymbol{T}_{3\ell-1}$ and $\Delta\boldsymbol{T}_{3\ell}$ are not collinear, i.e., $D_{3\ell-1,3\ell} \neq 0$, we may express the tangent directions as

$$\begin{aligned}
\boldsymbol{d}_0 &:= \boldsymbol{d}_0(\xi_0) := (\xi_0 - 1)\Delta\boldsymbol{T}_1 + \xi_0\Delta\boldsymbol{T}_0,\\
\boldsymbol{d}_{3\ell} &:= \boldsymbol{d}_{3\ell}(\xi_\ell) := \sigma_{3\ell}(1-\xi_\ell)\Delta\boldsymbol{T}_{3\ell-1} + \sigma_{3\ell-1}\xi_\ell\Delta\boldsymbol{T}_{3\ell}, \quad \ell = 1, \ldots, m-1, \\
\boldsymbol{d}_{3m} &:= \boldsymbol{d}_m(\xi_m) := (1-\xi_m)\Delta\boldsymbol{T}_{3m-1} - \xi_m\Delta\boldsymbol{T}_{3m-2},
\end{aligned} \tag{3.15}$$

with

$$\sigma_k := \operatorname{sign}\left(\frac{D_{k-1,k}}{D_{k,k+1}}\right).$$

The tangents introduced in (3.15) are not normalized, but by Remark 3.1 this is not important. Further, the definition (3.15) implies that some constants defined in (3.11) become explicit functions of ξ_ℓ. In particular,

$$\begin{aligned}
\delta_{2\ell-1} &= \delta_{2\ell-1}(\xi_{\ell-1}) = \frac{1}{1-\xi_{\ell-1}}\left|\frac{D_{3\ell-3,3\ell-2}}{D_{3\ell-4,3\ell-3}}\right|, \quad \ell = 2, 3, \ldots, m,\\
\delta_{2\ell} &= \delta_{2\ell}(\xi_\ell) = \frac{1}{\xi_\ell}\left|\frac{D_{3\ell-2,3\ell-1}}{D_{3\ell-1,3\ell}}\right|, \quad \ell = 1, 2, \ldots, m-1,\\
\delta_1 &= \delta_1(\xi_0) = \frac{1}{1-\xi_0}, \quad \delta_{2m} = \delta_{2m}(\xi_m) = \frac{1}{\xi_m},
\end{aligned} \tag{3.16}$$

show that requirements $\delta_{2\ell}(\xi_\ell) > 0$, $\delta_{2\ell+1}(\xi_\ell) > 0$ pins down ξ_ℓ to $(0, 1)$ as can be seen in Figure 3.7. The constants $\mu_{2\ell-1}$, $\mu_{2\ell}$ turn out as

$$\begin{aligned}
\mu_{2\ell-1} &= \mu_{2\ell-1}(\xi_{\ell-1}) = \sigma_{3\ell-4}\xi_{\ell-1}\delta_{2\ell-1}(\xi_{\ell-1}) + \frac{D_{3\ell-4,3\ell-2}}{D_{3\ell-4,3\ell-3}}, \quad \ell = 2, 3, \ldots, m,\\
\mu_{2\ell} &= \mu_{2\ell}(\xi_\ell) = \sigma_{3\ell}\left(1-\xi_\ell\right)\delta_{2\ell}(\xi_\ell) + \frac{D_{3\ell-2,3\ell}}{D_{3\ell-1,3\ell}}, \quad \ell = 1, 2, \ldots, m-1,\\
\mu_1 &= \mu_1(\xi_0) = \frac{\xi_0}{1-\xi_0}, \quad \mu_{2m} = \mu_{2m}(\xi_m) = \frac{1-\xi_m}{\xi_m}.
\end{aligned} \tag{3.17}$$

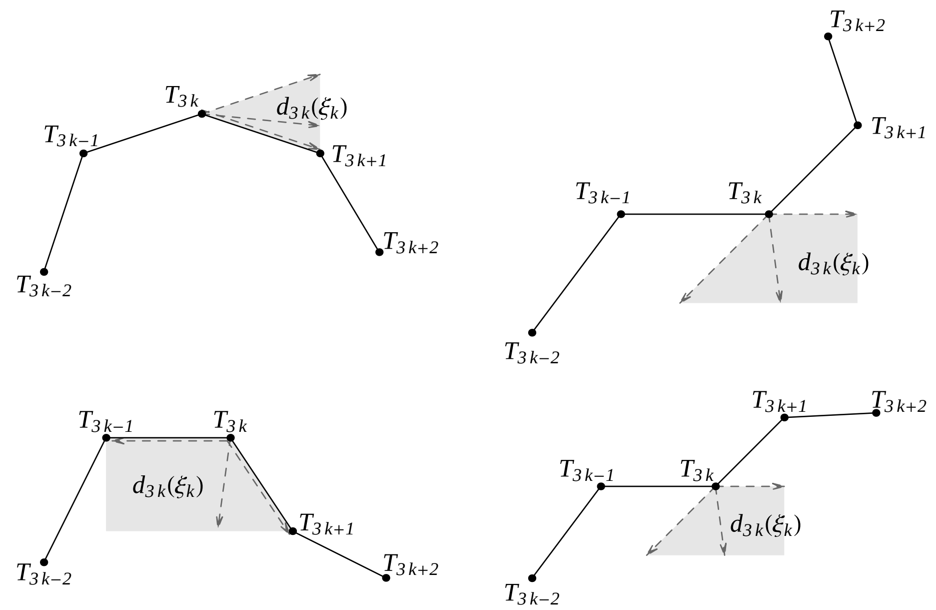


Figure 3.7: The tangent directions $\boldsymbol{d}_{3k}(\xi_k)$ for $\xi_k \in (0,1)$ (gray area).

In view of Theorem 3.6 or Theorem 3.7 it is necessary to determine for which $\xi_\ell \in (0,1)$ the functions $\mu_{2\ell}$ and $\mu_{2\ell+1}$ are both positive. Let us recall the notation

$$f(\mathcal{I}) := \{f(x);\ x \in \mathcal{I}\}, \quad f^{-1}(\mathcal{I}) := \{x;\ f(x) \in \mathcal{I}\}.$$

Then

$$\begin{aligned}
\mathcal{I}_0 &:= \mu_1^{-1}((0,\infty)) \cap (0,1) = (0,1),\\
\mathcal{I}_\ell &:= \mu_{2\ell}^{-1}((0,\infty)) \cap \mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1), \quad \ell = 1,2,\ldots,m-1,\\
\mathcal{I}_m &:= \mu_{2m}^{-1}((0,\infty)) \cap (0,1) = (0,1),
\end{aligned}$$

are the required subintervals, with $\mathcal{I}_\ell \neq \emptyset$ still to be assured. Let us restrict the study to the interval $(0,1)$ only. It is easy to see that $\mu_{2\ell}$ and $\mu_{2\ell+1}$ are both monotone as functions of ξ_ℓ. Moreover,

$$\begin{aligned}
&\lim_{\xi\downarrow 0} \mu_{2\ell}(\xi_\ell) = \sigma_{3\ell} \cdot \infty, \quad \mu_{2\ell}(1) = \frac{D_{3\ell-2,3\ell}}{D_{3\ell-1,3\ell}},\\
&\mu_{2\ell+1}(0) = \frac{D_{3\ell-1,3\ell+1}}{D_{3\ell-1,3\ell}}, \quad \lim_{\xi\uparrow 1} \mu_{2\ell+1}(\xi_\ell) = \sigma_{3\ell-1} \cdot \infty.
\end{aligned}$$

Therefrom it is easy to see that $\mu_{2\ell}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ iff

$$\sigma_{3\ell} = -1, \quad D_{3\ell-2,3\ell} D_{3\ell-1,3\ell} \leq 0, \tag{3.18}$$

and $\mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ iff

$$\sigma_{3\ell-1} = -1, \quad D_{3\ell-1,3\ell+1} D_{3\ell-1,3\ell} \leq 0. \tag{3.19}$$

Now, if conditions (3.18) and (3.19) are not fulfilled, each of the above intervals is nonempty, but that does not imply that the intersection is nonempty too. In this case it is easy to check that $\mu_{2\ell}^{-1}((0,\infty)) \cap \mu_{2\ell+1}^{-1}((0,\infty)) \cap (0,1) = \emptyset$ if and only if

$$\begin{aligned} &D_{3\ell-2,3\ell-1}D_{3\ell,3\ell+1} > 0, \quad && D_{3\ell-2,3\ell}D_{3\ell-1,3\ell+1} > 0, \\ &D_{3\ell-2,3\ell}D_{3\ell,3\ell+1} < 0, && \sigma_{3\ell}\,\mu_{2\ell}^{-1}(0) \leq \sigma_{3\ell}\,\mu_{2\ell+1}^{-1}(0). \end{aligned} \tag{3.20}$$

Let us summarize this discussion in the following theorem.

THEOREM 3.11. *Suppose that data points* (3.1) *satisfy*

$$\begin{aligned} &\lambda_{2\ell-1} > 0,\ \lambda_{2\ell} > 0 \ \text{ or } \ \lambda_{2\ell-1}\lambda_{2\ell} < 0, \quad \ell = 1,2,\ldots,m, \\ &D_{3\ell-1,3\ell} \neq 0, \quad \ell = 1,2,\ldots,m-1, \end{aligned}$$

and in addition none of the relations (3.18), (3.19) *or* (3.20) *is fulfilled. Further, let the tangents be given by* (3.15), *and the rest of the constants determined by* (3.16) *and* (3.17). *Then for every $\xi_\ell \in \mathcal{I}_\ell$, $\ell = 0,1,\ldots,m$, the suppositions of either Theorem 3.6 or Theorem 3.7 are fulfilled on the ℓ-th segment. Further, the algorithm* ForwardSweep *determines the admissible intervals for parameters ξ_ℓ.*

Only the algorithm is left to be constructed. We choose it to be a simple backtracking procedure that traverses the data (3.1) in a forward sweep $\boldsymbol{T}_0 \rightarrow \boldsymbol{T}_{3m}$ and determines an intermediate result

$$\Xi_\ell \subset \mathcal{I}_\ell, \quad \ell = 0,1,\ldots,m,$$

in such a way that for any $\xi_\ell \in \Xi_\ell$ there exists a choice

$$\xi_i \in \Xi_i, \quad i = 0,1,\ldots,\ell-1,$$

such that $(\xi_0, \xi_1, \ldots, \xi_\ell)$ is admissible as far as data $\boldsymbol{T}_i$, $i = 0,1,\ldots,3\ell$, are concerned. A backward sweep $\boldsymbol{T}_{3m} \rightarrow \boldsymbol{T}_0$ shrinks the temporary Ξ_ℓ, $\ell = m-1, m-2, \ldots, 0$, so that for any $\xi_\ell \in \Xi_\ell$ there exists a choice

$$\xi_i \in \Xi_i, \quad i = 0,1,\ldots,\ell-1,\ell+1,\ldots,m,$$

such that $(\xi_0, \xi_1, \ldots, \xi_m)$ is admissible for all data. The induction step $\Xi_{\ell-1} \rightarrow \Xi_\ell$ or $\Xi_{\ell-1} \rightarrow \Xi_\ell$ has two forms (Figure 3.8), based upon Theorem 3.6 and Theorem 3.7 respectively. The case $\lambda_{2\ell-1} > 0$ and $\lambda_{2\ell} > 0$ is easy to handle since the restrictions on tangent directions depend only on data points, more precisely on

$$\phi_{1,\ell-1} := \phi_1(\lambda_{2\ell-1}, \lambda_{2\ell}), \quad \phi_{2,\ell} := \phi_2(\lambda_{2\ell-1}, \lambda_{2\ell}), \quad \ell = 1,2,\ldots,m.$$

The case $\lambda_{2\ell-1}\lambda_{2\ell} < 0$ is more complex since the existence conditions connect left and right tangent direction. For this reason, we introduce two additional maps, $\mathcal{R}_{1,\ell}(\mathcal{I})$ and $\mathcal{R}_{2,\ell}(\mathcal{I})$, where $\mathcal{I}$ is an open or closed interval with endpoints a and b. For $\lambda_{2\ell-1} > 0$

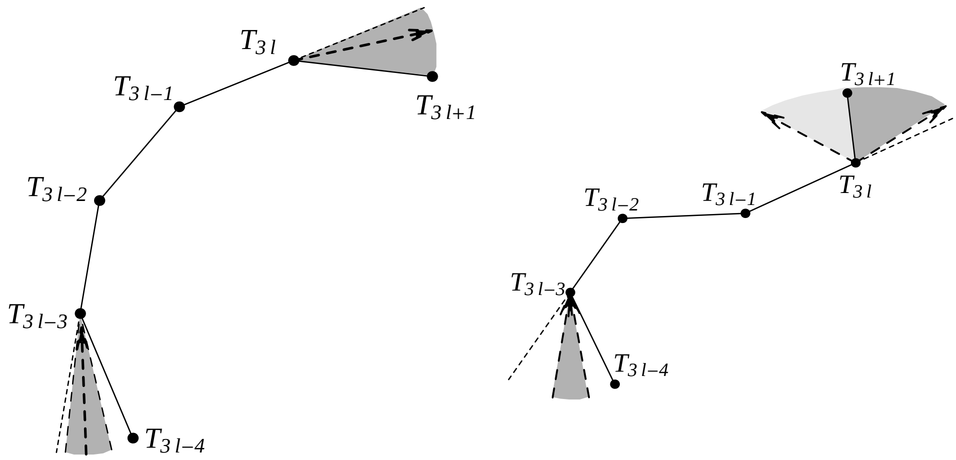


Figure 3.8: Induction step: $\lambda_{2\ell-1} > 0, \lambda_{2\ell} > 0$ (left), and $\lambda_{2\ell-1}\lambda_{2\ell} < 0$ (right).

and $\lambda_{2\ell} < 0$ the definition reads

$$\mathcal{R}_{1,\ell}(I) := \mathcal{R}_{1,\ell}(I; \lambda_{2\ell-1}, \lambda_{2\ell})$$
$$:= \begin{cases} \emptyset; & b \leq 0 \ \vee \ I = \emptyset, \\ (\phi_3(\lambda_{2\ell-1}, \lambda_{2\ell}, (a)_+), \phi_4(\lambda_{2\ell-1}, \lambda_{2\ell}, b)); & b > 0, \end{cases}$$

$$\mathcal{R}_{2,\ell}(I) := \mathcal{R}_{2,\ell}(I; \lambda_{2\ell-1}, \lambda_{2\ell})$$
$$:= \begin{cases} \emptyset; & b \leq 0 \ \vee \ a \geq \frac{1+2\lambda_{2\ell-1}}{\lambda_{2\ell-1}^2} \ \vee \ I = \emptyset, \\ (\phi_3(\lambda_{2\ell}, \lambda_{2\ell-1}, (a)_+), \phi_4(\lambda_{2\ell}, \lambda_{2\ell-1}, b)); & b < \frac{1}{\lambda_{2\ell-1}}, \\ (\phi_3(\lambda_{2\ell}, \lambda_{2\ell-1}, (a)_+), \infty); & b \geq \frac{1}{\lambda_{2\ell-1}}, \end{cases}$$

and for $\lambda_{2\ell-1} < 0$, $\lambda_{2\ell} > 0$ is given as

$$\mathcal{R}_{1,\ell}(I) := \mathcal{R}_{2,\ell}(I; \lambda_{2\ell}, \lambda_{2\ell-1}),$$
$$\mathcal{R}_{2,\ell}(I) := \mathcal{R}_{1,\ell}(I; \lambda_{2\ell}, \lambda_{2\ell-1}).$$

Recall Theorem 3.7 and Lemma 3.5. The meaning of $\mathcal{R}_{1,\ell}$ and $\mathcal{R}_{2,\ell}$ is the following. Suppose that $\mu_{2\ell-1}$, $\mu_{2\ell}$ are confined to intervals, i.e., $\mu_{2\ell-1} \in (a_1, b_1)$ and $\mu_{2\ell} \in (a_2, b_2)$. Then for every $\mu_{2\ell-1} \in (a_1, b_1) \cap \mathcal{R}_{2,\ell}((a_2, b_2))$ there exists at least one admissible $\mu_{2\ell} \in (a_2, b_2)$. Equivalently, for every $\mu_{2\ell} \in (a_2, b_2) \cap \mathcal{R}_{1,\ell}((a_1, b_1))$ there is at least one admissible $\mu_{2\ell-1} \in (a_1, b_1)$ (Figure 3.9). Now, we can write the algorithm that should be called as

1. solution := $\emptyset$;
2. $\Xi := (\mathcal{I}_0)$;
3. ForwardSweep(m, Ξ, 1, solution);

procedure ForwardSweep(m, Ξ, ℓ, solution)
 1. V := ForwardSplit(Ξ, ℓ);

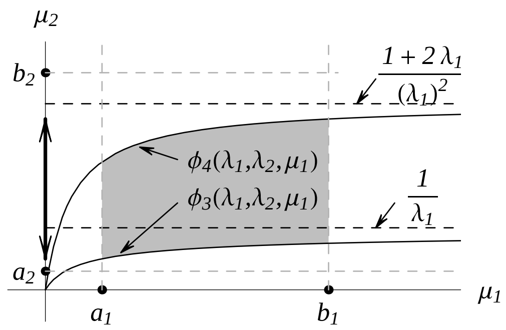


Figure 3.9: Geometric interpretation of $\mathcal{R}_{1,\ell}$ and $\mathcal{R}_{2,\ell}$ for $\ell = 1$. Every point (μ_1, μ_2) in the gray area is admissible.

2. **for** $i = 1$, $i \leq \text{length}(V)$, $i = i + 1$
3. $\Xi_\ell := V_i$;
4. **if** $\ell = m$ **then** BackwardSweep(m, Ξ, ℓ, solution);
5. **else** ForwardSweep(m, Ξ, $\ell + 1$, solution);

procedure BackwardSweep(m, Ξ, ℓ, solution)
1. $\Xi_{\ell-1}$:= BackwardSplit(Ξ, ℓ);
2. **if** $\Xi_{\ell-1} \neq \emptyset$
3. **if** $\ell = 1$ **then** solution := solution $\cup \{\Xi\}$;
4. **else** BackwardSweep(m, Ξ, $\ell - 1$, solution);

procedure ForwardSplit(Ξ, ℓ)
1. $V := \emptyset$; $\mathcal{I} := \mu_{2\ell-1}(\Xi_{\ell-1})$; $\mathcal{J} := \emptyset$;
2. **if** $\lambda_{2\ell-1} > 0$ **and** $\lambda_{2\ell} > 0$ **then**
3. **if** $\mathcal{I} \leq \phi_{1,\ell-1}$ **then** $\mathcal{J} := \{(0, \phi_{2,\ell})\}$;
4. **else if** $\mathcal{I} \geq \phi_{1,\ell-1}$ **then** $\mathcal{J} := \{\, (\phi_{2,\ell}, \infty) \,\}$;
5. **else** $\mathcal{J} := \{\, (0, \phi_{2,\ell}), (\phi_{2,\ell}, \infty) \,\}$;
6. **else if** $\lambda_{2\ell-1} \cdot \lambda_{2\ell} < 0$ **then**
7. $\mathcal{J} := \{\mathcal{R}_{1,\ell}\,(\mathcal{I}; \lambda_{2\ell-1}, \lambda_{2\ell})\}$;
8. **for** $i = 1$, $i \leq \text{length}(\mathcal{J})$, $i = i + 1$
9. **if** $\mathcal{I} := \mu_{2\ell}^{-1}(\mathcal{J}_i) \cap \mathcal{I}_\ell \neq \emptyset$ **then** $V = V \cup \{\mathcal{I}\}$;
10. **return** V

procedure BackwardSplit(Ξ, ℓ)
1. $\mathcal{I} := \mu_{2\ell}(\Xi_\ell)$; $\mathcal{J} := \emptyset$;
2. **if** $\lambda_{2\ell-1} > 0$ **and** $\lambda_{2\ell} > 0$ **then**
3. **if** $\mathcal{I} \leq \phi_{2,\ell}$ **then** $\mathcal{J} := (0, \phi_{1,\ell-1})$;
4. **if** $\mathcal{I} \geq \phi_{2,\ell}$ **then** $\mathcal{J} := (\phi_{1,\ell-1}, \infty)$;
5. **else if** $\lambda_{2\ell-1} \cdot \lambda_{2\ell} < 0$ **then**
6. $\mathcal{J} := \mathcal{R}_{2,\ell}\,(\mathcal{I}; \lambda_{2\ell-1}, \lambda_{2\ell})$;
7. **return** $\mu_{2\ell-1}^{-1}(\mathcal{J}) \cap \Xi_{\ell-1}$;

The result of the algorithm *ForwardSweep* is a set called *solution*. It may be empty, if no admissible directions were found. If not, the elements of *solution* are vectors $\Xi = (\Xi_\ell)_{\ell=0}^m$, where each Ξ gives at least one admissible set of parameters $\xi_\ell \in \Xi_\ell$, $\ell = 0, 1, \ldots, m$. A brief look at Theorem 3.6 reveals that the result in the convex case is much stronger.

COROLLARY 3.12. *Suppose that the assumptions of Theorem 3.11 hold. Let*

$$\lambda_{2\ell-1} > 0, \;\; \lambda_{2\ell} > 0, \quad \ell = 1, 2, \ldots, m,$$

and let Ξ be a vector of intervals returned by ForwardSweep. Any choice of parameters

$$(\xi_0, \xi_1, \ldots, \xi_m), \quad \xi_\ell \in \Xi_\ell,$$

is admissible.

Even in the general case, there is a natural way to generate admissible choices, based upon the following consequence.

COROLLARY 3.13. *Suppose that the assumptions of Theorem 3.11 hold. Let Ξ be a vector of intervals, returned by ForwardSweep. For any r, $0 \le r \le m$, and any chosen $\xi_r \in \Xi_r$, one can find at least one admissible selection $(\xi_0, \ldots, \xi_{r-1}, \xi_r, \xi_{r+1}, \ldots, \xi_m)$, $\xi_\ell \in \Xi_\ell$.*

Let us now look at a graphical interpretation of the algorithm for convex and nonconvex data. In Figure 3.10 *ForwardSweep* for convex data is shown. Gray areas show tangent directions that correspond to intervals for ξ_ℓ. If the direction such that $\phi_i(\lambda_{2\ell-1}, \lambda_{2\ell}) = \mu_{2\ell-2+i}$ is in the interval obtained from the previous step the solution branches out into two parts. The part of the solution that is further considered corresponds to dark gray areas and the part of the solution that is excluded in *BackwardSweep* corresponds to light gray areas. The final result of the algorithm for these data are five different sets of intervals for ξ_ℓ, shown in Figure 3.11. In Figure 3.12 *ForwardSweep* and *BackwardSweep* for nonconvex data are shown. In *BackwardSweep* gray areas for tangent directions are reduced to smaller ones in order to satisfy the result of the Corollary 3.13. The final result is shown in Figure 3.13.

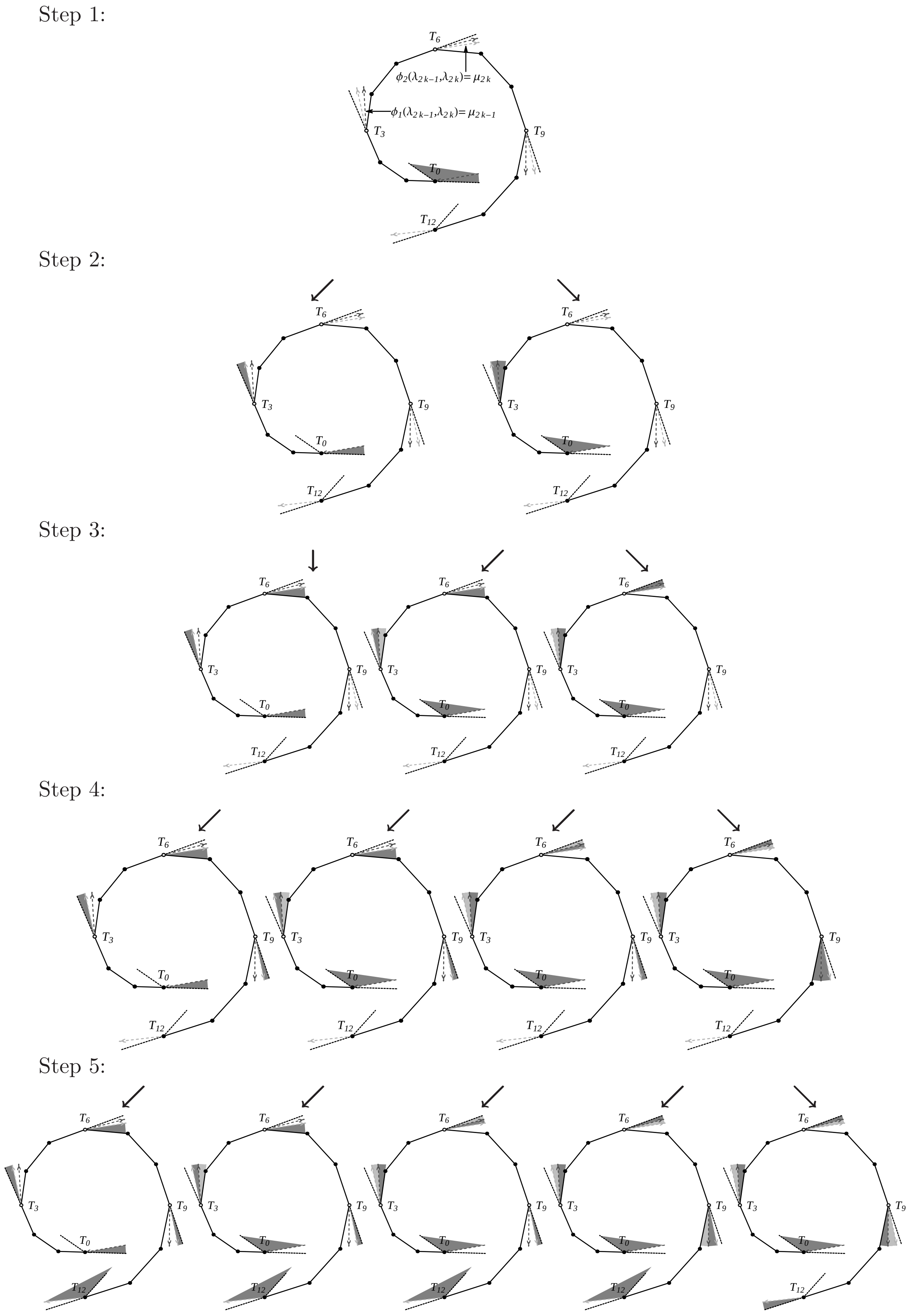


Figure 3.10: Graphical interpretation of the algorithm for convex data.

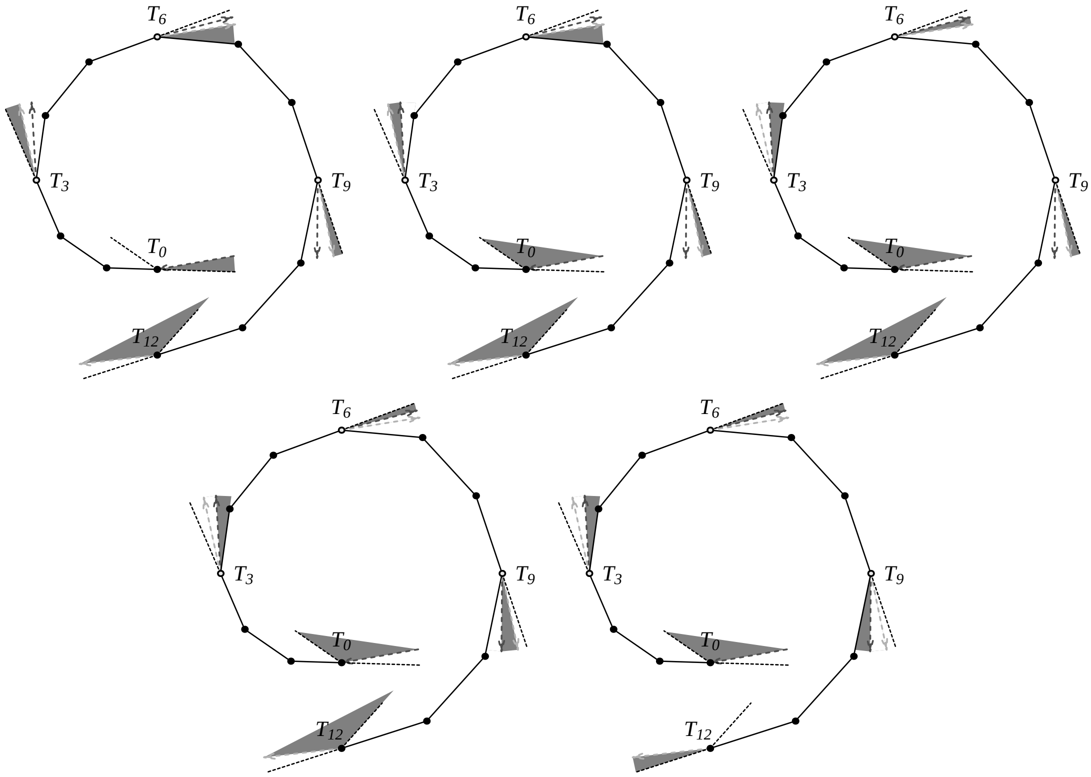
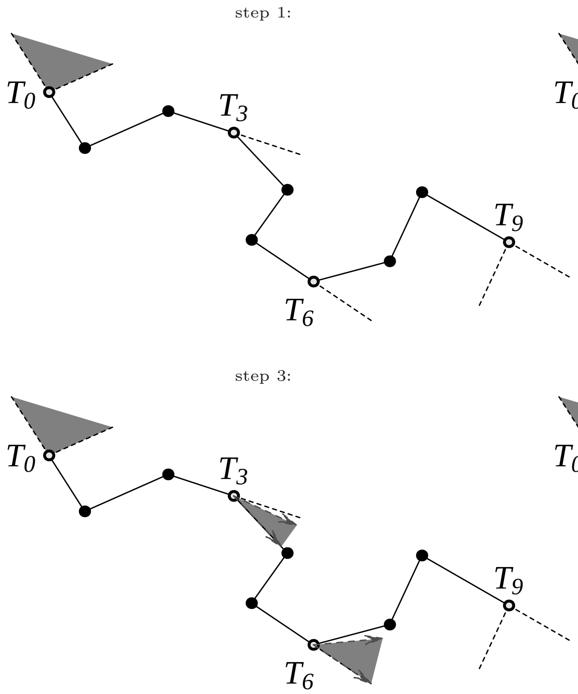


Figure 3.11: All possible sets of intervals for ξ_ℓ obtained by the algorithm.

ForwardSweep:



BackwardSweep:

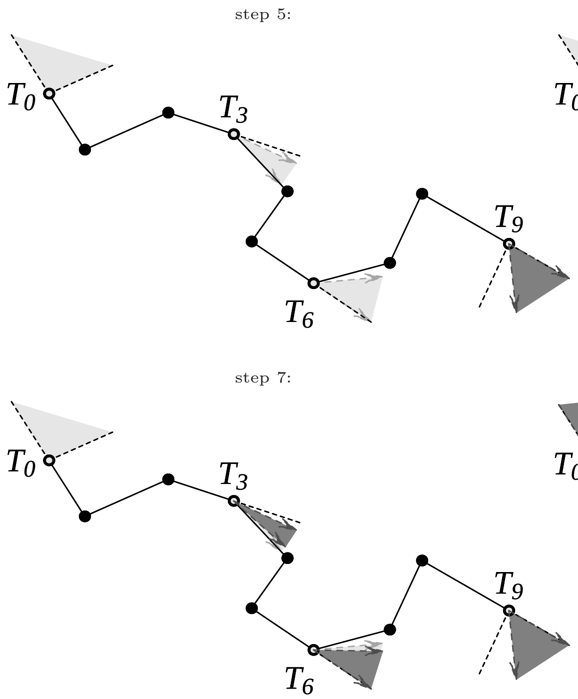


Figure 3.12: Graphical interpretation of the algorithm for nonconvex data.

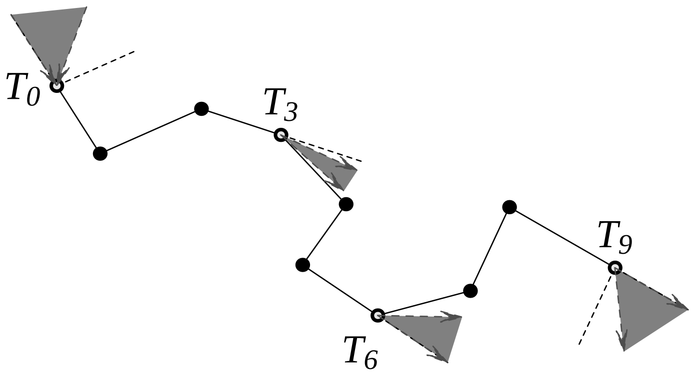


Figure 3.13: The result of the algorithm for nonconvex data.

Once the bounds Ξ have been determined, one has to choose the actual tangent directions. Let us pick one vector $\Xi \in$ *solution*. Then choose r, $0 \le r \le m$, and choose $\xi_r \in \Xi_r$. This means that Ξ_r has been in Ξ replaced by $[\xi_r, \xi_r]$. Corollary 3.13 for this new Ξ does not necessarily hold. But a call

$$\text{BackwardSweep}(r, \Xi, r, \text{solution})$$

properly shrinks the intervals $\Xi_{r-1}, \Xi_{r-2}, \ldots, \Xi_0$, and so does the mirror image of *BackwardSweep* on the intervals $\Xi_{r+1}, \Xi_{r+2}, \ldots, \Xi_m$. This brings the property of vector Ξ, described in Corollary 3.13, to each of its parts $(\Xi_\ell)_{\ell=0}^{r}$ and $(\Xi_\ell)_{\ell=r}^{m}$. So the whole step can be repeated on both parts separately. This *divide and conquer* procedure can be repeated until we are left with the admissible selection. It adds at most a factor $\mathcal{O}(m)$ to the complexity of *ForwardSweep*. The algorithm is the following.

$V := (0, 0, \ldots, 0)$;
procedure DivideAndConquer(Ξ, m, V)
choose r, $0 \le r \le m$, choose $\xi_r \in \Xi_r$;
$V_r := \xi_r$;
$\Xi_r := [\xi_r, \xi_r]$, $\Xi^L := \emptyset$, $\Xi^R := \emptyset$;
BackwardSweep(r, $(\Xi_0, \Xi_1, \ldots, \Xi_r)$, r, Ξ^L);
BackwardSweep($m-r$, $(\Xi_m, \Xi_{m-1}, \ldots, \Xi_r)$, $m-r$, Ξ^R);
DivideAndConquer(Ξ^L, r, $(V_0, V_1, \ldots, V_r)$);
DivideAndConquer(Reverse(Ξ^R), $m-r$, $(V_r, V_{r+1}, \ldots, V_m)$);

Vector V is an admissible selection. Most often r is chosen as

$$r = \left\lfloor \frac{m}{2} \right\rfloor, \quad r = \left\lceil \frac{m}{2} \right\rceil, \quad r = 0 \quad \text{or} \quad r = m.$$

This procedure can also be used to check if the interpolation problem (3.2) and (3.3) has a solution when the actual tangent directions are prescribed in advance. Then in each step it only needs to be checked if the prescribed ξ_r is in Ξ_r. There are many ways for choosing tangent directions (choosing $\xi_r \in \Xi_r$). One can approximate them as

$$\boldsymbol{d}_{3r} = \boldsymbol{d}_{3r}\left(\gamma_r \underline{\xi}_r + (1-\gamma_r)\,\overline{\xi}_r\right), \quad \Xi_r = \left(\underline{\xi}_r, \overline{\xi}_r\right) \quad \text{or} \quad \Xi_r = \left[\underline{\xi}_r, \overline{\xi}_r\right],$$

where γ_r can simply be chosen as a constant or determined by some local approximation scheme from the data (3.1). Namely, interpolating polynomials with respect to certain parameterization like chord length or uniform may determine the directions in the following way. Let $\boldsymbol{p}_r$ denote the interpolating polynomial through points $\boldsymbol{T}_{3r-j}, \dots, \boldsymbol{T}_{3r-1}$, $\boldsymbol{T}_{3r}$, $\boldsymbol{T}_{3r+1}, \dots, \boldsymbol{T}_{3r+j}$ at some chosen parameter values

$$u_{3r-j}, \dots, u_{3r-1}, u_{3r}, u_{3r+1}, \dots, u_{3r+j}.$$

Then γ_r can be determined so that $\boldsymbol{d}_{3r}$ is the derivative of $\boldsymbol{p}_r$ at u_{3r}. But it can happen that $\gamma_r \underline{\xi}_r + (1-\gamma_r)\,\overline{\xi}_r$ falls out of the interval Ξ_r. If that happens γ_r must be chosen in some other way, and in that case the admissible selection may depend on the r we choose in the *DivideAndConquer* procedure.

Also, with the help of Ξ, one may look for a G^2 spline curve with $\boldsymbol{d}_{3\ell}$ determined implicitly as a solution of the system (3.4).

3.5. Examples

Let us conclude this chapter with some numerical examples. As the first example let the data points be given as

$$\begin{aligned}
&\boldsymbol{T}_0 = (7,-24)^T, \quad \boldsymbol{T}_1 = (20,-18)^T, \quad \boldsymbol{T}_2 = (24,-6)^T, \quad \boldsymbol{T}_3 = (17,4)^T,\\
&\boldsymbol{T}_4 = (4,9)^T, \quad \boldsymbol{T}_5 = (-15,9)^T, \quad \boldsymbol{T}_6 = (-23,17)^T, \quad \boldsymbol{T}_7 = (-24,29)^T,\\
&\boldsymbol{T}_8 = (-13,40)^T, \quad \boldsymbol{T}_9 = (8,43)^T, \quad \boldsymbol{T}_{10} = (24,37)^T, \quad \boldsymbol{T}_{11} = (28,28)^T,\\
&\boldsymbol{T}_{12} = (38,25)^T, \quad \boldsymbol{T}_{13} = (51,29)^T, \quad \boldsymbol{T}_{14} = (56,37)^T, \quad \boldsymbol{T}_{15} = (70,42)^T,\\
&\boldsymbol{T}_{16} = (83,39)^T, \quad \boldsymbol{T}_{17} = (91,25)^T, \quad \boldsymbol{T}_{18} = (88,11)^T.
\end{aligned}$$

Further, let the first and the last tangent direction be prescribed as

$$\boldsymbol{d}_0 = \frac{1}{2\sqrt{82}}(18,-2)^T, \quad \boldsymbol{d}_{18} = \frac{1}{\sqrt{218}}(-13,-7)^T.$$

Algorithm *ForwardSweep* returns the set *solution* with two admissible vectors of intervals

$$\begin{aligned}
&\mathit{solution} = \{\Xi^1, \Xi^2\},\\
&\Xi^1 := \{[0.625698, 0.625698], (0.295393, 0.893511), (0, 0.634404), (0.355009, 0.987411),\\
&\qquad (0,1), (0.00770851, 0.684009), [0.403509, 0.403509]\}\\
&\Xi^2 := \{[0.625698, 0.625698], (0.295393, 0.806989), (0.634404, 1), (0, 0.355009)\\
&\qquad (0.0656466, 1), (0.00770851, 0.684009), [0.403509, 0.403509]\}.
\end{aligned}$$

By the *DivideAndConquer* procedure where r is chosen as $\left\lfloor \frac{m}{2} \right\rfloor$ and ξ_r as the middle value of the interval Ξ_r the admissible selection reads

$$\begin{aligned}
&\xi_0 = 0.625698, \quad \xi_1 = 0.594452, \quad \xi_2 = 0.317202, \quad \xi_3 = 0.67121,\\
&\xi_4 = 0.509024, \quad \xi_5 = 0.40432, \quad \xi_6 = 0.403509,
\end{aligned} \tag{3.21}$$

for the vector Ξ^1 and

$$\xi_0 = .625698, \quad \xi_1 = 0.551191, \quad \xi_2 = 0.817202, \quad \xi_3 = 0.177505, \qquad (3.22)$$
$$\xi_4 = 0.546479, \quad \xi_5 = 0.398809, \quad \xi_6 = 0.403509,$$

for the vector Ξ^2. As described in the previous subsection the parameters ξ_ℓ may also be determined by some local approximation scheme. Table 3.4 shows selections obtained by using interpolating polynomials of degrees $2, 4, 6$ based upon uniform and chord length parameterization. Note that all selections are admissible. The best choice would probably be to compute the parameters so that the spline is not only G^1 but also G^2. By solving the system (3.4) one obtains

	uniform parameterization			*chord length* parameterization		
	degree 2	degree 4	degree 6	degree 2	degree 4	degree 6
ξ_0	0.625698	0.625698	0.625698	0.625698	0.625698	0.625698
ξ_1	0.5	0.50634	0.480129	0.434402	0.483794	0.494794
ξ_2	0.5	0.527159	0.563559	0.468864	0.466898	0.468311
ξ_3	0.5	0.525746	0.524849	0.606469	0.531615	0.395345
ξ_4	0.5	0.515385	0.518351	0.370748	0.398726	0.404478
ξ_5	0.5	0.432277	0.414843	0.553885	0.55851	0.591284
ξ_6	0.403509	0.403509	0.403509	0.403509	0.403509	0.403509

Table 3.4: Admissible selections determined by interpolating polynomials based upon uniform and chord length parameterization.

$$\xi_0 = 0.625698, \quad \xi_1 = 0.496926, \quad \xi_2 = 0.539584, \quad \xi_3 = 0.448597,$$
$$\xi_4 = 0.38032, \quad \xi_5 = 0.646376, \quad \xi_6 = 0.403509.$$

Figure 3.14 shows the comparison between the G^2 spline (dashed) and G^1 splines with tangent directions given by (3.21) (light gray) and (3.22) (dark gray). One can check that the difference between G^1 splines obtained by interpolating polynomials of different degrees with the same parameterization (uniform or chord length) is almost imperceptible. A comparison between G^2 spline (dashed) and G^1 spline with tangent directions obtained by quadratic interpolating polynomials is shown in Figure 3.15.

For the next example let the data points be given as

$$\boldsymbol{T}_0 = (-45,-14)^T, \quad \boldsymbol{T}_1 = (-36,-11)^T, \quad \boldsymbol{T}_2 = (-28,-14)^T,$$
$$\boldsymbol{T}_3 = (-18,-15)^T, \quad \boldsymbol{T}_4 = (-9,-11)^T, \quad \boldsymbol{T}_5 = (-4,-1)^T,$$
$$\boldsymbol{T}_6 = (9,3)^T, \quad \boldsymbol{T}_7 = (21,0)^T, \quad \boldsymbol{T}_8 = (27,-8)^T, \quad \boldsymbol{T}_9 = (37,-15)^T,$$
$$\boldsymbol{T}_{10} = (48,-15)^T, \quad \boldsymbol{T}_{11} = (63,-10)^T, \quad \boldsymbol{T}_{12} = (65,-1)^T,$$

and the first and the last tangent direction prescribed as

$$\boldsymbol{d}_0 = \frac{1}{\sqrt{61}}(6,5)^T, \quad \boldsymbol{d}_{12} = \frac{1}{\sqrt{89}}(-5,8)^T.$$

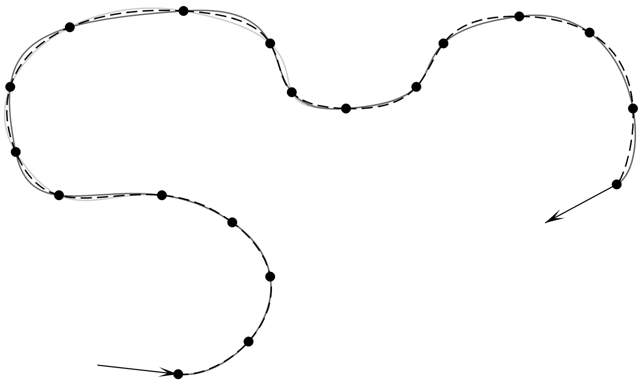


Figure 3.14: A comparison between G^2 (dashed) and G^1 spline curves with tangent directions given by (3.21) (light gray) and (3.22) (dark gray).

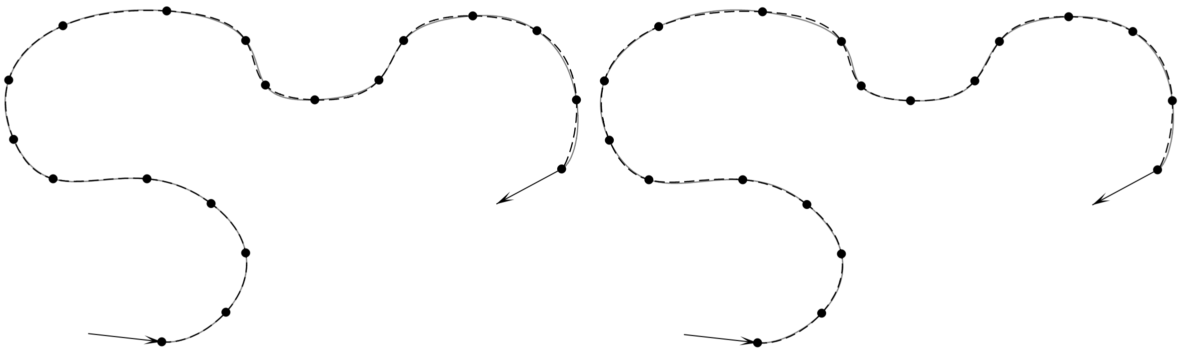


Figure 3.15: A comparison between G^2 (dashed) and G^1 (gray) spline curves with tangent directions obtained by quadratic interpolating polynomials based upon uniform (left) and chord length (right) parameterization.

Algorithm *ForwardSweep* now returns the set with one admissible vector of intervals

$$\Xi := \{[0.682353, 0.682353], (0.187934, 0.976309), (0, 1), (0.11408, 0.908464), [0.296117, 0.296117]\}.$$

By *DivideAndConquer* procedure with r chosen as $\lfloor \frac{m}{2} \rfloor$ and ξ_r as the middle value of the interval Ξ_r the admissible selection reads

$$\xi_0 = 0.682353, \quad \xi_1 = 0.534038, \quad \xi_2 = 0.5, \quad \xi_3 = 0.541808, \quad \xi_4 = 0.296117. \quad (3.23)$$

Selections obtained by using interpolating polynomials of degrees $2, 4, 6$ based upon uniform and chord length parameterization are shown in Table 3.5. Again all the selections are admissible. By choosing

$$\xi_0 = 0.682353, \quad \xi_1 = 0.410077, \quad \xi_2 = 0.61075, \quad \xi_3 = 0.74838, \quad \xi_4 = 0.296117,$$

one obtains a G^2 spline. Figure 3.16 shows the comparison between G^2 spline (dashed) and G^1 spline (gray) with tangent directions given by (3.23). A comparison between G^2

	uniform parameterization			*chord length* parameterization		
	degree 2	degree 4	degree 6	degree 2	degree 4	degree 6
ξ_0	0.682353	0.682353	0.682353	0.682353	0.682353	0.682353
ξ_1	0.5	0.423453	0.391853	0.510101	0.463167	0.468158
ξ_2	0.5	0.51706	0.533981	0.547337	0.544507	0.557311
ξ_3	0.5	0.454741,	0.451198	0.551852	0.542437	0.57377
ξ_4	0.296117	0.296117	0.296117	0.296117	0.296117	0.296117

Table 3.5: Admissible selections determined by interpolating polynomials based upon uniform and chord length parameterization.

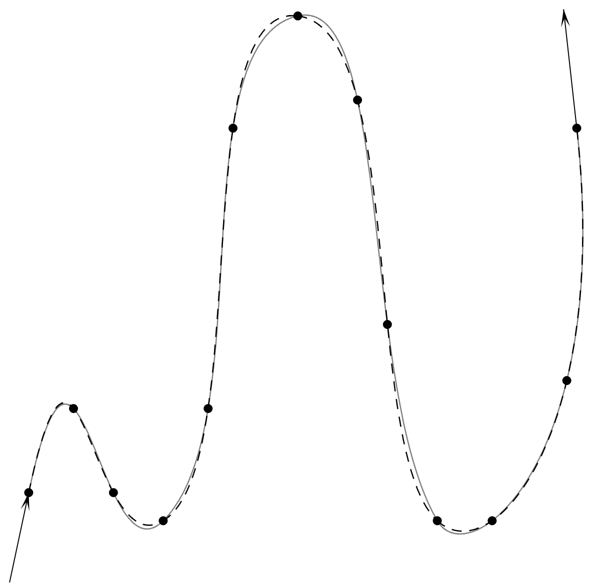


Figure 3.16: A comparison between G^2 (dashed) and G^1 (gray) spline curves with tangent directions given by (3.23).

spline (dashed) and G^1 spline with tangent directions obtained by quadratic interpolating polynomials is shown in Figure 3.17.

For the last example the data points are chosen as

$$\begin{aligned}
&\boldsymbol{T}_0 = (-44,-12)^T, \quad \boldsymbol{T}_1 = (-42,-4)^T, \quad \boldsymbol{T}_2 = (-37,4)^T, \\
&\boldsymbol{T}_3 = (-28,11)^T, \quad \boldsymbol{T}_4 = (-17,10)^T, \quad \boldsymbol{T}_5 = (-11,2)^T, \\
&\boldsymbol{T}_6 = (-2,-4)^T, \quad \boldsymbol{T}_7 = (7,-1)^T, \quad \boldsymbol{T}_8 = (11,7)^T, \quad \boldsymbol{T}_9 = (19,12)^T, \\
&\boldsymbol{T}_{10} = (31,10)^T, \quad \boldsymbol{T}_{11} = (37,3)^T, \quad \boldsymbol{T}_{12} = (40,-7)^T,
\end{aligned}$$

and the first and the last tangent direction prescribed as

$$\boldsymbol{d}_0 = (0,1)^T, \quad \boldsymbol{d}_{12} = \frac{1}{\sqrt{104}}(2,-10)^T.$$

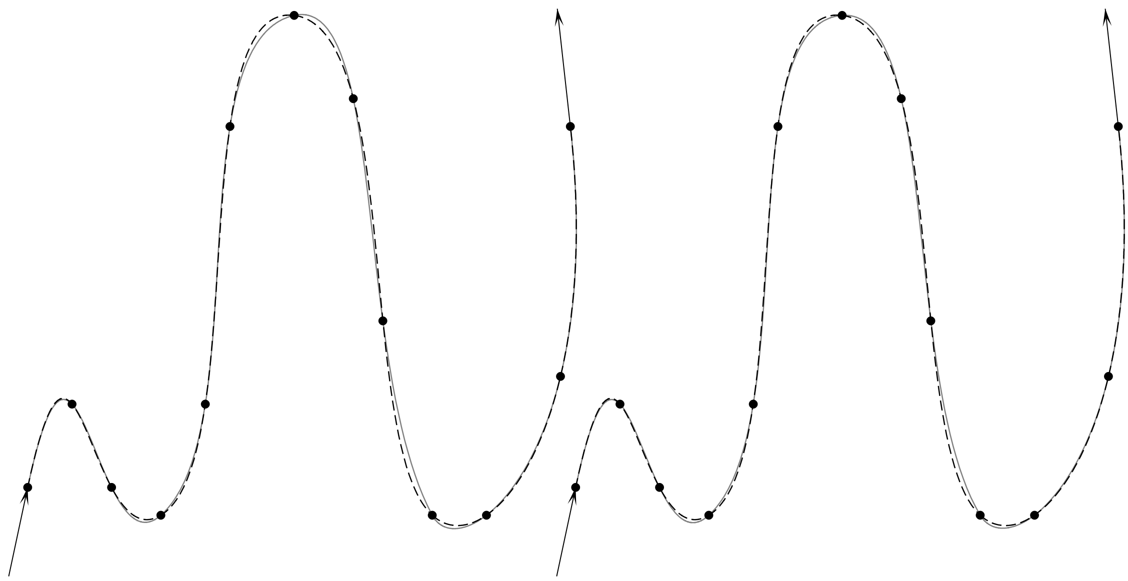


Figure 3.17: A comparison between G^2 (dashed) and G^1 (gray) spline curves with tangent directions obtained by quadratic interpolating polynomials based upon uniform (left) and chord length parameterization.

Algorithm *ForwardSweep* again returns the set with one admissible vector of intervals

$$\Xi := \{[0.714286, 0.714286], (0.247924, 1), (0.10384, 0.9800461), \\ (0.463815, 1), [0.178571, 0.178571]\},$$

and *DivideAndConquer* procedure gives the admissible selection

$$\xi_0 = 0.714286, \quad \xi_1 = 0.623962, \quad \xi_2 = 0.541943, \quad \xi_3 = 0.731908, \quad \xi_4 = 0.178571. \tag{3.24}$$

Admissible selections obtained by interpolating polynomials of degrees 2, 4, 6 based upon

	degree 2	degree 4	degree 6
ξ_0	0.714286	0.714286	0.714286
ξ_1	0.5	0.47043	0.456363
ξ_2	0.5	0.485632	0.489807
ξ_3	0.5	0.494949	0.509434
ξ_4	0.178571	0.178571	0.178571

Table 3.6: Admissible selections determined by interpolating polynomials based upon uniform parameterization.

uniform parameterization are given in Table 3.6. But interpolating polynomials based on chord length parameterization do not give admissible tangent directions. Furthermore, a G^2 spline curve can not be found. A comparison between the G^1 spline curve with tangent directions given by (3.24) and the G^1 spline curve with tangent directions obtained by quadratic interpolating polynomials based upon uniform parameterization is shown in Figure 3.18.

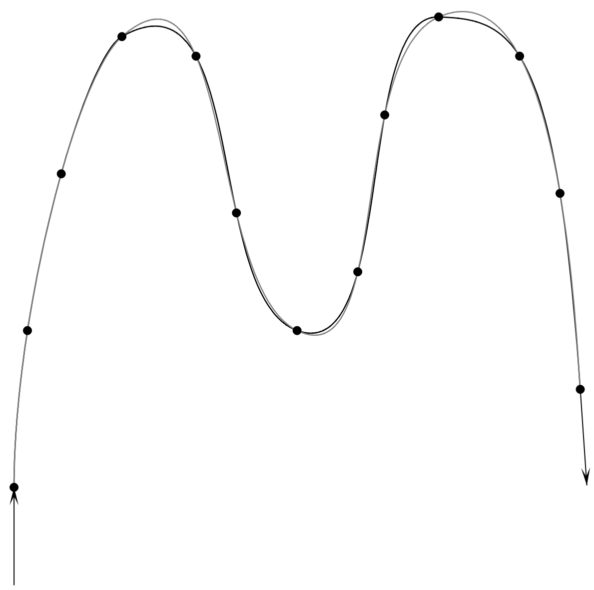


Figure 3.18: The G^1 spline curve with tangent directions given by (3.24) (black) and the G^1 spline curve with tangent directions obtained by quadratic interpolating polynomials based upon uniform parameterization (gray).

Chapter 4

Hermite geometric interpolation by cubic G^1 splines

In this chapter Hermite geometric interpolation by planar cubic G^1 splines is studied. Three data points and three tangent directions are interpolated per each polynomial segment. Sufficient conditions for the existence of such G^1 spline are determined that cover most of the cases encountered in practical applications. The existence requirements are based only upon geometric properties of data and can easily be verified in advance. The optimal approximation order six is confirmed, too.

4.1. Interpolation problem

The problem considered is the following. Suppose that $2m + 1$ points and tangent directions

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad \boldsymbol{d}_i \in \mathbb{R}^2, \qquad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \quad \|\boldsymbol{d}_i\|_2 = 1, \quad i = 0, 1, \ldots, 2m,$$

are given. Find a cubic G^1 spline curve $\boldsymbol{S} : [0, 1] \to \mathbb{R}^2$ with prescribed breakpoints $(t_{2i})_{i=0}^m$,

$$0 =: t_0 < t_1 < \cdots < t_{2m-1} < t_{2m} := 1,$$

that interpolates the data points $\boldsymbol{T}_i$ and tangent directions $\boldsymbol{d}_i$ at parameters t_i,

$$\boldsymbol{S}(t_i) = \boldsymbol{T}_i, \quad \frac{1}{\|\boldsymbol{S}'(t_i)\|_2}\boldsymbol{S}'(t_i) = \boldsymbol{d}_i, \quad i = 0, 1, \ldots, 2m, \tag{4.1}$$

where $(t_{2i-1})_{i=1}^m$ are the unknowns. Note that (4.1) makes sense even if $\boldsymbol{S}'$ jumps at the breakpoint t_{2i} since the tangent direction is continuous. This interpolation scheme is quite clearly local. Namely, the change of one point or one tangent direction affects only those segments that the point or the direction belongs to. So all the analysis and estimations can be done locally.

For a motivation, let us consider some numerical examples. First, suppose that the data are sampled from an exponential and logarithmic spiral

$$\boldsymbol{f}_1(t) := \exp\left(\frac{t}{4}\right)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \quad t \in [0, 3\pi], \qquad \boldsymbol{f}_2(t) := \log(1+t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \quad t \in [0, 4\pi],$$

at equidistantly chosen parameters in the parameter domain. In Figure 4.1 interpolating G^1 spline curves composed of five segments, i.e., $m = 5$, are shown for each curve $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$. The parametric error estimates between $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and their interpolants are

$$\operatorname{dist}(\boldsymbol{S}, \boldsymbol{f}_1) = 0.007915, \quad \operatorname{dist}(\boldsymbol{S}, \boldsymbol{f}_2) = 0.051094.$$

Considering a single segment case, Table 4.1 numerically suggests that the approxima-

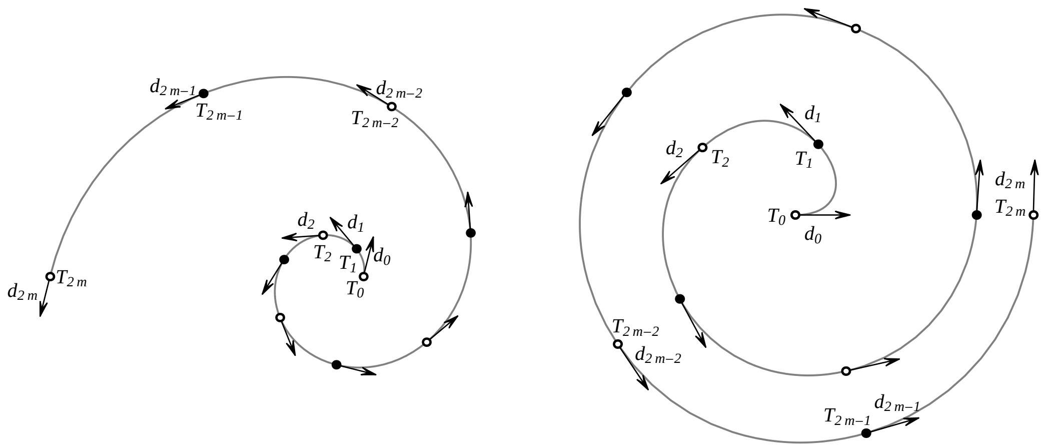


Figure 4.1: The interpolating G^1 spline curves $\boldsymbol{S}$ for data obtained from curves $\boldsymbol{f}_1$ (left) and $\boldsymbol{f}_2$ (right).

tion order, measured in the parametric distance, is optimal, i.e., 6. However, the data

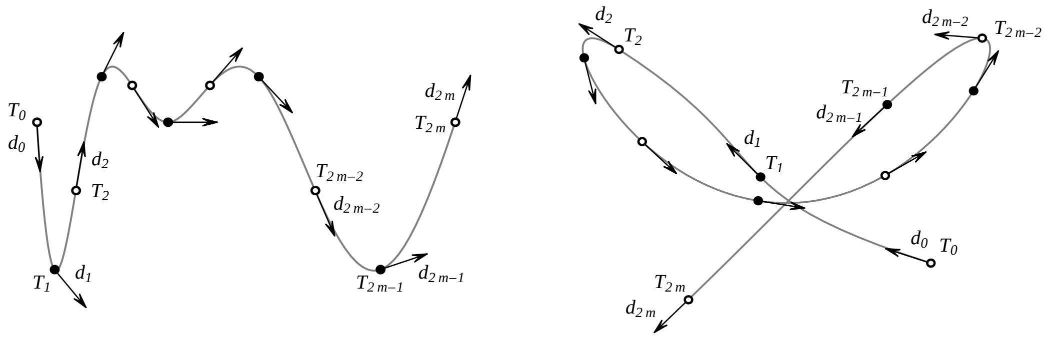


Figure 4.2: Cubic G^1 spline curves $\boldsymbol{S}$ for given data points and tangent directions.

do not need to be sampled from smooth curves only, they can be provided in some other way, maybe given by the user for design purposes, obtained from some other application, etc. The data do not need to be convex either. Figure 4.2 shows some more examples

Interval	Approximation error		Decay exponent	
	$\boldsymbol{f}_1$	$\boldsymbol{f}_2$	$\boldsymbol{f}_1$	$\boldsymbol{f}_2$
$[0, \pi]$	3.3754×10^{-2}	2.2251×10^{-1}	/	/
$\left[0, \frac{9\pi}{10}\right]$	1.6644×10^{-2}	1.0817×10^{-1}	6.71	6.85
$\left[0, \frac{8\pi}{10}\right]$	7.6630×10^{-3}	5.1067×10^{-2}	6.59	6.37
$\left[0, \frac{7\pi}{10}\right]$	3.2233×10^{-3}	2.2485×10^{-2}	6.49	6.14
$\left[0, \frac{6\pi}{10}\right]$	1.2017×10^{-3}	8.8151×10^{-3}	6.40	6.07
$\left[0, \frac{5\pi}{10}\right]$	3.7917×10^{-4}	2.8939×10^{-3}	6.33	6.11
$\left[0, \frac{4\pi}{10}\right]$	9.3807×10^{-5}	7.2252×10^{-4}	6.26	6.22
$\left[0, \frac{3\pi}{10}\right]$	1.5773×10^{-5}	1.4194×10^{-4}	6.20	6.39

Table 4.1: The errors between curves $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ and their polynomial geometric interpolants.

and, as one can see, the spline follows the shape of the data quite nicely.

The next two theorems give the main results of this chapter.

THEOREM 4.1. *Suppose that $2m+1$ points and tangent directions*

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad \boldsymbol{d}_i \in \mathbb{R}^2, \qquad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \quad \|\boldsymbol{d}_i\|_2 = 1, \quad i = 0, 1, \ldots, 2m,$$

are given. If on each segment $[t_{2\ell-2}, t_{2\ell}]$, $\ell = 1, 2, \ldots, m$, one of the sufficient conditions prescribed by Theorem 4.10, 4.11, 4.13 or 4.15 is fulfilled, then a cubic G^1 spline curve $\boldsymbol{S}$ that satisfies (4.1) *exists.*

THEOREM 4.2. *Suppose that the data are sampled from a smooth convex regular parametric curve $\boldsymbol{f} : [a, b] \to \mathbb{R}^2$ at parameter values s_i, $a = s_0 < s_1 < \cdots < s_{2m} = b$,*

$$\boldsymbol{T}_i = \boldsymbol{f}(s_i), \quad \boldsymbol{d}_i = \frac{1}{\|\boldsymbol{f}'(s_i)\|_2}\boldsymbol{f}'(s_i), \quad i = 0, 1, \ldots, 2m,$$

and let $h := \max\{\Delta s_i : \ i = 1, 2, \ldots, 2m-1\}$. Then one can find constants $h_0 > 0$ and $C > 0$ such that for all h, $0 < h \leq h_0$, a cubic G^1 spline curve $\boldsymbol{S}$ that satisfies (4.1) *exists, and approximates $\boldsymbol{f}$ with the optimal approximation order six, i.e.,* $\operatorname{dist}(\boldsymbol{f}, \boldsymbol{S}) \leq C\, h^6$.

The outline of this chapter is as follows. In Subsection 4.2 a system of equations is derived for a single segment case. The next subsection provides the conditions that imply the solution to attain the values that are not allowed. These results together with the number of solutions for some particular data lead to the existence theorems of Subsection 4.4. They are proved in Subsection 4.5 by the help of convex homotopy and Brouwer's degree argument. The last subsection deals with asymptotic analysis and contains the proof of Theorem 4.2.

4.2. Single segment case

Since the G^1 interpolation scheme (4.1) is local, all the properties can be determined from the simplest case $m = 1$. So from now on to the end of this chapter we assume $m = 1$. For $m = 1$ let the spline $\boldsymbol{S}$ be denoted by $\boldsymbol{P}$. The equations (4.1) simplify to

$$\boldsymbol{P}(t_i) = \boldsymbol{T}_i, \qquad \boldsymbol{P}'(t_i) = \alpha_i \boldsymbol{d}_i, \quad \alpha_i > 0, \qquad i = 0, 1, 2, \tag{4.2}$$

with $0 := t_0 < t_1 < t_2 := 1$. One is thus left with twelve equations for eight unknown coefficients of $\boldsymbol{P}$, and four unknown parameters t_1, α_0, α_1, and α_2.

REMARK 4.3. *If the tangent directions are not normalized that does not effect the existence of the solution of* (4.2)*. Only the magnitudes of* α_i*,* $i = 0, 1, 2$*, change.*

The first step is to separate the unknown coefficients from the rest of the unknowns. For any t_1, α_0, α_1 and α_2 there exists a unique polynomial $\boldsymbol{p}_5$ of degree ≤ 5 that solves the interpolation problem (4.2). But this $\boldsymbol{p}_5$ will be of degree three, i.e., $\boldsymbol{p}_5 = \boldsymbol{P}$, iff the coefficients at powers 4 and 5 are zero. This is true iff

$$[t_0, t_0, t_1, t_1, t_2]\boldsymbol{p}_5 = 0, \quad [t_0, t_1, t_1, t_2, t_2]\boldsymbol{p}_5 = 0, \tag{4.3}$$

which gives the system of four equations for four unknowns t_1, α_0, α_1, α_2, that must lie in an open set

$$\mathcal{U} := \{t_1;\ 0 < t_1 < 1\} \times \{(\alpha_0, \alpha_1, \alpha_2);\ \alpha_i > 0,\ i = 0, 1, 2\}.$$

Establishing these parameters is the only nonlinear part of the problem. The coefficients of $\boldsymbol{P}$ are then obtained by using any standard interpolation scheme componentwise. Since $t_0 = 0$ and $t_2 = 1$ the equations (4.3) simplify to

$$\begin{aligned}
\frac{\alpha_0}{t_1^2}\boldsymbol{d}_0 + \frac{\alpha_1}{(1-t_1)\,t_1^2}\boldsymbol{d}_1 - \frac{(2+t_1)}{t_1^3}\Delta\boldsymbol{T}_0 - \frac{1}{(1-t_1)^2}\Delta\boldsymbol{T}_1 = 0,\\
\frac{\alpha_1}{(1-t_1)^2\,t_1}\boldsymbol{d}_1 + \frac{\alpha_2}{(1-t_1)^2}\boldsymbol{d}_2 - \frac{1}{t_1^2}\Delta\boldsymbol{T}_0 + \frac{(t_1-3)}{(1-t_1)^3}\Delta\boldsymbol{T}_1 = 0.
\end{aligned} \tag{4.4}$$

To simplify the analysis it will be assumed from now on that the points $\boldsymbol{T}_0$, $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are not collinear. Using $\det(\cdot, \Delta\boldsymbol{T}_0)$ and $\det(\cdot, \Delta\boldsymbol{T}_1)$ on (4.4) one obtains

$$\begin{aligned}
\frac{\alpha_0}{t_1^2}\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_0) - \frac{\alpha_1}{(1-t_1)\,t_1^2}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1) + \frac{1}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0,\\
\frac{\alpha_0}{t_1^2}\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_1) + \frac{\alpha_1}{(1-t_1)\,t_1^2}\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1) - \frac{(2+t_1)}{t_1^3}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0,\\
\frac{\alpha_1}{(1-t_1)^2\,t_1}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1) + \frac{\alpha_2}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_2) + \frac{(t_1-3)}{(1-t_1)^3}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0,\\
\frac{\alpha_1}{(1-t_1)^2\,t_1}\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1) - \frac{\alpha_2}{(1-t_1)^2}\det(\Delta\boldsymbol{T}_1, \boldsymbol{d}_2) - \frac{1}{t_1^2}\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) = 0.
\end{aligned} \tag{4.5}$$

Let us define the constants that are determined by the data as

$$\begin{aligned}
\lambda_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_0)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \lambda_2 := \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)},\\
\lambda_3 &:= \frac{\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \lambda_4 := \frac{\det(\Delta\boldsymbol{T}_1, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)},\\
\mu_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \quad \mu_2 := \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}.
\end{aligned} \tag{4.6}$$

Their signs have a nice geometric interpretation as one can see in Figure 4.3. Note again that the definition of the constants λ_i and μ_i differs from that in Chapter 2 and Chapter 3. With these constants, the equations (4.5) become

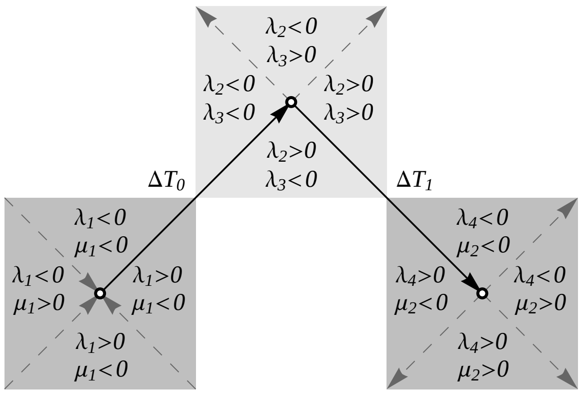


Figure 4.3: Geometric interpretation of signs of the constants λ_i and μ_i.

$$\boldsymbol{F}(t_1, \boldsymbol{\alpha}) := \boldsymbol{F}(t_1, \boldsymbol{\alpha}; \boldsymbol{\lambda}, \mu_1, \mu_2) := (F_i(t_1, \boldsymbol{\alpha}))_{i=1}^4 = \boldsymbol{0},$$

where

$$\begin{aligned}
F_1(t_1, \boldsymbol{\alpha}) &:= \lambda_1 \frac{\alpha_0}{t_1^2} - \lambda_2 \frac{\alpha_1}{t_1^2(1-t_1)} + \frac{1}{(1-t_1)^2} = 0,\\
F_2(t_1, \boldsymbol{\alpha}) &:= \mu_1 \frac{\alpha_0}{t_1^2} + \lambda_3 \frac{\alpha_1}{t_1^2(1-t_1)} - \frac{(2+t_1)}{t_1^3} = 0,\\
F_3(t_1, \boldsymbol{\alpha}) &:= \mu_2 \frac{\alpha_2}{(1-t_1)^2} + \lambda_2 \frac{\alpha_1}{(1-t_1)^2 t_1} - \frac{(3-t_1)}{(1-t_1)^3} = 0,\\
F_4(t_1, \boldsymbol{\alpha}) &:= \lambda_4 \frac{\alpha_2}{(1-t_1)^2} - \lambda_3 \frac{\alpha_1}{(1-t_1)^2 t_1} + \frac{1}{t_1^2} = 0.
\end{aligned} \tag{4.7}$$

Here $\boldsymbol{\alpha} := (\alpha_i)_{i=0}^2$, $\boldsymbol{\lambda} := (\lambda_i)_{i=1}^4$. Moreover, if

$$\lambda_1\lambda_3 + \lambda_2\mu_1 \neq 0, \quad \lambda_2\lambda_4 + \lambda_3\mu_2 \neq 0, \tag{4.8}$$

equations (4.7) can be rewritten as

$$\begin{aligned}
\alpha_0 &= \frac{(t_1^3 - 3t_1 + 2)\,\lambda_2 - t_1^3\lambda_3}{(t_1-1)^2\, t_1\, (\lambda_1\lambda_3 + \lambda_2\mu_1)},\\
\alpha_1 &= -\frac{\mu_1 t_1^3 + (t_1^3 - 3t_1 + 2)\,\lambda_1}{(t_1-1)\, t_1\, (\lambda_1\lambda_3 + \lambda_2\mu_1)},\\
\alpha_2 &= \frac{(t_1-3)\, t_1^2\lambda_3 - (t_1-1)^3\,\lambda_2}{(t_1-1)\, t_1^2\, (\lambda_2\lambda_4 + \lambda_3\mu_2)},
\end{aligned} \tag{4.9}$$

and

$$\frac{\mu_1 t_1^3 + (t_1^3 - 3t_1 + 2)\,\lambda_1}{\lambda_1\lambda_3 + \lambda_2\mu_1} + \frac{\mu_2\,(t_1-1)^3 + (t_1-3)\,t_1^2\lambda_4}{\lambda_2\lambda_4 + \lambda_3\mu_2} = 0. \tag{4.10}$$

The only nonlinear part that remains is (4.10) which is a cubic equation for t_1 that can easily be solved numerically. Since the unknowns must lie in $\mathcal{U}$, the next lemma follows directly from equations (4.7).

LEMMA 4.4. *A cubic polynomial curve $\boldsymbol{P}$ that satisfies* (4.2) *does not exist in any of the following cases:*

$$\begin{aligned}
&1.\ \lambda_2 \le 0 \quad \textit{and} \quad \lambda_1 \ge 0, \qquad &&2.\ \lambda_3 \le 0 \quad \textit{and} \quad \lambda_4 \ge 0,\\
&3.\ \lambda_3 \le 0 \quad \textit{and} \quad \mu_1 \le 0, \qquad &&4.\ \lambda_2 \le 0 \quad \textit{and} \quad \mu_2 \le 0.
\end{aligned}$$

As can be seen in Figure 4.3 cases where one of λ_i is equal to zero are very exceptional and for the sake of simplicity it will be assumed from now on that $\lambda_i \neq 0$, $i = 1, 2, 3, 4$. The first step to the existence of $\boldsymbol{P}$ is to find the relations between the data that force the solution of (4.7) to approach the boundary $\partial\mathcal{U}$. This analysis is given in the next subsection.

4.3. Relations, implying the solution to approach the boundary

If the solution $(t_1, \boldsymbol{\alpha})$ touches the boundary $\partial\mathcal{U}$, it attains the values that are not allowed. As it turns out this implies certain relations between data that could be used to avoid the parameter choices that are not admissible. The next two lemmas reveal the relations for $t_1 \to 0, 1$.

LEMMA 4.5. *Suppose that $\lambda_i \neq 0$, $i = 1, 2, 3, 4$. Parameter t_1 tends to zero if*

$$\mu_2 \to \varphi_2(\boldsymbol{\lambda}, \mu_1) := \frac{2\lambda_1\lambda_2\lambda_4}{\lambda_2\mu_1 - \lambda_1\lambda_3},$$

and $\lambda_1\lambda_2 > 0$, and one of the three following conditions hold:

$$\begin{aligned}
&\lambda_3 > 0, \quad \lambda_4 > 0, \quad -\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2}, \quad \textit{or}\\
&\lambda_3 > 0, \quad \lambda_4 < 0, \quad \frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1, \quad \textit{or}\\
&\lambda_3 < 0, \quad \lambda_4 < 0, \quad -\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1.
\end{aligned}$$

Under these conditions $\lambda_1\varphi_2(\boldsymbol{\lambda}, \mu_1) < 0$.

LEMMA 4.6. *Suppose that* $\lambda_i \neq 0$, $i = 1, 2, 3, 4$. *Parameter* t_1 *tends to one if*

$$\mu_1 \to \varphi_1(\boldsymbol{\lambda}, \mu_2) := \frac{2\lambda_1\lambda_3\lambda_4}{\lambda_3\mu_2 - \lambda_2\lambda_4},$$

and $\lambda_3\lambda_4 > 0$, *and one of the three following conditions hold:*

$$\begin{aligned} &\lambda_1 > 0, \quad \lambda_2 > 0, \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3}, \quad \text{or} \\ &\lambda_1 < 0, \quad \lambda_2 > 0, \quad \frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2, \quad \text{or} \\ &\lambda_1 < 0, \quad \lambda_2 < 0, \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2. \end{aligned}$$

Under these conditions $\lambda_4\varphi_1(\boldsymbol{\lambda}, \mu_2) < 0$.

Proof. From the symmetry between λ_i, λ_{5-i}, $i = 1, 2$, and μ_1, μ_2, and the symmetry in equations (4.7) it is enough to prove Lemma 4.5 only. If $\lambda_1\lambda_3 + \lambda_2\mu_1 = 0$, the first two equations in (4.7) become

$$\alpha_0 = \frac{-t_1^2 + (1 - t_1)\,\alpha_1\lambda_2}{(1 - t_1)^2\,\lambda_1}, \qquad \frac{\lambda_3}{(1 - t_1)^2\,\lambda_2} - \frac{t_1 + 2}{t_1^3} = 0.$$

Therefore it is clear that t_1 cannot approach zero or one. The same is true when $\lambda_2\lambda_4 + \lambda_3\mu_2 = 0$. Thus let us assume that conditions (4.8) hold. Equation (4.10) can be rewritten as

$$\frac{2\lambda_1}{\lambda_1\lambda_3 + \lambda_2\mu_1} - \frac{\mu_2}{\lambda_2\lambda_4 + \lambda_3\mu_2} + \mathcal{O}(t_1) = 0,$$

and $t_1 \to 0$ implies $\mu_2 \to \varphi_2(\boldsymbol{\lambda}, \mu_1)$. Moreover,

$$\begin{aligned} &\lim_{t_1\to 0}(t_1\alpha_0) = \frac{2\lambda_2}{\lambda_1\lambda_3 + \lambda_2\mu_1}, \quad \lim_{t_1\to 0}(t_1\alpha_1) = \frac{2\lambda_1}{\lambda_1\lambda_3 + \lambda_2\mu_1}, \\ &\lim_{t_1\to 0}\left(t_1^2\alpha_2\right) = -\frac{\lambda_2}{\lambda_2\lambda_4 + \lambda_3\mu_2}. \end{aligned}$$

It is now easy to check that $\boldsymbol{\alpha}$ is positive iff the conditions in the lemma are fulfilled, and further that $\lambda_1\varphi_2(\boldsymbol{\lambda}, \mu_1) < 0$, which completes the proof. □

The relations implied by $\alpha_i = 0$ are given in the next lemma.

LEMMA 4.7. *Parameter* $\alpha_0 = 0$ *if* $\lambda_i > 0$ *for* $i = 2, 3, 4$, *and* $\mu_2 = \psi_2(\boldsymbol{\lambda})$, *where*

$$\psi_2(\boldsymbol{\lambda}) := \lambda_4\frac{\tau(\lambda_2, \lambda_3)^2(3 - 2\tau(\lambda_2, \lambda_3))}{(1 - \tau(\lambda_2, \lambda_3))^2(1 + 2\tau(\lambda_2, \lambda_3))} > 0,$$

and $\tau(\lambda_2, \lambda_3)$ *is defined as a unique solution* t_1 *of the problem*

$$g(t_1; \lambda_2, \lambda_3) := \frac{1}{\lambda_2}\frac{t_1^2}{1 - t_1} - \frac{1}{\lambda_3}\frac{(1 - t_1)(2 + t_1)}{t_1} = 0, \quad 0 < t_1 < 1. \tag{4.11}$$

Similarly, parameter $\alpha_2 = 0$ *if* $\lambda_i > 0$ *for* $i = 1, 2, 3$, *and* $\mu_1 = \psi_1(\boldsymbol{\lambda})$, *where*

$$\psi_1(\boldsymbol{\lambda}) := \lambda_1 \frac{\tau(\lambda_3, \lambda_2)^2(3 - 2\tau(\lambda_3, \lambda_2))}{(1 - \tau(\lambda_3, \lambda_2))^2(1 + 2\tau(\lambda_3, \lambda_2))} > 0.$$

Moreover, parameter $\alpha_1 = 0$ *if* $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$ *and*

$$\mu_2 = \psi_3(\lambda_1, \lambda_4, \mu_1) := -\lambda_4 \frac{(3 - \tau(-\lambda_1, \mu_1))\, \tau(-\lambda_1, \mu_1)^2}{(1 - \tau(-\lambda_1, \mu_1))^3} > 0.$$

Proof. First let us prove that for $\lambda_2 > 0$ and $\lambda_3 > 0$, the equation (4.11) has a unique solution. Since

$$\lim_{t_1 \downarrow 0} g(t_1; \lambda_2, \lambda_3) = -\operatorname{sign}\left(\frac{1}{\lambda_3}\right) \cdot \infty = -\infty, \quad \lim_{t_1 \uparrow 1} g(t_1; \lambda_2, \lambda_3) = \operatorname{sign}\left(\frac{1}{\lambda_2}\right) \cdot \infty = \infty,$$
$$\lim_{t_1 \to -\infty} g(t_1; \lambda_2, \lambda_3) = \operatorname{sign}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) \cdot \infty, \quad \lim_{t_1 \to \infty} g(t_1; \lambda_2, \lambda_3) = -\operatorname{sign}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3}\right) \cdot \infty, \tag{4.12}$$

there exists at least one $t_1 \in (0, 1)$ that solves (4.11). It is straightforward to compute that when $\lambda_2 = \lambda_3$ the solution is unique, i.e., $\tau(\lambda_2, \lambda_3) := 2/3$. Now, for $\lambda_2 \neq \lambda_3$ the only possible solutions are

$$\tilde{t}_1 = \frac{\lambda_2(\lambda_3 - \lambda_2) - \sigma^{\frac{2}{3}}}{(\lambda_3 - \lambda_2)\sigma^{\frac{1}{3}}}, \tag{4.13}$$

$$\tilde{t}_1^{\pm} = \frac{(1 \pm i\sqrt{3})\lambda_2(\lambda_3 - \lambda_2) + (-1 \pm i\sqrt{3})\sigma^{\frac{2}{3}}}{2(\lambda_2 - \lambda_3)\sigma^{\frac{1}{3}}}, \tag{4.14}$$

where

$$\sigma := \lambda_2(\lambda_3 - \lambda_2)\left(\lambda_2 - \lambda_3 + \sqrt{\lambda_3(\lambda_3 - \lambda_2)}\right).$$

For $\lambda_3 > \lambda_2$ it is clear that σ is a real number, so solutions (4.14) are complex, thus (4.13) is a unique admissible solution and $\tau(\lambda_2, \lambda_3) := \tilde{t}_1$. For $\lambda_3 < \lambda_2$ the function g has three real zeros, but it follows from (4.12) that only one of them is in $(0, 1)$, namely $\tau(\lambda_2, \lambda_3) := \tilde{t}_1^{+}$.

Let us now analyse the case $\alpha_0 = 0$. From the first and the third equation in (4.7) one obtains

$$\lambda_2\alpha_1 = \frac{t_1^2}{1 - t_1}, \quad \mu_2 = \frac{1}{\alpha_2}\frac{3 - 2t_1}{(1 - t_1)}, \tag{4.15}$$

and from the remaining equations

$$\lambda_3\alpha_1 = \frac{(1 - t_1)(2 + t_1)}{t_1}, \quad \lambda_4\alpha_2 = \frac{(1 - t_1)(1 + 2t_1)}{t_1^2}. \tag{4.16}$$

Therefore it follows that the system (4.15)–(4.16) has an admissible solution $\alpha_1 > 0$, $\alpha_2 > 0$ and $t_1 \in (0, 1)$ iff $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$ and $\mu_2 = \psi_2(\boldsymbol{\lambda})$. The proof for the case $\alpha_2 = 0$ is symmetric to this one and it will be omitted.

Suppose now that $\alpha_1 = 0$. It is clear from (4.7) that neither λ_1 nor λ_4 can be equal to zero. From the first and the last equation in (4.7) one obtains

$$\alpha_0 = -\frac{t_1^2}{(t_1-1)^2\,\lambda_1}, \quad \alpha_2 = -\frac{(t_1-1)^2}{t_1^2\lambda_4}.$$

The remaining two equations then simplify to

$$-\frac{t_1+2}{t_1^3} - \frac{\mu_1}{(t_1-1)^2\,\lambda_1} = 0, \quad \frac{3-t_1}{(t_1-1)^3} - \frac{\mu_2}{t_1^2\lambda_4} = 0. \tag{4.17}$$

Since the solution must be in $\mathcal{U}$, it is clear that $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$ and $\mu_2 > 0$. Multiplying the first equation in (4.17) by $\frac{1}{\mu_1}t_1^2(1-t_1)$ the equation rewrites to $g(t_1; -\lambda_1, \mu_1) = 0$. Since $-\lambda_1 > 0$ and $\mu_1 > 0$ there exists a unique $\tau(-\lambda_1, \mu_1) \in (0,1)$ that solves it. From (4.17) it then follows that

$$\mu_1 > 0, \quad \mu_2 = \psi_3(\lambda_1, \lambda_4, \mu_1), \quad \text{or equivalently} \quad \mu_2 > 0, \quad \mu_1 = \psi_4(\lambda_1, \lambda_4, \mu_2),$$

where $\psi_4(\lambda_1, \lambda_4, \mu) := \psi_3(\lambda_4, \lambda_1, \mu)$. This completes the proof. □

The following properties of functions φ_i and ψ_i will be needed.

LEMMA 4.8. *If $\boldsymbol{\lambda} > 0$ then*

$$-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < -\frac{\lambda_2\lambda_4}{\lambda_3}, \tag{4.18}$$

$$-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < -\frac{\lambda_1\lambda_3}{\lambda_2}. \tag{4.19}$$

Also, if $\boldsymbol{\lambda} < 0$, then

$$\mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < -\frac{\lambda_2\lambda_4}{\lambda_3}, \tag{4.20}$$

$$\mu_2 > -\frac{\lambda_2\lambda_4}{\lambda_3} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < -\frac{\lambda_1\lambda_3}{\lambda_2}. \tag{4.21}$$

Furthermore, for $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$ and $\lambda_4 < 0$ the following implication holds:

$$\mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2} \quad \Longrightarrow \quad -\frac{\lambda_2\lambda_4}{\lambda_3} < \varphi_2(\boldsymbol{\lambda}, \mu_1) < \frac{\lambda_2\lambda_4}{\lambda_3} + \frac{2\lambda_1\lambda_4}{\mu_1} < \frac{\lambda_2\lambda_4}{\lambda_3}. \tag{4.22}$$

LEMMA 4.9. *For $\boldsymbol{\lambda} > 0$ functions ψ_i are bounded from below as*

$$\frac{\lambda_1\lambda_3}{\lambda_2} < \psi_1(\boldsymbol{\lambda}) \quad \text{and} \quad \frac{\lambda_2\lambda_4}{\lambda_3} < \psi_2(\boldsymbol{\lambda}). \tag{4.23}$$

Moreover, for $\boldsymbol{\lambda} < 0$

$$\varphi_2(\boldsymbol{\lambda}) < \psi_3(\lambda_1, \lambda_4, \mu_1) \quad \text{and} \quad \varphi_1(\boldsymbol{\lambda}) < \psi_4(\lambda_1, \lambda_4, \mu_2), \tag{4.24}$$

$$\frac{2\lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} < \psi_3(\lambda_1, \lambda_4, \mu_1), \tag{4.25}$$

$$\text{and} \quad \mu_1 > 0, \quad \mu_2 > \frac{2\lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} \quad \Longrightarrow \quad \varphi_1(\boldsymbol{\lambda}) < \mu_1. \tag{4.26}$$

The proofs of Lemma 4.8 and Lemma 4.9 are elementary, but very technical. They can easily be done by using a Computer Algebra system's symbolic facilities, thus they will be omitted.

4.4. Main theorems

In this subsection sufficient conditions that imply the existence of a cubic geometric interpolant $\boldsymbol{P}$ that satisfies (4.2) will be given. There are sixteen possibilities for the signs of λ_i as shown in Table 4.2. Lemma 4.4 shows that for the last seven options a solution of (4.7) does not exist. Other possibilities are considered in the following theorems.

sign(λ_1)	+	+	−	+	−	−	−	−	−	+	+	+	+	−	+	−
sign(λ_2)	+	+	+	+	−	+	+	−	−	+	−	−	−	+	−	−
sign(λ_3)	+	+	+	−	+	+	−	+	−	−	+	−	+	−	−	−
sign(λ_4)	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+

Table 4.2: Sixteen possibilities for the signs of λ_i.

THEOREM 4.10. *Suppose that the data* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0, 1, 2$, *satisfy* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$ *and* $\boldsymbol{\lambda} > 0$. *If one of the listed cases*

1. $\mu_1 > \psi_1(\boldsymbol{\lambda})$ *and* $\mu_2 > \psi_2(\boldsymbol{\lambda})$,
2. $\frac{\lambda_1\lambda_3}{\lambda_2} \leq \mu_1 < \psi_1(\boldsymbol{\lambda})$ *and* $\mu_2 < \psi_2(\boldsymbol{\lambda})$,
3. $\frac{\lambda_2\lambda_4}{\lambda_3} \leq \mu_2 < \psi_2(\boldsymbol{\lambda})$ *and* $\mu_1 < \psi_1(\boldsymbol{\lambda})$,
4. $-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2}$ *and* $\varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \psi_2(\boldsymbol{\lambda})$,
5. $-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3}$ *and* $\varphi_1(\boldsymbol{\lambda}, \mu_2) < \mu_1 < \psi_1(\boldsymbol{\lambda})$,

holds, then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

THEOREM 4.11. *Suppose that the data* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0, 1, 2$, *satisfy* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$, $\lambda_i > 0$, $i = 1, 2, 3$, *and* $\lambda_4 < 0$. *If*

$$\mu_1 > \psi_1(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_2 > \varphi_2(\boldsymbol{\lambda}, \mu_1), \quad \text{or}$$
$$\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \psi_1(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_2 < \varphi_2(\boldsymbol{\lambda}, \mu_1),$$

then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

REMARK 4.12. *The case* $\lambda_1 < 0$ *and* $\lambda_i > 0$, $i = 2, 3, 4$, *is symmetric to the one considered in Theorem 4.11. The result is the following. If*

$$\mu_2 > \psi_2(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_1 > \varphi_1(\boldsymbol{\lambda}, \mu_2), \quad \text{or}$$
$$\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \psi_2(\boldsymbol{\lambda}) \quad \text{and} \quad \mu_1 < \varphi_1(\boldsymbol{\lambda}, \mu_2),$$

then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

THEOREM 4.13. *Suppose that the data* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0, 1, 2$, *satisfy* $\det(\Delta \boldsymbol{T}_0, \Delta \boldsymbol{T}_1) \neq 0$ *and* $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\lambda_4 < 0$. *If*

$$-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 \quad \textit{and} \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3} + \frac{2\lambda_1\lambda_4}{\mu_1},$$

then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

REMARK 4.14. *The case where* $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$, $\lambda_4 > 0$ *is symmetric to the one considered in Theorem 4.13, and the result is the following. If*

$$-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 \quad \textit{and} \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2} + \frac{2\lambda_1\lambda_4}{\mu_2},$$

then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

THEOREM 4.15. *Suppose that the data* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0, 1, 2$, *satisfy* $\det(\Delta \boldsymbol{T}_0, \Delta \boldsymbol{T}_1) \neq 0$ *and* $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$. *If one of the following cases holds,*

1. $\lambda_2 > 0, \quad \lambda_3 > 0, \quad \mu_2 > \psi_3(\lambda_1, \lambda_4, \mu_1)$,
2. $\lambda_2 < 0, \quad \lambda_3 > 0, \quad 0 < \mu_1 \leq \frac{\lambda_1\lambda_3}{\lambda_2}, \quad \mu_2 > \psi_3(\lambda_1, \lambda_4, \mu_1)$,
3. $\lambda_2 > 0, \quad \lambda_3 < 0, \quad 0 < \mu_2 \leq \frac{\lambda_2\lambda_4}{\lambda_3}, \quad \mu_1 > \psi_4(\lambda_1, \lambda_4, \mu_2)$,
4. $\lambda_2 < 0, \quad \lambda_3 < 0, \quad \mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2}, \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \psi_3(\lambda_1, \lambda_4, \mu_1)$,
5. $\lambda_2 < 0, \quad \lambda_3 < 0, \quad 0 < \mu_1 \leq -\frac{\lambda_1\lambda_3}{\lambda_2}, \quad \frac{2\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} < \mu_2 < \psi_3(\lambda_1, \lambda_4, \mu_1)$,

then the cubic interpolating curve $\boldsymbol{P}$ *that satisfies* (4.2) *exists.*

REMARK 4.16. *The constants* $\boldsymbol{\lambda}$, μ_1 *and* μ_2 *change if lengths of* $\boldsymbol{d}_i$ *change, but all the relations in Theorems 4.10–4.15 remain the same.*

These theorems provide us with sufficient conditions that imply the existence of the interpolating polynomial. If none of these conditions is fulfilled the number of solutions is even, in most cases zero. Let us take for example Theorem 4.10 and choose data so that $\lambda_i = 1$, $i = 1, 2, 3, 4$, and $(\mu_1, \mu_2) = (3, 1)$, $(2, 3)$, $(0, \pm 3)$, $(\pm 3, 0)$, $(-3, \pm 3)$. In all of these examples the data do not satisfy any of the conditions of Theorem 4.10, and since the solutions can be computed analytically, one can easily check that there is no solution in $\mathcal{U}$.

Proofs of these theorems will be made in two steps and will be given as the next subsection. First the existence of the solution will be proved for the particular data. In the second step conclusions will be carried from the particular case to a general one by a homotopy and Brouwer's degree argument.

4.5. Proofs of main theorems

Step 1: Particular cases

Let the points be chosen as

$$\boldsymbol{T}_0 = (-1,-1)^T, \quad \boldsymbol{T}_1 = (0,0)^T, \quad \boldsymbol{T}_2 = (1,-1)^T, \tag{4.27}$$

and the tangent directions given in Table 4.3. Tables 4.4 and 4.5 show the constants for this data, and Table 4.6 gives the admissible solutions $(t_1, \boldsymbol{\alpha}) \in \mathcal{U}$. Note that there is a unique admissible solution in all the cases (Figure 4.4).

	$\boldsymbol{d}_0$	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$
data 1	$(3,5)^T$	$(2,0)^T$	$(3,-5)^T$
data 2	$(1,3)^T$	$(2,0)^T$	$(1,-3)^T$
data 3	$(-0.5,1.5)^T$	$(2,0)^T$	$(-0.5,-1.5)^T$
data 4	$(3,5)^T$	$(2,0)^T$	$(5,-3)^T$
data 5	$(1,3)^T$	$(2,0)^T$	$(-7,9)^T$
data 6	$(2,4)^T$	$(0,-2)^T$	$(1,1)^T$
data 7	$(7,5)^T$	$(2,0)^T$	$(7,-5)^T$
data 8	$(1.8,-0.2)^T$	$(0,2)^T$	$(41,-39)^T$
data 9	$(3,1)^T$	$(-2,0)^T$	$(3,-1)^T$
data 10	$(1.5,-0.5)^T$	$(-2,0)^T$	$(11,-9)^T$

Table 4.3: The tangent directions for different choices of data.

data	λ_1	λ_2	λ_3	λ_4	μ_1	μ_2
1	1	1	1	1	4	4
2	1	1	1	1	2	2
3	1	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$
4	1	1	1	-1	4	4
5	1	1	1	-1	2	-8
6	1	1	-1	-1	3	0
7	-1	1	1	-1	6	6
8	-1	-1	1	-1	0.8	40
9	-1	-1	-1	-1	2	2
10	-1	-1	-1	-1	0.5	10

Table 4.4: The constants for the particular data.

Step 2: Homotopy

In order to prove Theorems 4.10, 4.11, 4.13 and 4.15 one must show that the system (4.7) has a solution $(t_1, \boldsymbol{\alpha}) \in \mathcal{U}$. The conclusions for the particular data outlined in

data	$\psi_1(\boldsymbol{\lambda})$	$\psi_2(\boldsymbol{\lambda})$	$\varphi_1(\boldsymbol{\lambda},\mu_2)$	$\varphi_2(\boldsymbol{\lambda},\mu_1)$	$\psi_3(\boldsymbol{\lambda},\mu_1)$
1	2.8571	2.8571	/	/	/
2	2.8571	2.8571	/	/	/
3	2.8571	2.8571	-4	-4	/
4	2.8571	/	/	$-\frac{2}{3}$	/
5	2.8571	/	/	-2	/
6	/	/	2	$-\frac{1}{2}$	/
7	/	/	/	/	4.1724
8	/	/	/	/	36.2152
9	/	/	$\frac{2}{3}$	$\frac{2}{3}$	12.96
10	/	/	$\frac{2}{11}$	/	63.1769

Table 4.5: The constants for the particular data.

data	t_1	α_0	α_1	α_2
1	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{9}{10}$	$\frac{4}{5}$
2	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{7}{6}$	$\frac{4}{3}$
3	$\frac{1}{2}$	$\frac{8}{3}$	$\frac{11}{6}$	$\frac{8}{3}$
4	0.3624	1.2392	0.9961	0.3470
5	0.1902	3.8203	3.1384	1.6290
6	0.4462	3.0658	2.0574	6.1521
7	0.5	0.8	0.1	0.8
8	0.6860	4.8270	0.0169	0.1849
9	0.5	4	1.5	4
10	0.7620	13.2434	0.7134	1.0338

Table 4.6: The admissible solutions for the particular data.

Table 4.5 will be carried to the general case by the use of the homotopy and Brouwer's degree argument. Let the general data be denoted by $(\boldsymbol{\lambda}, \mu_1, \mu_2)$ and the particular one by $(\boldsymbol{\lambda}^*, \mu_1^*, \mu_2^*)$. A homotopy is defined as

$$\boldsymbol{H}(t_1, \boldsymbol{\alpha}; \zeta) := \boldsymbol{F}\left(t_1, \boldsymbol{\alpha}; \boldsymbol{\lambda}(\zeta), \mu_1(\zeta), \mu_2(\zeta)\right),$$

where

$$\boldsymbol{\lambda}(\zeta) := (1-\zeta)\boldsymbol{\lambda}^* + \zeta\boldsymbol{\lambda}, \quad \mu_i(\zeta) := q_i(\zeta; \mu_i^*, \mu_i), \quad i = 1, 2,$$

and $q_i := q_i(\cdot; \mu_i^*, \mu_i) : [0,1] \to \mathbb{R}$ will be chosen later on as continuous piecewise linear functions that satisfy $q_i(0; \mu_i^*, \mu_i) = \mu_i^*$, $q_i(1; \mu_i^*, \mu_i) = \mu_i$. The idea of the proof is to connect particular and general data so that a set of solutions

$$V := \{(t_1(\zeta), \boldsymbol{\alpha}(\zeta)) \in \mathcal{U}; \quad \boldsymbol{H}(t_1(\zeta), \boldsymbol{\alpha}(\zeta); \zeta) = \boldsymbol{0}, \quad \zeta \in [0,1]\}$$

stays away from the boundary $\partial\mathcal{U}$. If this can be done, one can find a compact set $K \subset \mathcal{U}$, such that

$$V \subset K \subset \mathcal{U}, \quad V \cap \partial K = \emptyset. \tag{4.28}$$

Therefore the map $\boldsymbol{H}$ does not vanish at the boundary ∂K, and Brouwer's degree of $\boldsymbol{H}$ on K is invariant for all $\zeta \in [0,1]$. But since it is odd for the particular map $\boldsymbol{F}(\cdot,\cdot;\boldsymbol{\lambda}^*,\mu_1^*,\mu_2^*)$, equations $\boldsymbol{F}(t_1,\boldsymbol{\alpha};\boldsymbol{\lambda},\mu_1,\mu_2)=\mathbf{0}$ must have at least one admissible solution.

One is now left to show how to choose q_1 and q_2, and to prove that (4.28) holds. Namely, it must be shown that solutions of $\boldsymbol{H}(t_1(\zeta),\boldsymbol{\alpha}(\zeta);\zeta)=\mathbf{0}$ satisfy

$$t_1(\zeta) \geq \text{const} > 0, \quad 1-t_1(\zeta) \geq \text{const} > 0, \quad \alpha_i(\zeta) \geq \text{const} > 0, \quad i=0,1,2, \tag{4.29}$$

for all $\zeta \in [0,1]$. The inequality

$$|\lambda_i(\zeta)| \geq \min_{\zeta\in[0,1]} \{|(1-\zeta)\lambda_i^* + \zeta\lambda_i|\} \geq \min\{|\lambda_i^*|,|\lambda_i|\} \geq \text{const} > 0$$

will be fulfilled, and the results of Lemma 4.5, Lemma 4.6 and Lemma 4.7 will be the main tool in the proof of all four theorems. Each of them will be analysed separately.

Theorem 4.10: There are five cases to be considered. Choosing data 1, 2 or 3 as particular data yields $\boldsymbol{\lambda}(\zeta) > 0$, and thus $\alpha_1(\zeta)$ cannot approach zero, i.e., $\alpha_1(\zeta) \geq \text{const} > 0$ for all $\zeta \in [0,1]$. In the first case where $\mu_i > \psi_i(\boldsymbol{\lambda})$, $i=1,2$, choose data 1 as the particular data. Since $\mu_i^* > \psi_i(\boldsymbol{\lambda}^*)$ and ψ_i does not depend on μ_i, there obviously exists q_i, such that

$$\mu_i(\zeta) > \psi_i(\boldsymbol{\lambda}(\zeta)), \quad \zeta \in [0,1], \quad i=1,2.$$

Therefore $\alpha_0(\zeta)$ and $\alpha_2(\zeta)$ cannot approach zero for any $\zeta \in [0,1]$, and by (4.23) the parameter $t_1(\zeta)$ cannot approach zero or one either.

Suppose now that $\dfrac{\lambda_1\lambda_3}{\lambda_2} \leq \mu_1 < \psi_1(\boldsymbol{\lambda})$ and $\mu_2 < \psi_2(\boldsymbol{\lambda})$ (case 2) and choose data 2. There clearly exists q_2 that satisfies $\mu_2(\zeta) < \psi_2(\boldsymbol{\lambda}(\zeta))$ for every $\zeta \in [0,1]$. Moreover, from (4.23) it follows that $\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \psi_1(\boldsymbol{\lambda}(\zeta))$, so one can find q_1 such that

$$\frac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \mu_1(\zeta) < \psi_1(\boldsymbol{\lambda}(\zeta)), \quad \zeta \in [0,1].$$

Now, it is clear that $\alpha_0(\zeta),\alpha_2(\zeta),t_1(\zeta) \geq \text{const} > 0$ for all $\zeta \in [0,1]$. Moreover, by (4.19) the parameter $t_1(\zeta)$ cannot approach 1 either which completes the proof. The case 3 is symmetric to the second one and will be omitted.

Consider now the case 4 and choose data 3 that satisfy

$$-\frac{\lambda_1^*\lambda_3^*}{\lambda_2^*} < \mu_1^* < \frac{\lambda_1^*\lambda_3^*}{\lambda_2^*} \quad \text{and} \quad \varphi_2(\boldsymbol{\lambda}^*,\mu_1^*) < \mu_2^* < \psi_2(\boldsymbol{\lambda}^*).$$

There obviously exists q_1 such that

$$-\frac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \mu_1(\zeta) < \frac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)}, \quad \zeta \in [0,1].$$

Since by (4.18) and (4.23)

$$\varphi_2(\boldsymbol{\lambda}(\zeta),\mu_1(\zeta)) < -\frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)} < \frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)} < \psi_2(\boldsymbol{\lambda}(\zeta)), \quad \zeta \in [0,1],$$

there exists q_2 that satisfies $\varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta)) < \mu_2(\zeta) < \psi_2(\boldsymbol{\lambda}(\zeta))$. Thus $\alpha_0(\zeta)$, $t_1(\zeta) \geq$ const > 0. Further, by (4.19) and (4.23), $\alpha_2(\zeta)$, $1 - t_1(\zeta) \geq$ const > 0, and (4.29) holds. The case 5 is symmetric to the case 4 and will be omitted. This completes the proof of Theorem 4.10.

Theorem 4.11: Particular data must be chosen so that $\lambda_i(\zeta) > 0$, $i = 1, 2, 3$, and $\lambda_4(\zeta) < 0$. It is then clear that there exists a constant 'const' such that $\alpha_i(\zeta) \geq$ const > 0, $i = 0, 1$, and $1 - t_1(\zeta) \geq$ const > 0, no matter how we define q_1 and q_2. By (4.23), $\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \psi_1(\boldsymbol{\lambda}(\zeta))$. Now, for the first case choose data 4, define q_1 so that $\mu_1(\zeta) > \psi_1(\boldsymbol{\lambda}(\zeta))$, and then choose such q_2 that $\mu_2(\zeta) > \varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta))$, $\zeta \in [0, 1]$. In the second case choose data 5, define q_1 so that

$$\frac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \mu_1(\zeta) < \psi_1(\boldsymbol{\lambda}(\zeta)), \quad \zeta \in [0, 1],$$

and choose such q_2 that $\mu_2(\zeta) < \varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta))$. Now, (4.29) obviously holds and Theorem 4.11 is proved.

Theorem 4.13: Let us choose data 6 as the particular data. Since $\lambda_1(\zeta) > 0$, $\lambda_2(\zeta) > 0$, $\lambda_3(\zeta) < 0$ and $\lambda_4(\zeta) < 0$ it is clear from Lemma 4.7 that $\alpha_i(\zeta) \geq$ const > 0, $i = 0, 1, 2$, for $\zeta \in [0, 1]$. Let q_1 be chosen in such a way that $-\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)} < \mu_1(\zeta)$ for $\zeta \in [0, 1]$. By using (4.22) one can find q_2 that satisfies

$$-\frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)} < \varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta)) < \mu_2(\zeta) < \frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)} + \frac{2\lambda_1(\zeta)\lambda_4(\zeta)}{\mu_1(\zeta)} < \frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)}.$$

Therefore $t_1(\zeta)$ cannot approach zero for any $\zeta \in [0, 1]$. Now, it can easily be proved that $\mu_1(\zeta) > \varphi_1(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta))$, thus the parameter $t_1(\zeta)$ cannot approach 1 either and the proof is completed.

Theorem 4.15: Particular data must be chosen so that $\lambda_1(\zeta) < 0$ and $\lambda_4(\zeta) < 0$. Then it is clear that $\alpha_i(\zeta) \geq$ const > 0, $i = 0, 2$, for all $\zeta \in [0, 1]$. In the case 1 choose data 7. It is clear that $t_1(\zeta)$ cannot approach zero or one. The only problem could be if $\alpha_1(\zeta)$ would go to zero. But for $q_1(\zeta; \mu_1^*, \mu_1) := (1 - \zeta)\mu_1^* + \zeta\mu_1$, and q_2 chosen so that $\mu_2(\zeta) > \psi_3(\lambda_1(\zeta), \lambda_4(\zeta), \mu_1(\zeta))$, this cannot happen.

In the second case choose data 8 and define q_1 so that $0 < \mu_1(\zeta) \leq \dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)}$. Now, $1 - t_1(\zeta), t_1(\zeta) \geq$ const > 0 for all $\zeta \in [0, 1]$. Choosing q_2 as in the previous case completes the proof. The next case is symmetric to this one and will be omitted.

In the case 4 choose data 9 and q_1 so that $\mu_1(\zeta) > -\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)}$. Now, by (4.20) and (4.24),

$$\varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta)) < -\frac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)} \text{ and } \varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta)) < \psi_3(\lambda_1(\zeta)\lambda_4(\zeta), \mu_1(\zeta)).$$

So there exists q_2 such that $\varphi_2(\boldsymbol{\lambda}(\zeta), \mu_1(\zeta)) < \mu_2(\zeta) < \psi_3(\lambda_1(\zeta), \lambda_4(\zeta), \mu_1(\zeta))$, and thus $t_1(\zeta), \alpha_1(\zeta) \geq$ const > 0 for $\zeta \in [0, 1]$. The parameter $t_1(\zeta)$ can approach 1 only for

$\mu_2(\zeta) > -\dfrac{\lambda_2(\zeta)\lambda_4(\zeta)}{\lambda_3(\zeta)}$, but in this case, by (4.21), $\varphi_1(\boldsymbol{\lambda}(\zeta), \mu_2(\zeta)) < -\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)}$, so $1 - t_1(\zeta) \geq \text{const} > 0$ for $\zeta \in [0,1]$.

For the last case choose data 10 and q_1 so that $0 < \mu_1(\zeta) \leq -\dfrac{\lambda_1(\zeta)\lambda_3(\zeta)}{\lambda_2(\zeta)}$. Moreover, by (4.25), q_2 can be chosen so that

$$\frac{2\lambda_1(\zeta)\lambda_3(\zeta)\lambda_4(\zeta) + \lambda_2(\zeta)\lambda_4(\zeta)\mu_1(\zeta)}{\lambda_3(\zeta)\mu_1(\zeta)} < \mu_2(\zeta) < \psi_3(\lambda_1(\zeta), \lambda_4(\zeta), \mu_1(\zeta)).$$

It is clear that $t_1(\zeta) \geq \text{const} > 0$ and $\alpha_1(\zeta) \geq \text{const} > 0$ for $\zeta \in [0,1]$. But, by (4.26), $\mu_1(\zeta) > \varphi_1(\boldsymbol{\lambda}(\zeta), \mu_2(\zeta))$, so $1 - t_1(\zeta) \geq \text{const} > 0$ too and the proof is completed.

4.6. Approximation order

In this subsection Theorem 4.2 will be proved. Recall the notation declared there. One needs to show that there exists $h_0 > 0$ small enough and a constant $C > 0$, so that for every $h = \max_{\ell} \Delta s_\ell$, $0 < h \leq h_0$, a G^1 spline exists as well as

$$\operatorname{dist}(\boldsymbol{f}, \boldsymbol{S}) = \operatorname{dist}(\boldsymbol{f}, \boldsymbol{S})_{[a,b]} = \inf_{\theta} \max_{t \in [a,b]} \|\boldsymbol{f}(t) - \boldsymbol{S}(\theta(t))\| \leq C\, h^6, \tag{4.30}$$

where $\theta : [a,b] \to [0,1]$ is a regular reparameterization. Since

$$\operatorname{dist}(\boldsymbol{f}, \boldsymbol{S}) \leq \max \left\{ \operatorname{dist}(\boldsymbol{f}, \boldsymbol{S})_{[s_{2\ell-2}, s_{2\ell}]};\ \ell = 1, 2, \ldots, m \right\},$$

it is enough to analyse the polynomial case $m = 1$ only. Again, for $m = 1$ let the spline $\boldsymbol{S}$ be denoted by $\boldsymbol{P}$. Without losing generality, one can assume $a = 0$, $\boldsymbol{f}(0) = (0,0)^T$, $\boldsymbol{f}'(0) = (1,0)^T$. Further, let h be redefined as $h := s_2 - s_0$. For h small enough, $\boldsymbol{f}$ can be parameterized by the first component,

$$\boldsymbol{f}(s) = \begin{pmatrix} s \\ y(s) \end{pmatrix}, \quad y(s) = \frac{1}{2} y''(0) s^2 + \frac{1}{3!} y^{(3)}(0) s^3 + \mathcal{O}\left(s^4\right), \quad y''(0) \neq 0.$$

Moreover, let $\eta_i := s_i / h$, $i = 0, 1, 2$. By Remark 4.3 the tangent directions can be redefined as $\boldsymbol{d}_i = h \boldsymbol{f}'(h\,\eta_i)$, $i = 0, 1, 2$. Since $\boldsymbol{f}$ is convex, $\det(\boldsymbol{T}_0, \boldsymbol{T}_1) \neq 0$ and constants (4.6) are well defined. It is straightforward to compute

$$\begin{aligned}
\lambda_1 &= \frac{\eta_1}{1 - \eta_1} + \mathcal{O}(h), \quad \lambda_2 = \frac{\eta_1}{1 - \eta_1} + \mathcal{O}(h), \quad \lambda_3 = \frac{1 - \eta_1}{\eta_1} + \mathcal{O}(h), \\
\lambda_4 &= \frac{1 - \eta_1}{\eta_1} + \mathcal{O}(h), \quad \mu_1 = 1 + \frac{1}{\eta_1} + \mathcal{O}(h), \quad \mu_2 = 1 + \frac{1}{1 - \eta_1} + \mathcal{O}(h),
\end{aligned}$$

and the nonlinear system (4.7) becomes

$$\boldsymbol{F}\left(t_1, \boldsymbol{\alpha}; \tilde{\boldsymbol{\lambda}}, 1 + \frac{1}{\eta_1}, 1 + \frac{1}{1 - \eta_1}\right) + \mathcal{O}(h) = 0, \tag{4.31}$$

where $\tilde{\boldsymbol{\lambda}} := \left(\frac{\eta_1}{1-\eta_1}, \frac{\eta_1}{1-\eta_1}, \frac{1-\eta_1}{\eta_1}, \frac{1-\eta_1}{\eta_1}\right)$. It is easy to check that the solution at the limit $h = 0$ is $t_1 = \eta_1$, $\alpha_i = 1$, $i = 0, 1, 2$. But unfortunately, the Jacobian at the limit solution is singular, and one can not make use of the Implicit Function Theorem. To show that the solution exists also for all h small enough Theorem 4.1 will be used. There obviously exists such h_0 that $\boldsymbol{\lambda} > 0$ for all $0 < h \le h_0$. With some elementary mathematics one can prove that the inequalities

$$1 + \frac{1}{\eta_1} > \psi_1(\tilde{\boldsymbol{\lambda}}), \quad 1 + \frac{1}{1-\eta_1} > \psi_2(\tilde{\boldsymbol{\lambda}})$$

hold. Furthermore,

$$\psi_i(\boldsymbol{\lambda}) = \psi_i(\tilde{\boldsymbol{\lambda}}) + \mathcal{O}(h), \quad i = 1, 2,$$

thus h_0 can be chosen so small that $\mu_i > \psi_i(\boldsymbol{\lambda})$, $i = 1, 2$, for all $0 < h \le h_0$. The existence of the solution is now guaranteed by Theorem 4.10 (case 1), which also shows that this is probably the most important existence result. The first part of Theorem 4.2 is thus proved.

From the previous analysis one can conclude only that the solution is of the form $t_1 = \eta_1 + \mathcal{O}(h)$ and $\alpha_i = 1 + \mathcal{O}(h)$, $i = 0, 1, 2$. But to prove that the approximation order is optimal, a more precise expansion is needed. It is enough to study only the unknown difference $t_1 - \eta_1$ as a function of h. So we may assume from now on that t_1 is given and η_1 is unknown. Let us introduce new unknowns $\boldsymbol{\xi} = (\xi_i)_{i=1}^4$ by

$$\begin{aligned}
\mu_1 &= t_1 + h\,\xi_1 t_1(t_1 - 1),\\
\alpha_i &= 1 + h\,\xi_1 \frac{d}{dt}(t(t-1))\Big|_{t=t_i} + h^2\,\xi_{i+2}\frac{d}{dt}(t(t-t_1)(t-1))\Big|_{t=t_i}, \quad i = 0, 1, 2.
\end{aligned}$$

Since $0 < t_1 < 1$, the change of variables $(\mu_1, \alpha_0, \alpha_1, \alpha_2) \to \boldsymbol{\xi}$ is one to one. Now, the system (4.3) can be rewritten in an equivalent form, with divided differences replaced by their linear combinations, i.e.,

$$[t_0, t_0, t_1, t_1, t_2]\boldsymbol{P} = 0, \quad [t_0, t_0, t_1, t_1, t_2, t_2]\boldsymbol{P} = 0. \tag{4.32}$$

The first and the third equation read

$$\frac{\xi_3 - \xi_2}{t_1} h^3 = 0, \quad \frac{(1-t_1)\xi_2 - \xi_3 + t_1\xi_4}{t_1(1-t_1)} h^3 = 0,$$

therefore a solution exists iff $\xi_2 = \xi_3 = \xi_4$. The remaining two equations then expand as

$$\begin{aligned}
&\frac{1}{24}\left(12\left(\left(\xi_1^2 + 2\xi_2\right)y''(0) + \xi_1 y^{(3)}(0)\right) + y^{(4)}(0)\right) + \mathcal{O}(h) = 0, \qquad (4.33)\\
&\frac{1}{120}\left(60y^{(3)}(0)\xi_1^2 + 20\left(6\xi_2 y''(0) + y^{(4)}(0)\right)\xi_1 + 60\xi_2 y^{(3)}(0) + y^{(5)}(0)\right) + \mathcal{O}(h) = 0.
\end{aligned}$$

Since the first equation in (4.33) is linear in ξ_2 and $y''(0) \neq 0$, the system can be reduced to a cubic equation

$$-\frac{y''(0)}{2}\xi_1^3 - \frac{y^{(3)}(0)}{4}\xi_1^2 + \left(\frac{y^{(4)}(0)}{8} - \frac{2y^{(3)}(0)^2}{8y''(0)}\right)\xi_1 + \left(\frac{y^{(5)}(0)}{120} - \frac{y^{(3)}(0)y^{(4)}(0)}{48y''(0)}\right) + \mathcal{O}(h) = 0,$$

which always has at least one real solution ξ_1. Therefore, by a backward substitution, (4.32) has a real solution $\boldsymbol{\xi}$ for all h small enough, and the solution satisfies $\xi_2 = \xi_3 = \xi_4$. So, if we define

$$\mu(t;\boldsymbol{\xi}) := t + h\,\xi_1 t(t-1) + h^2\,\xi_2 t(t-t_1)(t-1),$$

then

$$\mu_i = \mu(t_i;\boldsymbol{\xi}), \quad \alpha_i = \mu'(t_i;\boldsymbol{\xi}), \quad i=0,1,2, \quad \mu(0;\boldsymbol{\xi}) = 0, \quad \mu(1;\boldsymbol{\xi}) = 1.$$

Let us now estimate the error $\|\boldsymbol{P}\circ\theta - \boldsymbol{f}\|$ for some regular reparameterization θ that satisfies

$$(\boldsymbol{P}\circ\theta)(s_i) = \boldsymbol{f}(s_i), \quad (\boldsymbol{P}\circ\theta)'(s_i) = \boldsymbol{f}'(s_i), \quad i=0,1,2. \tag{4.34}$$

Now,

$$\|\boldsymbol{P}\circ\theta - \boldsymbol{f}\| \le \|\boldsymbol{P}\circ\theta - \boldsymbol{p}\| + \|\boldsymbol{p} - \boldsymbol{f}\|,$$

where $\boldsymbol{p}$ denotes a polynomial of degree ≤ 5 that satisfies $\boldsymbol{p}(s_i) = \boldsymbol{f}(s_i)$, $\boldsymbol{p}'(s_i) = \boldsymbol{f}'(s_i)$, $i = 0,1,2$. Since $\boldsymbol{f}$ is smooth, it follows from Newton's formula that $\|\boldsymbol{p}-\boldsymbol{f}\| = \mathcal{O}(h^6)$, but $\|\boldsymbol{P}\circ\theta - \boldsymbol{p}\| = \mathcal{O}(h^6)$ if the derivatives of $\boldsymbol{P}\circ\theta$ up to order six are bounded as $h\to 0$. The interpolant $\boldsymbol{P} = (P_1, P_2)^T$ can be given as

$$\begin{aligned}\boldsymbol{P}(t) &= \sum_{i=0}^{2}\mathcal{L}_{i,2}(t)\boldsymbol{T}_i + \omega_{0,2}(t)\left(\sum_{i=0}^{1}\frac{1}{\dot{\omega_{0,2}}(t_i)}\frac{\boldsymbol{T}_2-\boldsymbol{T}_i}{t_2-t_i} + \frac{\alpha_2\boldsymbol{d}_2}{\dot{\omega_{0,2}}(t_2)}\right)\\ &=: \mathcal{P}\left(t;(t_0,\boldsymbol{T}_0),(t_1,\boldsymbol{T}_1),(t_2,\boldsymbol{T}_2,\alpha_2\boldsymbol{d_2})\right),\end{aligned}$$

where $\mathcal{L}_{i,2}(t) = \prod_{j=0,j\ne i}^{2}\frac{t-t_j}{t_i-t_j}$ are the Lagrange basis polynomials, and $\omega_{0,2}$ is defined by (2.5). More precisely,

$$\begin{aligned}P_1(t) =& \mathcal{P}\left(t;(t_0,h\mu(t_0;\boldsymbol{\xi})),(t_1,h\mu(t_1;\boldsymbol{\xi})),(t_2,h\mu(t_2;\boldsymbol{\xi}),h\mu'(t_2;\boldsymbol{\xi}))\right) = h\mu(t;\boldsymbol{\xi}),\\ P_2(t) =& \mathcal{P}\left(t;(t_0,y(h\mu(t_0;\boldsymbol{\xi}))),(t_1,y(h\mu(t_1;\boldsymbol{\xi}))),(t_2,y(h\mu(t_2;\boldsymbol{\xi})),h\mu'(t_2;\boldsymbol{\xi})y'(h\mu(t_2;\boldsymbol{\xi})))\right)\\ =& \mathcal{P}\left(t;\left(t_0,\frac{1}{2}y''(0)h^2t_0^2+\mathcal{O}\left(h^3\right)\right),\left(t_1,\frac{1}{2}y''(0)h^2t_1^2+\mathcal{O}\left(h^3\right)\right),\right.\\ &\left.\left(t_2,\frac{1}{2}y''(0)h^2t_2^2+\mathcal{O}\left(h^3\right),y''(0)h^2t_2+\mathcal{O}\left(h^3\right)\right)\right) = \frac{1}{2}y''(0)h^2t^2 + h^3\sum_{i=0}^{3}c_i(h)t^i,\end{aligned} \tag{4.35}$$

where $c_i(h) = \mathcal{O}(1)$. If we define $\theta := P_1^{-1}$, then the conditions (4.34) are clearly fulfilled, and θ is a regular reparameterization since

$$P_1(0) = h\mu(0;\boldsymbol{\xi}) = 0, \quad P_1(1) = h\mu(1;\boldsymbol{\xi}) = h, \quad P_1(t) = ht + \mathcal{O}\left(h^2\right).$$

As already observed in [4], to show that the derivatives of $P_2\circ P_1^{-1}$ are bounded as $h\to 0$ it suffices to see that $P_1'(t) = c\,h + \mathcal{O}(h^2)$, $c \ne 0$, and

$$P_i^{(k)}(t) = \mathcal{O}\left(h^k\right), \quad i=1,2, \quad k=2,3,\ldots,6.$$

But, this follows directly from (4.35). Therefore the relation (4.30) holds and the proof is completed.

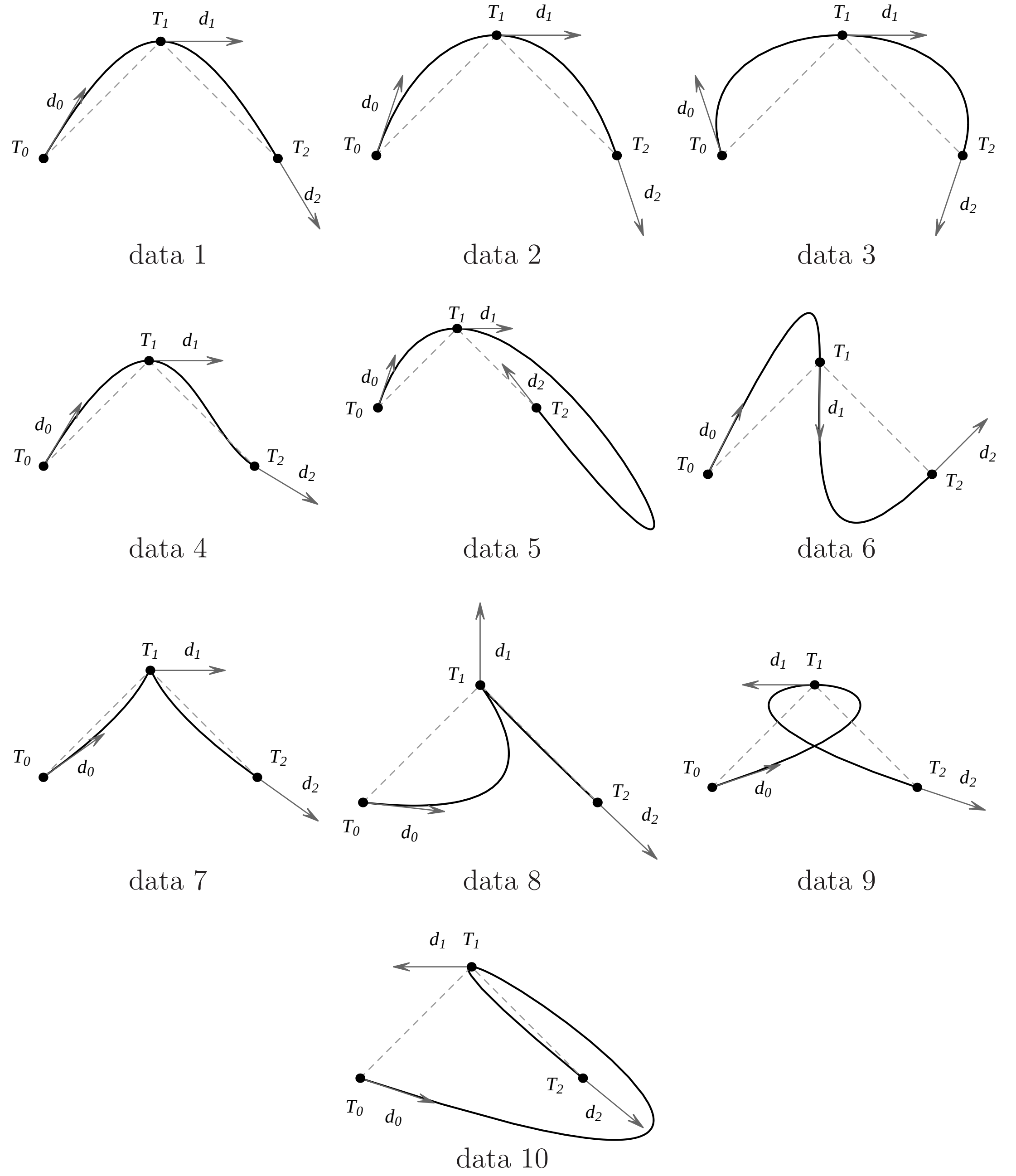


Figure 4.4: Cubic Hermite geometric interpolants for particular data points defined by (4.27) and tangent directions given in Table 4.3.

Chapter 5

Asymptotic analysis

From the cubic case considered in Chapter 2 one can conclude that in general the nonlinear system of equations (1.3) is impossible to analyse without some additional restrictions. In this chapter the asymptotic approach is applied which means that the data points are sampled from a smooth regular convex planar parametric curve. The conjecture that a parametric polynomial curve of degree $\leq n$ can interpolate $2n$ given points in $\mathbb{R}^2$ is confirmed for $n \leq 5$. This conclusion also implies the optimal asymptotic approximation order. More generally, the optimal order $2n$ can be achieved as soon as the interpolating curve exists.

5.1. Asymptotic approach

Recall the interpolation problem (1.3) introduced in Chapter 1. Let us assume that the points $\boldsymbol{T}_\ell$ are sampled from a smooth regular convex planar parametric curve

$$\boldsymbol{f} : [0, h] \to \mathbb{R}^2.$$

The length of the parameter interval h is supposed to be small enough so that a local expansion of $\boldsymbol{f}$ around 0 can be applied. Since affine transformations of the points $\boldsymbol{T}_\ell$ do not change the solution $\boldsymbol{t}$ of (1.3) one can assume $\boldsymbol{f}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{f}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and parameterize $\boldsymbol{f}$ by the first component

$$\boldsymbol{f}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix},$$

where y expands as

$$y(x) = \frac{1}{2} y''(0) x^2 + \frac{1}{3!} y^{(3)}(0) x^3 + \cdots + \frac{1}{(2\,n-1)!} y^{(2\,n-1)}(0) x^{2\,n-1} + \mathcal{O}(x^{2\,n}). \tag{5.1}$$

The curve is assumed to be convex, which implies $y''(0) > 0$. The values of $\boldsymbol{f}$ at small values of h will be considered, therefore the coordinate system needs an appropriate scaling which is done by the matrix

$$D_h := \operatorname{diag}\left(\frac{1}{h}, \frac{2}{y''(0)h^2}\right).$$

Now let $\boldsymbol{T}_\ell$ be the points on the curve $\boldsymbol{f}$, taken at different parameter values in $[0, h]$. Then for some η_ℓ,

$$\eta_0 := 0 < \eta_1 < \cdots < \eta_{2n-2} < \eta_{2n-1} := 1, \tag{5.2}$$

the data points are chosen as $\boldsymbol{T}_\ell = D_h \boldsymbol{f}(\eta_\ell h)$. Their expansion in h is

$$\boldsymbol{T}_\ell = \begin{pmatrix} \eta_\ell \\ \sum_{k=2}^{\infty} c_k h^{k-2} \eta_\ell^k \end{pmatrix}, \quad \ell = 0, 1, \ldots, 2n-1. \tag{5.3}$$

Here, the constants c_k depend on y, but not on η_ℓ or h, i.e.,

$$c_k = \frac{2}{k!} \frac{y^{(k)}(0)}{y''(0)}, \quad k = 2, 3, \ldots$$

5.2. Nonlinear system

The system of equations (1.3) should determine the unknown $\boldsymbol{P}_n$ as well as the parameters $\boldsymbol{t} \in \mathcal{D}_n$. But the two tasks can be separated if one can provide enough linearly independent functionals, depending on $\boldsymbol{t}$ only, that map $\boldsymbol{P}_n$ to zero. Divided differences, based upon $\geq\ n + 2$ values, are a natural choice. Let us apply the divided differences

$$[t_{j-1}, t_j, \ldots, t_{n+j}], \quad j = 1, 2, \ldots, n-1, \tag{5.4}$$

to both sides of (1.3). Since $\deg P_n \leq n$, the left side vanishes, and so should the right one. But the t_ℓ are distinct and this condition becomes

$$\sum_{\ell=j-1}^{n+j} \frac{\boldsymbol{T}_\ell}{\prod_{\substack{m=j-1 \\ m \neq \ell}}^{n+j} (t_\ell - t_m)} = 0, \quad j = 1, 2, \ldots, n-1. \tag{5.5}$$

This nonlinear system depends on the data $\boldsymbol{T}_\ell$ and the unknowns $\boldsymbol{t}$ only. For each j it provides two equations based upon the first and the second component of the data. The solution of the system (5.5) may or may not exist. The difficult part of the interpolation problem is to find it. If the unknowns $\boldsymbol{t}$ have been already determined it is straightforward to obtain the polynomial curve $\boldsymbol{P}_n$. One only has to take any $n+1$ distinct interpolating conditions in (1.3), and apply any standard interpolation scheme like Newton or Lagrange to $\boldsymbol{P}_n$ componentwise.

5.3. System of equations in asymptotic form

Let us now analyse the system (5.5) with data points given by (5.3) and h small enough. As far as the existence of the solution is concerned, one has to show that there exists $h_0 > 0$, such that the system (5.5) has a solution $\boldsymbol{t}$ for all h, $0 \le h \le h_0$. The solution is easy to guess at the limit value $h = 0$, i.e.,

$$\boldsymbol{t} = \boldsymbol{\eta} := (\eta_\ell)_{\ell=1}^{2n-2}, \tag{5.6}$$

since

$$\lim_{h\to 0} \boldsymbol{T}_\ell = \begin{pmatrix} \eta_\ell \\ \eta_\ell^2 \end{pmatrix}$$

and

$$[\eta_{j-1}, \eta_j, \ldots, \eta_{n+j}]_\eta \begin{pmatrix} \eta \\ \eta^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1, 2, \ldots, n-1.$$

In view of (5.6) it is important to study the unknown differences

$$\eta_\ell - t_\ell, \quad \ell = 1, 2, \ldots, 2n-2, \tag{5.7}$$

as functions of h. It does not matter if (5.7) is studied with η_ℓ given and t_ℓ unknown or vice versa. From now on it will be simpler to assume that $\boldsymbol{t}$ are given parameters and $\boldsymbol{\eta}$ are the unknowns, as in [37]. Furthermore, the system of equations (5.5) will be rewritten in an equivalent form, with divided differences (5.4) replaced by their linear combinations, i.e.,

$$[t_0, t_1, \ldots, t_{n+j}], \quad j = 1, 2, \ldots, n-1. \tag{5.8}$$

With the notation (2.5) the system (5.5) is transformed into

$$\sum_{\ell=0}^{n+j} \frac{1}{\dot\omega_{0,n+j}(t_\ell)} \boldsymbol{T}_\ell = 0, \quad j = 1, 2, \ldots, n-1. \tag{5.9}$$

Of course, the limit properties of the system are preserved, since the linear transformation from (5.4) to (5.8) is invertible.

Unfortunately, the implicit function theorem can not be applied to extend the limit solution $\boldsymbol{\eta} = \boldsymbol{t}$ continuously to $h > 0$ for $n > 2$. This is obvious from the following theorem.

THEOREM 5.1. *Let J be the Jacobian of the system* (5.9) *with respect to the unknowns $\boldsymbol{\eta}$ at $h = 0$. Then*

$$\dim\ker J = n - 2.$$

Proof. The Jacobian J is easily computed from

$$\left(\frac{\partial}{\partial \eta_m} \boldsymbol{T}_\ell\right)\bigg|_{\boldsymbol{\eta}=\boldsymbol{t}, h=0} = \begin{pmatrix} \delta_{\ell,m} \\ 2\delta_{\ell,m}\, t_\ell \end{pmatrix}, \quad \ell,\, m = 1, 2, \ldots, 2n-2, \tag{5.10}$$

and from the system (5.9). Let

$$\boldsymbol{x}_i^T := (\underbrace{0, \ldots, 0}_{2i-2}, -2, 0, -2t_{n+1+i}, 1, 0, 0, \ldots, 0)^T, \quad i = 1, 2, \ldots, n-2.$$

Observe that

$$\left(\boldsymbol{x}_i^T J\right)_m = \begin{cases} -\dfrac{2}{\dot\omega_{0,n+i}(t_m)} - \dfrac{2t_{n+1+i}}{\dot\omega_{0,n+i+1}(t_m)} + \dfrac{2t_m}{\dot\omega_{0,n+i+1}(t_m)} & = 0, \quad m = 1,2,\ldots,n+i, \\ -\dfrac{2t_{n+1+i}}{\dot\omega_{0,n+i+1}(t_m)} + \dfrac{2t_{n+1+i}}{\dot\omega_{0,n+i+1}(t_m)} & = 0, \quad m = n+1+i, \\ 0, & \quad \text{otherwise.} \end{cases}$$

But the $\boldsymbol{x}_i^T$ are linearly independent, hence $\dim\ker J \geq n-2$. Let

$$M := J \begin{pmatrix} 2 & 1 & 3 & 5 & \cdots & 2n-3 \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix} \cdot \operatorname{diag}(\dot\omega_{0,2n-1}(t_1), \dot\omega_{0,2n-1}(t_2), \ldots, \dot\omega_{0,2n-1}(t_n)).$$

It is easy to see that $M = (q_j(t_m))_{j=1,m=1}^{n,n}$, where the polynomials q_j are given as

$$q_1(t) := 2t \prod_{\ell=n+2}^{2n-1} (t - t_\ell), \quad q_j(t) = \prod_{\ell=n+j}^{2n-1} (t - t_\ell), \quad j = 2,3,\ldots,n.$$

This implies that M must be nonsingular. If not, its rows would be linearly dependent and there would exist a polynomial $\sum_{j=1}^n \gamma_j q_j$ of degree $\leq n-1$ with n roots t_m, $m = 1,2,\ldots,n$, an obvious contradiction. So rank$J \geq n$ and the result of the lemma follows. □

Thus a more refined existence analysis has to be applied. The system of equations (5.9) will now be split in two parts, the equations determined by the first components of the points $\boldsymbol{T}_\ell$, and the equations provided by the second ones. A proper reparameterization of the curve $\boldsymbol{f}$, the idea heavily leaned upon in [37], will yield a simple solution of the first part. Let us introduce new unknowns $\boldsymbol{\xi} := (\xi_\ell)_{\ell=1}^{2n-2}$ by a reparameterization η of the curve $\boldsymbol{f}$,

$$\eta \;\to\; \eta(t) \;:=\; \eta(t;\boldsymbol{\xi}), \tag{5.11}$$

given at t_ℓ as

$$\eta_\ell = \eta(t_\ell;\boldsymbol{\xi}) = t_\ell + u(t_\ell;\boldsymbol{\xi}) + \xi_{n-2+\ell} h^{n-1} p(t_\ell), \quad \ell = 1,2,\ldots,2n-2, \tag{5.12}$$

where $\xi_\ell := 0,\ \ell > 2n-2$. Furthermore, let

$$p(t) := (t - t_0) \prod_{\ell=n+1}^{2n-1} (t - t_\ell), \quad u(t;\boldsymbol{\xi}) := (t - t_0)(t - t_{2n-1}) \sum_{j=1}^{n-2} \xi_j h^j t^{j-1}. \tag{5.13}$$

The reparameterization (5.11) is quite clearly regular for $\boldsymbol{\xi}$ bounded independently of h and h small enough, since

$$\eta(t_0;\boldsymbol{\xi}) = t_0 = 0 = \eta_0, \quad \eta(t_{2n-1};\boldsymbol{\xi}) = t_{2n-1} = 1 = \eta_{2n-1}, \quad \eta'(t;\boldsymbol{\xi}) = 1 + \mathcal{O}(h).$$

The limit conditions $\eta_\ell = t_\ell$ at $h = 0$ are fulfilled too.

LEMMA 5.2. *The change of variables* $\boldsymbol{\eta} \to \boldsymbol{\xi}$ *introduced in* (5.12) *is one-to-one.*

Proof. Note that $p(t_\ell) = 0,\ \ell = n+1, n+2, \ldots, 2n-2$. So (5.12) provides the conditions $\eta_\ell = t_\ell + u(t_\ell; \boldsymbol{\xi}),\ \ell = n+1, n+2, \ldots, 2n-2$, that uniquely determine the polynomial

$$\sum_{j=1}^{n-2} \xi_j h^j t^{j-1}$$

of degree $< n-2$ since t_ℓ are distinct. But then the rest of the new unknowns $(\xi_\ell)_{\ell=n-1}^{2n-2}$ are obtained from (5.12) by choosing $\ell = 1, 2, \ldots, n$. □

Let $\boldsymbol{T}_\ell$ be given by (5.3), with the reparameterization (5.12) applied. The system (5.9) can now be written as

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{\xi}; h) &:= (F_j(\boldsymbol{\xi}; h))_{j=1}^{n-1} = \boldsymbol{0}, \\ \boldsymbol{G}(\boldsymbol{\xi}; h) &:= (G_j(\boldsymbol{\xi}; h))_{j=1}^{n-1} = \boldsymbol{0}, \end{aligned} \tag{5.14}$$

where

$$F_j(\boldsymbol{\xi}; h) := \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(t_\ell + u(t_\ell; \boldsymbol{\xi}) + \xi_{n-2+\ell} h^{n-1} p(t_\ell)\right), \tag{5.15}$$

and

$$G_j(\boldsymbol{\xi}; h) := \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(\sum_{k=2}^{\infty} c_k h^{k-2} \left(t_\ell + u(t_\ell; \boldsymbol{\xi}) + \xi_{n-2+\ell} h^{n-1} p(t_\ell)\right)^k\right).$$

The following result has been conjectured from some numerical experiments.

THEOREM 5.3. *The unknowns* $\boldsymbol{\xi}$ *can solve* (5.14) *if and only if*

$$\xi_{n-1} = \xi_n = \xi_{n+1} = \cdots = \xi_{2n-2}. \tag{5.16}$$

Proof. A divided difference is a linear functional, so the functions F_j, defined in (5.15), can be simplified to

$$\begin{aligned} F_j(\boldsymbol{\xi}; h) &= [t_0, t_1, \ldots, t_{n+j}]_t \,(t + u(t; \boldsymbol{\xi})) + h^{n-1} \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \xi_{n-2+\ell}\, p(t_\ell) \\ &= h^{n-1} \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \xi_{n-2+\ell}\, p(t_\ell), \end{aligned} \tag{5.17}$$

since the polynomial $t + u(t; \boldsymbol{\xi})$ is of degree $\leq n-1$ in t. Further, the polynomial p is of the particular form (5.13) and $\deg p = n$, so it follows that (5.15) reads

$$\begin{aligned} 0 = \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \xi_{n-2+\ell}\, p(t_\ell) &= \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \xi_{n-2+\ell}\, p(t_\ell) - \xi_{n-1} \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)}\, p(t_\ell) \\ &= \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(\xi_{n-2+\ell} - \xi_{n-1}\right) p(t_\ell) \\ &= \sum_{\ell=2}^{n} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(\xi_{n-2+\ell} - \xi_{n-1}\right) p(t_\ell). \end{aligned}$$

It is easy to verify that the square matrix

$$A := \left(\frac{1}{\dot{\omega}_{0,n+j}(t_\ell)}\right)_{j=1;\ell=2}^{n-1;n}$$

is nonsingular by finding a closed form of $\det A$ (see [32], e.g.). So it can map only the trivial vector to $\mathbf{0}$. Since $p(t_\ell) \neq 0$, $\ell = 2, 3, \ldots, n$, the term $\xi_{n-2+\ell} - \xi_{n-1}$ should vanish for all ℓ concerned, and the claim (5.16) follows. □

Theorem 5.3 pins down the choice $\xi_j = \xi_{n-1}$, $j = n, n+1, \ldots, 2n-2$, that will be assumed from now on. The rest of the unknowns $(\xi_\ell)_{\ell=1}^{n-1}$ should be determined by the second part of equations (5.14). But (5.16) simplifies the reparameterization (5.11) to a polynomial

$$\eta(t; \boldsymbol{\xi}) = t + u(t; \boldsymbol{\xi}) + \xi_{n-1} h^{n-1} p(t), \tag{5.18}$$

and further $G_j(\boldsymbol{\xi}; h)$ to

$$G_j(\boldsymbol{\xi}; h) = [t_0, t_1, \ldots, t_{n+j}] \sum_{k=2}^{\infty} c_k h^{k-2} \eta(\cdot\,; \boldsymbol{\xi})^k. \tag{5.19}$$

In order to study (5.19) further the following lemma is needed.

LEMMA 5.4. *Let*

$$q(t; \boldsymbol{\xi}) := t + \sum_{\ell=1}^{n-1} \xi_\ell h^\ell t^{\ell+1}.$$

Then

$$[t_0, t_1, \ldots, t_{n+j}]\eta(\cdot\,; \boldsymbol{\xi})^k = [t_0, t_1, \ldots, t_{n+j}]q(\cdot\,; \boldsymbol{\xi})^k + \mathcal{O}(h^{n+j+1-k}),$$

and

$$[t_0, t_1, \ldots, t_{n+j}]q(\cdot\,; \boldsymbol{\xi})^k = \mathcal{O}(h^{n+j-k}).$$

Proof. Let us introduce some new notation. If r is a polynomial in variables t and h, then let

$$\operatorname{termdeg}_t(r) \leq \operatorname{termdeg}_h(r) \tag{5.20}$$

denote that for every term $t^{\alpha_i} h^{\beta_i}$ of r, the exponents α_i and β_i satisfy the relation $\alpha_i \leq \beta_i$.

From (5.13) it is straightforward to verify that

$$\operatorname{termdeg}_t(\eta) \leq \operatorname{termdeg}_h(\eta) + 1,$$

where η is given by (5.18). Furthermore, the difference $\eta - q$ turns out as

$$\begin{aligned}\eta(t; \boldsymbol{\xi}) - q(t; \boldsymbol{\xi}) &= (-(t_0 + t_{2n-1})t + t_0 t_{2n-1}) \sum_{j=1}^{n-2} \xi_j h^j t^{j-1} \\ &\quad + \xi_{n-1} h^{n-1} \left((t - t_0) \prod_{\ell=n+1}^{2n-1} (t - t_\ell) - t^n \right),\end{aligned}$$

and clearly $\text{termdeg}_t(\eta - q) \le \text{termdeg}_h(\eta - q)$. But

$$\eta(t;\boldsymbol{\xi})^k = q(t;\boldsymbol{\xi})^k + \sum_{j=1}^{k} \binom{k}{j} \left(\eta(t;\boldsymbol{\xi}) - q(t;\boldsymbol{\xi})\right)^j q(t;\boldsymbol{\xi})^{k-j},$$

and $q(t;\boldsymbol{\xi})^k$ satisfies

$$\text{termdeg}_t(q^k) = \text{termdeg}_h(q^k) + k. \tag{5.21}$$

On the other hand, a brief look at the remaining sum yields

$$\text{termdeg}_t(\eta^k - q^k) \le \text{termdeg}_h(\eta^k - q^k) + k - 1. \tag{5.22}$$

The divided difference $[t_0, t_1, \ldots, t_{n+j}]_t$ maps polynomials in t of degree $< n+j$ to zero. So the monomials with degree $= n+j$ will provide the leading term of the error. But then (5.21) and (5.22) confirm the lemma. □

Lemma 5.4 simplifies the functions (5.19) to

$$G_j(\boldsymbol{\xi};h) = [t_0, t_1, \ldots, t_{n+j}] \sum_{k=2}^{\infty} c_k h^{k-2} q(\,\cdot\,;\boldsymbol{\xi})^k + \mathcal{O}(h^{n+j-1}), \tag{5.23}$$

and the following conclusion provides the final form of the system (5.14).

THEOREM 5.5. *The expansion referenced in* (5.23) *could be rewritten as*

$$\sum_{k=2}^{\infty} c_k h^{k-2} q(t;\boldsymbol{\xi})^k = \sum_{k=2}^{\infty} C_k(\boldsymbol{\xi}) h^{k-2} t^k, \tag{5.24}$$

where

$$C_k(\boldsymbol{\xi}) := \frac{2}{k!\, h^k y''(0)} \left(\frac{d^k}{dx^k}\, y\left(h\, q(x;\boldsymbol{\xi})\right) \right) \bigg|_{x=0}.$$

The polynomials $C_k(\boldsymbol{\xi})$ depend on $\boldsymbol{\xi}$ only, but not on h nor the parameters $\boldsymbol{t}$. So the final form of the system (5.14)*, for h small enough, is given as*

$$C_{n+j}(\boldsymbol{\xi}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \ldots, n-1. \tag{5.25}$$

Proof. Let us recall the proof of Lemma 5.4 and the notation (5.20). A close look reveals that

$$\text{termdeg}_t(h^{k-2}q^k) = \text{termdeg}_h(h^{k-2}q^k) + 2,$$

hence (5.24) follows. But then

$$\frac{1}{h^{n+j-2}}\, G_j(\boldsymbol{\xi};h) = C_{n+j}(\boldsymbol{\xi}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \ldots, n-1.$$

□

Let us sum up all the asymptotic conclusions.

THEOREM 5.6. *If there exists $h_0 > 0$, such that the system of nonlinear equations (5.25) has a real solution for all h, $0 \le h \le h_0$, then the interpolating polynomial parametric curve $\boldsymbol{P}_n$ exists and approximates $\boldsymbol{f}$ with the optimal approximation order, i.e., $2n$.*

Proof. The proof will follow the path already applied in [4]. If the interpolating curve $\boldsymbol{P}_n$ is reparameterized by a regular reparameterization $\theta : [0, h] \to [0, 1]$ in such a way that

$$(\boldsymbol{P}_n \circ \theta)\,(h\eta_\ell) = \boldsymbol{f}(h\eta_\ell), \quad \ell = 0, 1, \ldots, 2n-1, \tag{5.26}$$

the error analysis can be applied to each component separately, using the standard approach for the function case. But this implies that the optimal approximation order $2n$ is achieved, provided that $\| \left(\boldsymbol{P}_n \circ \theta\right)^{(2n)} \|$ remains bounded for all h small enough.

By assumption the system (5.25) has a real solution and the unknown parameters $\boldsymbol{t}$ exist. Thus one can represent the curve $\boldsymbol{P}_n$ in the Lagrange form,

$$\boldsymbol{P}_n = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \sum_{\ell=0}^{n} \boldsymbol{f}(h\eta_\ell)\,\mathcal{L}_{\ell,n}, \quad \mathcal{L}_{\ell,n}(t) := \prod_{\substack{j=0\\ j\neq \ell}}^{n} \frac{t - t_j}{t_\ell - t_j}\,. \tag{5.27}$$

For the particular coordinate system, chosen in subsection 5.1, a reparameterization $\theta := P_1^{-1}$ is a proper choice. Indeed, from (5.27) and (5.12) it follows

$$P_1(t) = \sum_{\ell=0}^{n} (h\eta_\ell)\; \mathcal{L}_{\ell,n}(t) = h \sum_{\ell=0}^{n} \eta(t_\ell; \boldsymbol{\xi})\,\mathcal{L}_{\ell,n}(t) = h\eta(t; \boldsymbol{\xi}), \tag{5.28}$$

since the polynomial $\eta(\cdot\,; \boldsymbol{\xi})$, defined in (5.18), is of degree $\le n$. Note that

$$P_1(0) = h\eta(0; \boldsymbol{\xi}) = 0, \quad P_1(1) = h\eta(1; \boldsymbol{\xi}) = h, \quad P_1(t) = ht + \mathcal{O}(h^2).$$

So P_1 is a diffeomorphism $[0, 1] \to [0, h]$ for h small enough. The interpolation conditions (5.26) are satisfied, since

$$\left(\boldsymbol{P}_n \circ P_1^{-1}\right)(h\eta_\ell) \;=\; \boldsymbol{P}_n(t_\ell) \;=\; \boldsymbol{f}(h\eta_\ell), \quad \ell = 0, 1, \ldots, 2n-1,$$

and $\theta = P_1^{-1}$ is the required reparameterization. In order to prove the boundedness of $(\boldsymbol{P}_n \circ \theta)^{(2n)}$ we apply the chain rule derivation to $\boldsymbol{P}_n \circ P_1^{-1}$. As already observed in [4] for the cubic case, and in [16] for a general n, it suffices to see that $P_1'(t) = ch + \mathcal{O}(h^2)$, $c \neq 0$, and

$$P_i^{(k)}(t) = \mathcal{O}(h^k), \quad i = 1, 2, \quad k = 2, 3, \ldots, 2n.$$

Obviously $P_1'(t) = h + \mathcal{O}(h^2)$. Since $\deg P_i \le n$, it is enough to consider $2 \le k \le n$ only. The case $i = 1$ follows immediately from (5.18) and (5.28). As to the other,

$$P_2(t) = \sum_{\ell=0}^{n} y(h\eta_\ell)\,\mathcal{L}_{\ell,n}(t) = \sum_{\ell=0}^{n} y(h\eta(t_\ell; \boldsymbol{\xi}))\,\mathcal{L}_{\ell,n}(t).$$

Let us recall the expansion (5.1) from which we observe that the sums involved are

$$\sum_{\ell=0}^{n} (h\eta(t_\ell;\boldsymbol{\xi}))^m \mathcal{L}_{\ell,n}(t) = \sum_{\ell=0}^{n} h^m t_\ell^m \mathcal{L}_{\ell,n}(t) + \mathcal{O}(h^{m+1}), \quad m \geq 2.$$

Since the interpolation is a projection on the space of polynomials of degree $\leq n$, the proof is complete. ☐

THEOREM 5.7. *The system of nonlinear equations* (5.25) *has a real solution for* $n \leq 5$, *and* h *small enough.*

Proof. If $n = 2$, the system (5.25) simplifies to one linear equation for ξ_1,

$$2\,\xi_1 + c_3 + \mathcal{O}(h) = 0,$$

which obviously has a real solution. The case $n = 3$ is easy to analyse too, since the nonlinear system in this case is

$$\begin{aligned} \xi_1^2 + 3\,c_3\,\xi_1 + 2\,\xi_2 + c_4 + \mathcal{O}(h) &= 0, \\ 3\,c_3\,\xi_1^2 + 2\,\xi_1\,(\xi_2 + 2\,c_4) + 3\,c_3\,\xi_2 + c_5 + \mathcal{O}(h) &= 0. \end{aligned} \tag{5.29}$$

The first equation is always linear in $\xi_{n-1} = \xi_2$. So (5.29) can be reduced to a cubic equation for ξ_1,

$$\xi_1^3 + \frac{3}{2}\,c_3\,\xi_1^2 + \left(\frac{9}{2}\,c_3^2 - 3\,c_4\right)\xi_1 + \frac{3}{2}\,c_3\,c_4 - c_5 + \mathcal{O}(h) = 0,$$

and the conclusion follows. The proof for the case $n = 4$ and $n = 5$ will require quite a lot of technical details, and will be given in the next two subsections. ☐

5.4. The case $n = 4$.

The idea of the proof will be borrowed from [46], but it will require some additional steps. For $n = 4$ the system (5.25) is given as

$$\begin{aligned} 3\,c_3\,\xi_1^2 + \xi_1\,(2\,\xi_2 + 4\,c_4) + 3\,c_3\,\xi_2 + 2\,\xi_3 + c_5 + \mathcal{O}(h) &= 0, \\ c_3\,\xi_1^3 + 6\,c_4\,\xi_1^2 + \xi_1\,(6\,c_3\,\xi_2 + 2\,\xi_3 + 5\,c_5) + \xi_2^2 + 4\,c_4\,\xi_2 + 3\,c_3\,\xi_3 + c_6 + \mathcal{O}(h) &= 0, \\ 4\,c_4\,\xi_1^3 + \xi_1^2\,(3\,c_3\,\xi_2 + 10\,c_5) + \xi_1\,(12\,c_4\,\xi_2 + 6\,c_3\,\xi_3 + 6\,c_6) + 3\,c_3\,\xi_2^2 & \\ + \xi_2\,(2\,\xi_3 + 5\,c_5) + 4\,c_4\,\xi_3 + c_7 + \mathcal{O}(h) &= 0. \end{aligned}$$

The unknown ξ_3 appears linearly again, and this system reduces to

$$\begin{aligned} \xi_2^2 + \xi_2\left(-2\,\xi_1^2 - \frac{9}{2}\,c_3^2 + 4\,c_4\right) - 2\,c_3\,\xi_1^3 + \xi_1^2\left(-\frac{9}{2}\,c_3^2 + 2\,c_4\right) & \\ + \xi_1\,(-6\,c_3\,c_4 + 4\,c_5) - \frac{3}{2}\,c_3\,c_5 + c_6 + \mathcal{O}(h) &= 0, \end{aligned} \tag{5.30}$$

$$\begin{aligned} (-9\,c_3^2 + 4\,c_4)\,\xi_1^3 + \xi_1^2\,(-6\,c_3\,\xi_2 - 18\,c_3\,c_4 + 10\,c_5) & \\ + \xi_1\,\left(-2\,\xi_2^2 + (-9\,c_3^2 + 4\,c_4)\,\xi_2 - 3\,c_3\,c_5 - 8\,c_4^2 + 6\,c_6\right) & \\ + (-6\,c_3\,c_4 + 4\,c_5)\,\xi_2 - 2\,c_4\,c_5 + c_7 + \mathcal{O}(h) &= 0. \end{aligned} \tag{5.31}$$

The equation (5.30) is quadratic in ξ_2, and its solution gives ξ_2 as a function of ξ_1,

$$\xi_2 = \xi_1^2 + \frac{9}{4}\,c_3^2 - 2\,c_4 \,\pm\, \frac{1}{4}\,\sqrt{R(\xi_1)} + \mathcal{O}(h), \tag{5.32}$$

where

$$\begin{aligned} R(\xi_1) := {}& 16\,\xi_1^4 + 32\,c_3\,\xi_1^3 + (144\,c_3^2 - 96\,c_4)\,\xi_1^2 + (96\,c_3\,c_4 - 64\,c_5)\,\xi_1 \\ &+ 81\,c_3^4 - 144\,c_3^2\,c_4 + 24\,c_3\,c_5 + 64\,c_4^2 - 16\,c_6. \end{aligned}$$

Inserting (5.32) into the equation (5.31) yields

$$H_\pm(\xi_1) := p_5(\xi_1) \mp \frac{1}{64} R'(\xi_1)\sqrt{R(\xi_1)} + \mathcal{O}(h) = 0, \tag{5.33}$$

where

$$\begin{aligned} p_5(\xi_1) = {}& -4\,\xi_1^5 - 10\,c_3\,\xi_1^4 + (-45\,c_3^2 + 28\,c_4)\,\xi_1^3 + \left(-\frac{27}{2}\,c_3^3 - 24\,c_3\,c_4 + 22\,c_5\right)\,\xi_1^2 \\ &+ \left(-\frac{81}{2}\,c_3^4 + 63\,c_3^2\,c_4 - 6\,c_3\,c_5 - 32\,c_4^2 + 8\,c_6\right)\,\xi_1 - \frac{27}{2}\,c_3^3\,c_4 + 9\,c_3^2\,c_5 \\ &+ 12\,c_3\,c_4^2 - 10\,c_4\,c_5 + c_7. \end{aligned}$$

Let $D := (-\infty, \infty)$ if the polynomial R has only complex zeros. Otherwise, let z_m and z_M denote its smallest and largest real zero respectively, and choose $D := (-\infty, z_m] \cup [z_M, \infty)$. Since R is obviously nonnegative on D, the functions $H_\pm$ are well defined and continuous on D. If we prove that the ranges $H_\pm(D)$ of $H_\pm$ satisfy

$$H_-(D) \cup H_+(D) = \mathbb{R}, \tag{5.34}$$

then the equation (5.33) has a real solution for any right hand side, with a proper choice of $\pm$. Note that

$$H_-(\xi_1) \approx (-5\,c_3^2 + 4\,c_4)\,\xi_1^3, \quad \text{and} \quad H_+(\xi_1) \approx -8\,\xi_1^5, \quad |\xi_1| \to \infty.$$

When R has no real zeros, the relation (5.34) is confirmed since $D = \mathbb{R}$ and

$$\lim_{\xi_1 \to -\infty} H_+(\xi_1) = \infty, \quad \lim_{\xi_1 \to \infty} H_+(\xi_1) = -\infty.$$

Suppose now that the polynomial R has real zeros. Quite clearly

$$H_-(z_m) = H_+(z_m), \quad H_-(z_M) = H_+(z_M).$$

If $c_4 > \frac{5}{4}\,c_3^2$ the condition (5.34) is fulfilled too since

$$\lim_{\xi_1 \to \infty} H_+(\xi_1) = -\infty, \quad \lim_{\xi_1 \to \infty} H_-(\xi_1) = \infty,$$

(see Figure 5.1, left). In the case $c_4 < \frac{5}{4}\,c_3^2$ a more refined analysis is needed since

$$\lim_{\xi_1 \to -\infty} H_\pm(\xi_1) = \infty, \quad \lim_{\xi_1 \to \infty} H_\pm(\xi_1) = -\infty.$$

But if one can find values $\xi_L \leq z_m$ and $z_M \leq \xi_R$ such that $H_-(\xi_L) \leq H_-(\xi_R)$, then the equation (5.34) is verified again since then $H_-(D) = \mathbb{R}$. The proof of this fact will be given as Lemma 5.8. We are then left with some particular cases. First, the choice $c_4 = \frac{5}{4} c_3^2$ implies that

$$H_-(\xi_1) \approx -\frac{5}{2}(7\,c_3^3 - 4\,c_5)\,\xi_1^2, \quad |\xi_1| \to \infty,$$

and (5.34) follows if $c_5 \neq \frac{7}{4} c_3^3$. Furthermore, $c_4 = \frac{5}{4} c_3^2$, $c_5 = \frac{7}{4} c_3^3$ leads to

$$H_-(\xi_1) \approx -\frac{3}{4}(21\,c_3^4 - 8\,c_6)\,\xi_1, \quad |\xi_1| \to \infty.$$

It is straightforward to verify that $c_6 < \frac{21}{8} c_3^4$ implies only complex zeros of R, and (5.34) holds also for these particular constants. The possibility $c_4 = \frac{5}{4} c_3^2$, $c_5 = \frac{7}{4} c_3^3$, $c_6 = \frac{21}{8} c_3^4$ is covered too since

$$H_-(\xi_1) \approx -\frac{33}{8} c_3^5 - c_7, \quad |\xi_1| \to \infty,$$

and $R(\xi_1) = (2\,\xi_1 + c_3)^4$. An additional choice $c_7 = \frac{33}{8} c_3^5$ gives an infinite number of solutions of the system without terms $\mathcal{O}(h)$. The following lemma will conclude the proof for the case $n = 4$.

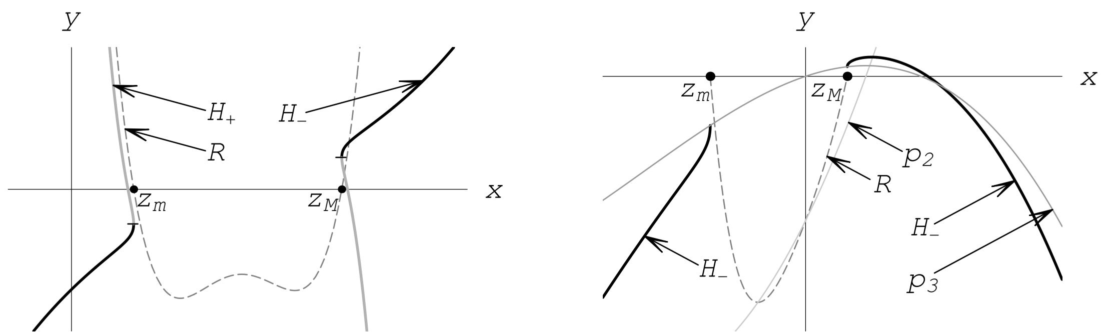


Figure 5.1: Functions $H_\pm$ and R (left), polynomials p_2 and p_3 together with H_- and R (right).

LEMMA 5.8. *Suppose that $c_4 < \frac{5}{4} c_3^2$ and polynomial R has real roots. Then there exist points ξ_L and ξ_R such that $H_-(\xi_L) \leq H_-(\xi_R)$.*

Proof. Let us introduce the constants

$$\begin{aligned}
d_1 &:= -\frac{1}{2} c_3, \\
d_2 &:= 24\,\left(5\,{c_3}^2 - 4\,c_4\right), \\
d_3 &:= -64\,\left(2\,{c_3}^3 - 3\,c_3\,c_4 + c_5\right), \\
d_4 &:= 2\,\left(57\,{c_3}^4 - 108\,{c_3}^2\,c_4 + 28\,c_3\,c_5 + 32\,{c_4}^2 - 8\,c_6\right),
\end{aligned}$$

and let us shift the origin $\xi_1 \to \xi_1 + d_1$. The polynomial R is simplified to

$$R(\xi_1) = 16\,\xi_1^4 + d_2\,\xi_1^2 + d_3\,\xi_1 + d_4,$$

where $d_2 > 0$ by the assumption. So $R''(\xi_1) = 192\,\xi_1^2 + 2\,d_2 > 0$, and R is strictly convex with exactly two real roots z_m and z_M. Let ξ_* denote the abscissa of the minimum of R. By the assumption $R(\xi_*) \le 0$. It is straightforward to verify

$$\begin{aligned} R(\xi_1) &= \frac{1}{4}\,\xi_1\,R'(\xi_1) + p_2(\xi_1),\\ p_5(\xi_1) &= -\frac{1}{4}\,\xi_1\,R(\xi_1) + p_3(\xi_1),\end{aligned}$$

where $p_2(\xi_1) = \frac{1}{2}\,d_2\,\xi_1^2 + \frac{3}{4}\,d_3\,\xi_1 + d_4$ and p_3 is a cubic polynomial that satisfies $p_3' = -\frac{1}{4}\,p_2$. The polynomial p_2 agrees with R precisely at points $\xi_1 = 0$, $\xi_1 = \xi_*$ and $p_2 > R$ only on the interval $(\min(0,\xi_*), \max(0,\xi_*))$. Suppose $z_m \le 0 \le z_M$. Then $p_2(z_m) \le 0$, $p_2(z_M) \le 0$, and since the parabola p_2 has a positive leading coefficient, $p_2 \le 0$ on $[z_m, z_M]$. But then

$$H_-(z_M) - H_-(z_m) = p_3(z_M) - p_3(z_m) = -\frac{1}{4}\int_{z_m}^{z_M} p_2(\xi)\,d\xi \ge 0,$$

and the choice $\xi_L := z_m$, $\xi_R := z_M$ will satisfy the claim of the lemma (see Figure 5.1, right).

Suppose now that $z_M < 0$. The constants involved are now all positive since $R(0) = d_4 > 0$ and $R'(0) = d_3 > 0$. The only part in R that can be negative is $d_3\,\xi_1$. Therefore the condition $R(\xi_*) \le 0$ gives a lower bound on the size of d_3. The polynomial R has a double zero in ξ_* if and only if the resultant of R and R' is zero, namely

$$1024\,\left(-108\,d_3^4 + d_3^2\,\left(576\,d_2\,d_4 - d_2^3\right) + 4\,d_2^4\,d_4 - 512\,d_2^2\,d_4^2 + 16384\,d_4^3\right) = 0.$$

This is a quadratic equation in d_3^2 with one positive solution which implies

$$d_3^2 \ge \frac{1}{216}\left(-d_2^3 + 576\,d_2\,d_4 + \left(d_2^2 + 192\,d_4\right)^{\frac{3}{2}}\right) \tag{5.35}$$

and a rough estimation yields

$$d_3^2 \ge \frac{8}{3}\,d_2\,d_4. \tag{5.36}$$

Let us show that the choice

$$\xi_L := -\frac{5}{2}\,\frac{d_3}{d_2}, \quad \xi_R := 0$$

is appropriate here. Since $H_-(0) = \frac{1}{64}\,d_3\sqrt{d_4} \ge 0$, it is sufficient to show $H_-(\xi_L) \le 0$. Now

$$\begin{aligned} H_-(\xi_L) = -\frac{d_3}{384\,d_2^5}\Bigg(&\left(12\,d_2^3 + 3000\,d_3^2\right)\sqrt{15\,d_2^3\,d_3^2 + 4\,d_2^4\,d_4 + 2500\,d_3^4}\\ &- \left(480\,d_2^4\,d_4 + 925\,d_2^3\,d_3^2 + 150000\,d_3^4\right)\Bigg)\end{aligned}$$

is negative if

$$\left(12\, d_2^3 + 3000\, d_3^2\right)^2 \left(15\, d_2^3\, d_3^2 + 4\, d_2^4\, d_4 + 2500\, d_3^4\right) - \left(480\, d_2^4\, d_4 + 925\, d_2^3\, d_3^2 + 150000\, d_3^4\right)^2$$

is positive or equivalently

$$1500000\, d_2^3\, d_3^4 \left(25\, d_3^2 - 72\, d_2\, d_4\right) + 25\, d_2^6 \left(125\, d_3^2 \left(187\, d_3^2 - 192\, d_2\, d_4\right) - 9216\, d_2^2\, d_4^2\right) + 576\, d_2^{10}\, d_4 + 2160\, d_2^9\, d_3^2 \,\geq\, 0.$$

The term $25\, d_3^2 - 72\, d_2\, d_4$ is estimated from below using (5.35). The square root free form of the estimate is

$$\frac{1}{216^2} \left(625 \left(d_2^2 + 192\, d_4\right)^3 - \left(25\, d_2^3 + 1152\, d_2\, d_4\right)^2\right) = \frac{1}{27}\, d_4 \left(175\, d_2^4 + 39232\, d_2^2\, d_4 + 2560000\, d_4^2\right) \geq 0.$$

Furthermore, the second term gives

$$25\, d_2^6 \left(125\, d_3^2 \left(187\, d_3^2 - 192\, d_2\, d_4\right) - 9216\, d_2^2\, d_4^2\right) \geq 25\, d_2^7\, d_4 \left(\frac{115000}{3}\, d_3^2 - 9216\, d_2\, d_4\right) \geq \frac{20926400}{9}\, d_2^8\, d_4^2 \geq 0,$$

where (5.36) was used twice. Since the rest of the terms are nonnegative the result follows. The case $0 < z_m$ is symmetric, and the proof is completed. □

5.5. The case $n = 5$.

Let us now consider the case $n = 5$. After the elimination of the variable $\xi_4 = \xi_{n-1}$ the system (5.25) becomes

$$B_i(\xi_1, \xi_2, \xi_3) + \mathcal{O}(h) = 0, \quad i = 1, 2, 3, \tag{5.37}$$

with

$$
\begin{aligned}
&B_1(\xi_1,\xi_2,\xi_3) := \\
&\quad -c_3\,\xi_1^4 - \left(\tfrac{3}{2}\,c_3^2 + 2\,c_4\right)\,\xi_1^3 - (9\,c_3\,c_4 - 5\,c_5)\,\xi_1^2 - \left(\tfrac{15}{2}\,c_3\,c_5 - 5\,c_6\right)\,\xi_1 \\
&\quad + \left(\tfrac{3}{2}\,c_3 - \xi_1\right)\,\xi_2^2 + (6\,c_3\,c_4 - 5\,c_5 + (9\,c_3^2 - 8\,c_4)\,\xi_1 - 3\,c_3\,\xi_1^2)\,\xi_2 \\
&\quad - \left(\tfrac{9}{2}\,c_3^2 - 4\,c_4 + 2\,\xi_1^2 - 2\,\xi_2\right)\,\xi_3 - \tfrac{3}{2}\,c_3\,c_6 + c_7, \\
&B_2(\xi_1,\xi_2,\xi_3) := \\
&\quad -(3\,c_3^2 - c_4)\,\xi_1^4 - (20\,c_3\,c_4 - 10\,c_5)\,\xi_1^3 - (12\,c_4^2 + 15\,c_3\,c_5 - 15\,c_6)\,\xi_1^2 \\
&\quad - (10\,c_4\,c_5 + 3\,c_3\,c_6 - 7\,c_7)\,\xi_1 - 2\,c_4\,c_6 + c_8 - \xi_2^3 - 6\,c_3\,\xi_1\,\xi_2^2 \\
&\quad - (8\,c_4^2 - 5\,c_6 + (24\,c_3\,c_4 - 15\,c_5)\,\xi_1 + (18\,c_3^2 - 6\,c_4)\,\xi_1^2 - c_3\,\xi_1^3)\,\xi_2 \\
&\quad + \xi_3^2 - (6\,c_3\,c_4 - 5\,c_5 + (9\,c_3^2 - 8\,c_4)\,\xi_1 + 3\,c_3\,\xi_1^2 - (3\,c_3 - 2\,\xi_1)\,\xi_2)\,\xi_3, \\
&B_3(\xi_1,\xi_2,\xi_3) := \\
&\quad -\tfrac{3}{2}\,c_3^2\,\xi_1^5 - (15\,c_3\,c_4 - 5\,c_5)\,\xi_1^4 - (36\,c_4^2 + 10\,c_3\,c_5 - 20\,c_6)\,\xi_1^3 \\
&\quad - \left(45\,c_4\,c_5 + \tfrac{3}{2}\,c_3\,c_6 - 21\,c_7\right)\,\xi_1^2 - \left(\tfrac{25}{2}\,c_5^2 + 6\,c_4\,c_6 - 8\,c_8\right)\,\xi_1 - \tfrac{5}{2}\,c_5\,c_6 + c_9 \\
&\quad -2\,c_3\,\xi_2^3 - \left(12\,c_3\,c_4 - \tfrac{15}{2}\,c_5 + (18\,c_3^2 - 6\,c_4)\,\xi_1 + \tfrac{3}{2}\,c_3\,\xi_1^2\right)\,\xi_2^2 \\
&\quad - (10\,c_4\,c_5 + 3\,c_3\,c_6 - 7\,c_7 + (24\,c_4^2 + 30\,c_3\,c_5 - 30\,c_6)\,\xi_1 \\
&\qquad - (60\,c_3\,c_4 - 30\,c_5)\,\xi_1^2 + (12\,c_3^2 - 4\,c_4)\,\xi_1^3)\,\xi_2 \\
&\quad -2\,\xi_1\,\xi_3^2 - \left(\tfrac{15}{2}\,c_3\,c_5 - 5\,c_6 + (18\,c_3\,c_4 - 10\,c_5)\,\xi_1 + \left(\tfrac{9}{2}\,c_3^2 + 6\,c_4\right)\,\xi_1^2\right. \\
&\qquad \left. -4\,c_3\,\xi_1^3 - (9\,c_3^2 - 8\,c_4 + 6\,c_3\,\xi_1)\,\xi_2 - \xi_2^2\right)\,\xi_3.
\end{aligned}
$$

The terms $\mathcal{O}(h)$ in (5.37) will be neglected for the moment. Let $\mathbb{R}[\xi_1,\xi_2,\ldots,\xi_i]$ denote the ring of polynomials in variables $\xi_1,\xi_2,\ldots,\xi_i$ over $\mathbb{R}$. A straightforward approach to the system (5.37) is right at hand: compute the Gröbner basis of the ideal

$$\mathcal{I} := \langle B_1, B_2, B_3 \rangle \subset \mathbb{R}[\xi_1,\xi_2,\xi_3], \tag{5.38}$$

and study the properties of the zeros of this basis, i.e., the variety $\mathcal{V}(\mathcal{I})$, the set of common zeros of B_i, $i = 1,2,3$. But this approach is computationally too complex, and some ad hoc simplification is needed. The key conclusion is summarized in the following lemma.

LEMMA 5.9. *Let $\mathcal{I}$ be the ideal given in* (5.38) *and let $\mathcal{I}_2 := \mathcal{I} \cap \mathbb{R}[\xi_1]$ denote the second elimination ideal, obtained from $\mathcal{I}$ after elimination of ξ_3 and ξ_2. Then $\mathcal{V}(\mathcal{I}_2) = \mathcal{V}(Q)$ where Q is a polynomial of degree $\leq$ 25 in ξ_1, given as*

$$
\begin{aligned}
Q(\xi_1) \;=\; & \frac{14641}{65536}\left(5\,c_3^2 - 4\,c_4\right)^5 \xi_1^{25} \\
& - \frac{3025}{131072}\left(5\,c_3^2 - 4\,c_4\right)^4 \left(1043\,c_3^3 - 1988\,c_3\,c_4 + 824\,c_5\right)\,\xi_1^{24} + \ldots
\end{aligned}
$$

Proof. Consider the system (5.37). The first equation is linear in ξ_3, and can be written as

$$B_1(\xi_1,\xi_2,\xi_3) = \psi_1(\xi_1,\xi_2) + \psi_2(\xi_1,\xi_2)\,\xi_3 = 0, \quad \psi_1,\ \psi_2 \in \mathbb{R}[\xi_1,\xi_2]. \tag{5.39}$$

Similarly, the modified third equation

$$B_3(\xi_1, \xi_2, \xi_3) + 2\,\xi_1 B_2(\xi_1, \xi_2, \xi_3) \;=\; \chi_1(\xi_1, \xi_2) + \chi_2(\xi_1, \xi_2)\,\xi_3 \;=\; 0, \tag{5.40}$$

with $\chi_1, \chi_2 \in \mathbb{R}[\xi_1, \xi_2]$, turns out to be linear in ξ_3 too. One can now use the equation (5.39) to eliminate ξ_3, and the system (5.37) becomes

$$\begin{aligned} B_{12}(\xi_1, \xi_2) &:= \psi_2(\xi_1, \xi_2)^2\, B_2\left(\xi_1, \xi_2, -\frac{\psi_1(\xi_1, \xi_2)}{\psi_2(\xi_1, \xi_2)}\right) &=& \;0, \\ B_{13}(\xi_1, \xi_2) &:= \chi_1(\xi_1, \xi_2)\,\psi_2(\xi_1, \xi_2) - \chi_2(\xi_1, \xi_2)\,\psi_1(\xi_1, \xi_2) &=& \;0. \end{aligned} \tag{5.41}$$

Finally, the resultant of B_{12} and B_{13} with respect to ξ_2 is a single equation

$$R_1(\xi_1) := \mathrm{Res}\,(B_{12}(\xi_1, \xi_2), B_{13}(\xi_1, \xi_2); \xi_2) = 0.$$

The variety $\mathcal{V}(R_1) \supseteq \mathcal{V}(\mathcal{I}_2)$ may include some extraneous zeros introduced by the factor $\psi_1(\xi_1, \xi_2)^2$ in (5.41) or by the resultant Res. Also, the variety $\mathcal{V}$ does not precisely keep track of the multiple zeros, and the number of zeros of R_1 counting multiplicities could be greater than the number of elements in $\mathcal{V}(R_1)$, i.e., $\#\mathcal{V}(R_1)$. The elimination procedure described provides also the extension path: if $\xi_1 \in \mathcal{V}(I_2)$, the equations (5.41) determine ξ_2, and (5.39) finally ξ_3, except when $\psi_2(\xi_1, \xi_2) = 0$, since then (5.39) leaves ξ_3 undefined. But then, at the first elimination step, one may choose the equation (5.40) rather than (5.39) to eliminate ξ_3. The equation (5.41) would be replaced by

$$B_{32}(\xi_1, \xi_2) := \chi_2(\xi_1, \xi_2)^2\, B_2\left(\xi_1, \xi_2, -\frac{\chi_1(\xi_1, \xi_2)}{\chi_2(\xi_1, \xi_2)}\right) \;=\; 0,$$

and one would finally be left with

$$R_2(\xi_1) := \mathrm{Res}\,(B_{32}(\xi_1, \xi_2), B_{13}(\xi_1, \xi_2); \xi_2) = 0.$$

Thus any $\xi_1 \in \mathcal{V}\,(R_1) \cap \mathcal{V}\,(R_2)$ that is not extraneous can be extended to the complete solution of the system (5.37) provided $\psi_2(\xi_1, \xi_2) \neq 0$ or $\chi_2(\xi_1, \xi_2) \neq 0$. Using a computer algebra system (Mathematica, Maple, ...), the polynomials R_1 and R_2 can be factorized as

$$R_1(\xi_1) = \nu_1(\xi_1)^2\, Q(\xi_1), \quad R_2(\xi_1) = -\nu_2(\xi_1)^2\, Q(\xi_1), \tag{5.42}$$

where ν_i form a basis of the elimination ideals

$$\langle \nu_1 \rangle = \langle \psi_1, \psi_2 \rangle \cap \mathbb{R}[\xi_1], \quad \nu_1(\xi_1) = 4\,\xi_1^5 + 10\,c_3\,\xi_1^4 + \left(60\,c_3^2 - 40\,c_4\right)\,\xi_1^3 + \ldots,$$

and

$$\langle \nu_2 \rangle = \langle \chi_1, \chi_2 \rangle \cap \mathbb{R}[\xi_1], \quad \nu_2(\xi_1) = \left(5\,c_3^2 - 4\,c_4\right)\,\xi_1^{12} + \ldots,$$

and Q has the form as written in Lemma 5.9. If ν_1 and ν_2 have no common divisors, they are obviously extraneous factors in the equation (5.42), and

$$\mathcal{V}(\mathcal{I}_2) \subseteq \mathcal{V}(Q). \tag{5.43}$$

If $\nu_1(\xi_1) = 0,\ \nu_2(\xi_1) = 0$ for some $\xi_1 \in \mathbb{C}$ then $\operatorname{Res}(\nu_1(\xi_1), \nu_2(\xi_1); \xi_1) = 0$ gives a tremendous, but polynomial relation between the constants c_i that has to be satisfied. So the measure of the set of constants

$$\{(c_3, c_4, \ldots, c_9) \in \mathbb{R}^7; \quad \nu_1(\xi_1) = 0,\ \nu_2(\xi_1) = 0\}$$

is zero, and one may extend (5.43) to all possible constants by the continuity. Note that the inclusion (5.43) also shows that if $\xi_1 = \xi_1(c_3, c_4, \ldots, c_9) \in \mathcal{V}(\mathcal{I}_2)$ grows unboundedly or decreases from infinity as a function of the constants c_i so must the corresponding $\xi_1 \in \mathcal{V}(Q)$. Thus the difference of the number of solutions

$$\#\mathcal{V}(Q) - \#\mathcal{V}(\mathcal{I}_2) \geq 0$$

is independent of the constants c_i. But for a particular choice of the constants

$$c_3 = 1,\ c_4 = \frac{1}{2},\ c_5 = -1,\ c_6 = 1,\ c_7 = 0,\ c_8 = -\frac{1}{2},\ c_9 = 1$$

it is straightforward to verify that $\mathcal{V}(\mathcal{I}_2)$ is equal to $\mathcal{V}(Q)$ since $\#\mathcal{V}(\mathcal{I}_2) = \#\mathcal{V}(Q) = 25$. □

Thus only the roots of Q have to be considered. If the degree of Q is odd, then Q has at least one real root ξ_1 of odd multiplicity, which can be extended to a real element of $\mathcal{V}(\mathcal{I})$. Since $\xi_1 \in \mathbb{R}$ is of odd multiplicity, a perturbation $\mathcal{O}(h)$ preserves the existence of a real solution, and the system (5.37) has at least one real solution for all h small enough.

There remains to verify that $Q \not\equiv 0$ is of odd degree. The first step is obvious from Lemma 5.9 since Q is of degree 25 unless the leading term vanishes. In this case,

$$c_4 = \frac{5}{4}c_3^2, \quad Q(\xi_1) = -\frac{1953125}{1048576}\left(7\,c_3^3 - 4\,c_5\right)^4 \xi_1^{23} \ + \ \ldots$$

and further,

$$c_4 = \frac{5}{4}c_3^2,\ c_5 = \frac{7}{4}c_3^3, \quad Q(\xi_1) = -\frac{1953125}{65536}\left(21\,c_3^4 - 8\,c_6\right)^4 \xi_1^{19} + \ldots$$

The additional assumptions

$$c_6 = \frac{21}{8}\,c_3^4,\ c_7 = \frac{33}{8}\,c_3^5,\ c_8 = \frac{429}{64}\,c_3^6$$

reduce the degree of Q to 15, 11 and 7, respectively. Note that regardless of the particular constants c_i the polynomial Q remains to be odd. Finally, the choice $c_9 = \dfrac{715}{64}\,c_3^7$ gives $Q \equiv 0$, and any $\xi_1 \in \mathbb{R}$ is suitable. It is not difficult to guess where this particular data curve comes from. One can easily verify that in this case

$$\sum_{k=2}^{9} c_k h^{k-2} x^k$$

is the Taylor polynomial of the function

$$y(x) = \frac{1}{2c^2} - \frac{x}{c} - \frac{\sqrt{1-4cx}}{2c^2}, \quad c := \frac{h}{2} c_3.$$

A straightforward computation leads to one of the possible reparameterizations of $\boldsymbol{f}$

$$\boldsymbol{f}(z) = \begin{pmatrix} cz - c^3 z^2 \\ c^2 z^2 \end{pmatrix}, \quad z := z(c) := \frac{1 - \sqrt{1 - 4\,c\,x}}{2c^2}.$$

Thus $\boldsymbol{f}$ is a quadratic parametric polynomial. This remains true also as $c_3 \to 0$, since $y(x) \to x^2$ in this case. Of course, the quintic geometric interpolation reproduces the quadratic parametric polynomial. The proof is concluded.

Chapter 6

Circle-like curves

In the previous chapter the conjecture that a parametric polynomial curve of degree $\leq n$ can interpolate $2n$ given points in $\mathbb{R}^2$ has been confirmed for $n \leq 5$ under certain natural restrictions. Furthermore, the optimal asymptotic approximation order $2n$ has been confirmed provided that an interpolating polynomial curve exists. But its existence for a general n has been an open challenge for quite a while. In this chapter the existence of an interpolating curve for a general n is established, provided the data are sampled from a smooth curve sufficiently close to a circular arc a so called *circle-like* curve. Numerical examples confirm the results of the main theorem and suggest that when the whole circle is approximated the error decreases exponentially with growing n.

6.1. Circle-like curves

The approximation of circular arcs is an important task in Computer Aided Geometric Design (CAGD), Computer Aided Design (CAD) and Computer Aided Manufacturing (CAM). Though a circle arc can be exactly represented by a rational quadratic Bézier curve (or, generally, by a rational parametric curve of a low degree, see [13], e.g.), some CAD/CAM systems require a polynomial representation of circular segments. Also, some important algorithms, such as lofting and blending can not be directly applied to rational curves. On the other hand, circular arcs can not be represented by polynomials exactly, thus interpolation or approximation has to be used to represent them accurately.

There are several papers dealing with good approximation of circular segments with radial error as the parametric distance. In [10] the authors study the existence of a cubic Bézier Hermite type interpolant which is sixth-order accurate, and in [19] a similar problem with various boundary conditions is presented. In [35] the problem of approximation of circle segments by quadratic Bézier curves is considered. Lyche and Mørken (see [34]) give an excellent explicit approximation of circular segments by odd degree parametric polynomial curves, and they conjecture that the same problem with even degree could be a tough task. Their method is based on Taylor-type approximation and explicitly provides parametric polynomials of odd degree n with high asymptotic approximation order, i.e., $2\,n$. Probably the most general results on Hermite type polynomial approximation of conic sections by parametric polynomial curves of odd degree are given in [17] and [18]. Several new special types of Hermite interpolation schemes are also given in [11] and [12]. All these results include odd degree interpolating curves as a rule and do not extend to Lagrange type of interpolation directly.

In this chapter geometric interpolation of circular segments and, more generally, so called *circle-like* curves by parametric polynomial curves is studied. *Circle-like* curves are defined in the following way. Let $\boldsymbol{\mathcal{A}}$ be a circular arc of an arclength $h > 0$. Since the term circle-like necessarily involves a comparison of two curves, the arclength parametric representation $\boldsymbol{\mathcal{A}} : [0, h] \mapsto \mathbb{R}^2$ is perhaps the most convenient tool. Suppose that a convex curve $\boldsymbol{f} \approx \boldsymbol{\mathcal{A}}$ is parameterized by the same parameter as $\boldsymbol{\mathcal{A}}$. The curve $\boldsymbol{f}$ will be called *circle-like*, if it agrees twice with $\boldsymbol{\mathcal{A}}$ at 0, has the curvature of the same sign at 0 as well, i.e.,

$$\boldsymbol{f}(0) = \boldsymbol{\mathcal{A}}(0),\ \boldsymbol{f}'(0) = \boldsymbol{\mathcal{A}}'(0),\ \det\left(\boldsymbol{f}'(0), \boldsymbol{f}''(0)\right) \det\left(\boldsymbol{\mathcal{A}}'(0), \boldsymbol{\mathcal{A}}''(0)\right) > 0, \tag{6.1}$$

and its smooth correction $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{\mathcal{A}}$ expands as

$$\boldsymbol{g}(s) = \frac{1}{2!}\,\boldsymbol{g}''(0)\,s^2 + \frac{1}{3!}\,\boldsymbol{g}^{(3)}(0)\,s^3 + \dots \tag{6.2}$$

In order to make a distinction among the circle-like curves, we introduce a constant M,

$$\max_{2 \le r \le 2n-1} \left\|\boldsymbol{g}^{(r)}(0)\right\|_\infty \le M, \tag{6.3}$$

that bounds the magnitudes of derivatives at 0. For any particular M, the corresponding set of circle-like curves will be denoted by $\mathbb{F}_M$.

As a motivation, let us consider the following numerical example. Let

$$\boldsymbol{f}(t) = \exp(t/4)\begin{pmatrix}\sin t\\ 1-\cos t\end{pmatrix}, \quad t \in \left[0, \frac{\pi}{2}\right], \tag{6.4}$$

be a particular exponential spiral, geometrically interpolated by a polynomial parametric curve of degree $n = 6$ and $n = 7$, respectively, at $2\,n$ points obtained by the equidistant splitting of the parameter interval. The curve (6.4) is clearly a circle-like one. Figure 6.1 (left) shows the curve (6.4) and the circular arc, and Table 6.1 gives numerical evidence of the approximation error measured as a parametric distance between the curve (6.4) and its geometric interpolant. A simple error analysis indicates that the asymptotic approximation order is $\mathcal{O}(h^{2n})$, at least for $n = 6,\ 7$.

6.2. Interpolation problem

The interpolation problem considered is the problem (1.3) introduced in Chapter 1 and further considered in Chapter 5. But here the data points $\boldsymbol{T}_j$ are sampled from a smooth circle-like curve

$$\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} : [0, h] \to \mathbb{R}^2.$$

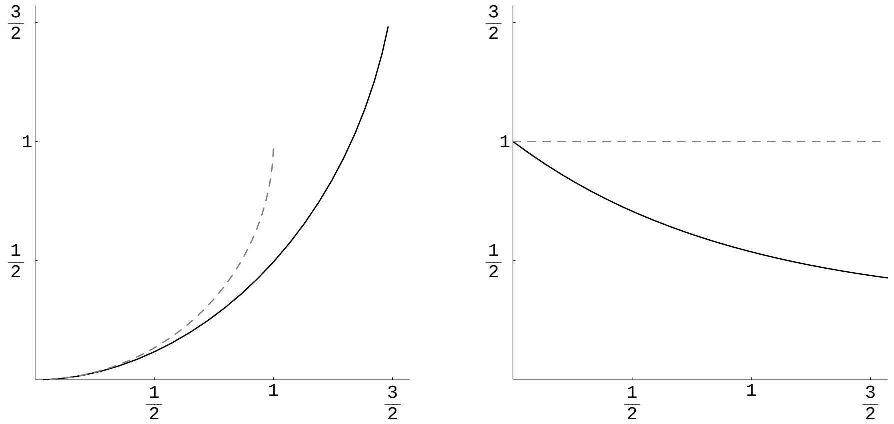


Figure 6.1: The exponential spiral (6.4) and the circular arc (left) and their curvatures (right).

Interval	Approximation error		Decay exponent	
	$n=6$	$n=7$	$n=6$	$n=7$
$\left[0,\frac{\pi}{2}\right]$	1.7783×10^{-11}	2.4704×10^{-13}	—	—
$\left[0,\frac{7\pi}{16}\right]$	2.9789×10^{-11}	3.6024×10^{-14}	3.86	- 14.42
$\left[0,\frac{6\pi}{16}\right]$	4.3754×10^{-12}	3.9342×10^{-15}	- 12.44	- 14.37
$\left[0,\frac{5\pi}{16}\right]$	4.5808×10^{-13}	2.8953×10^{-16}	- 12.38	- 14.31
$\left[0,\frac{4\pi}{16}\right]$	2.9377×10^{-14}	2.4957×10^{-18}	- 12.31	- 21.30
$\left[0,\frac{3\pi}{16}\right]$	8.6811×10^{-16}	4.4201×10^{-20}	- 12.24	- 14.02
$\left[0,\frac{2\pi}{16}\right]$	6.2403×10^{-18}	6.5502×10^{-22}	- 12.17	- 10.39
$\left[0,\frac{\pi}{16}\right]$	1.4208×10^{-21}	3.7763×10^{-26}	- 12.10	- 14.08

Table 6.1: The error of geometric interpolation of the exponential spiral (6.4).

Since affine transformations of data points do not change the solution of equations (5.5), one can place the origin of a coordinate system at $\boldsymbol{f}(0)=\boldsymbol{\mathcal{A}}(0)$, and choose $\boldsymbol{\mathcal{A}}$ to be the unit circle, centered at $(0,1)^T$. Thus

$$\boldsymbol{\mathcal{A}}(t) := (\sin t, 1-\cos t)^T.$$

But then the expansion (6.2) implies $\boldsymbol{f}'(0)=\boldsymbol{\mathcal{A}}'(0)=(1,0)^T$. Therefore, for h small enough, $\boldsymbol{f}=(f_i)_{i=1}^2$ can be reparameterized as

$$\boldsymbol{f}(s) := \begin{pmatrix} s \\ u(s) \end{pmatrix} := \begin{pmatrix} s \\ \alpha(s)+\gamma(s) \end{pmatrix}, \tag{6.5}$$

where

$$\alpha(s) := 1-\sqrt{1-s^2}$$

is a circular arc, parameterized by the first component, and

$$\gamma(s) = f_2\left(f_1^{-1}(s)\right)-\alpha(s) = \frac{\gamma''(0)}{2!}s^2+\frac{\gamma^{(3)}(0)}{3!}s^3+\ldots. \tag{6.6}$$

The coefficients $\gamma^{(i)}(0)$ in (6.6) are polynomials in the components of $\boldsymbol{g}^{(r)}(0)$, but with the constant term equal to 0. Indeed, with $\nu(s) := f_1^{-1}(s) - \arcsin s$, and an obvious conclusion $\nu(0) = 0$ as well as $\nu'(0) = 0$ since

$$f_1^{-1}(0) = 0, \quad \frac{d}{ds} f_1^{-1}(s)\Big|_{s=0} = \frac{1}{f_1'(0)} = 1,$$

we obtain

$$\begin{aligned}\gamma(s) &= 1 - \cos(f_1^{-1}(s)) + g_2(f_1^{-1}(s)) - \alpha(s)\\ &= \sqrt{1-s^2}\,(1 - \cos(\nu(s))) + s\sin(\nu(s)) + g_2(f_1^{-1}(s))\\ &= \frac{1}{2!}\,g_2''(0)s^2 + \left(\frac{1}{2!}\,(1 + g_2''(0))\,\nu''(0) + \frac{1}{3!}g_2^{(3)}(0)\right)s^3 + \ldots,\end{aligned}$$

and the claim will be confirmed if $\nu^{(i)}(0)$ are polynomials in $g_1^{(r)}(0)$ without the constant term. This fact could be formally verified by an application of Faa di Bruno's formula to the implicit definition of ν, i.e., $f_1(\nu(s) + \arcsin s) - s = 0$ and the induction, but the following expansion is even more convincing,

$$\begin{aligned}0 = f_1(\nu(s) + \arcsin s) - s =& \frac{1}{2!}\,(g_1''(0) + \nu''(0))\,s^2\\ &+ \frac{1}{3!}\left(g_1^{(3)}(0) + 3g_1''(0)\nu''(0) + \nu^{(3)}(0)\right)s^3\\ &+ \frac{1}{4!}\Big(g_1^{(4)}(0) + 6\,g_1^{(3)}(0)\nu''(0)\\ &+ g_1''(0)\left(4 + 3\,(\nu''(0))^2 + 4\,\nu^{(3)}(0)\right)\\ &- 6\,\nu''(0) + \nu^{(4)}(0)\Big)\,s^4 + \ldots.\end{aligned}$$

So one can find a bound $c(M)$,

$$|\gamma^{(i)}(0)| \le c(M), \quad i = 2, 3, \ldots, 2n-1,$$

depending only on M that was introduced in (6.3). This bound can be chosen as a nondecreasing continuous function of M, starting with $c(0) = 0$, since $\boldsymbol{g} \equiv \boldsymbol{0}$ implies $\gamma = 0$. This proves the following lemma.

LEMMA 6.1. *With a proper choice of M, a circle-like curve $\boldsymbol{f} \in \mathbb{F}_M$ has the correction γ and its derivatives arbitrary small.*

Now, since $\boldsymbol{f}$ is of the form (6.5), the expansion (6.6) and assumptions (6.1) imply

$$u(0) = u'(0) = 0, \quad u''(0) > 0.$$

With this assumption the analysis carried out in Chapter 5 shows that the asymptotic existence of the solution of the interpolation problem (1.3) is in general equivalent to a fact that a system of nonlinear equations

$$C_{n+j}(\boldsymbol{a}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \ldots, n-1, \tag{6.7}$$

has a real solution $\boldsymbol{a} := (a_\ell)_{\ell=1}^{n-1}$ for all h small enough.

For circle-like curves (6.5), the functions C_{n+j} simplify to

$$C_{n+j}(\boldsymbol{a}) := \frac{1}{(n+j)!}\frac{d^{n+j}}{dt^{n+j}}\left(\alpha\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)+\gamma\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)\right)\Bigg|_{t=0}.$$

Further discussion will prove that the system

$$\frac{1}{(n+j)!}\frac{d^{n+j}}{dt^{n+j}}\left(\alpha\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)\right)\Bigg|_{t=0} = 0, \quad j=1,2,\ldots,n-1, \tag{6.8}$$

has a real solution and that the Jacobian at that solution is nonsingular. So the Implicit Function Theorem implies the existence of constant $c(M)$ for M small enough such that the equations

$$C_{n+j}(\boldsymbol{a}) = 0, \quad j=1,2,\ldots,n-1,$$

for circle-like curves that satisfy (6.3) for this particular M also have a real solution with a nonsingular Jacobian, by Lemma 6.1. But then, again by the Implicit Function Theorem, the system (6.7) has a real solution for h small enough too, which proves the next theorem.

6.3. The main results

THEOREM 6.2. *Let $\boldsymbol{\mathcal{A}} : [0,h] \mapsto \mathbb{R}^2$ be a circular arc, parameterized by the arclength. There exist positive constants M and h_0 with $h_0 \leq h$, such that for any $h_1 \leq h_0$, any circle-like curve $\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} \in \mathbb{F}_M$ can be geometrically interpolated by a polynomial parametric curve of degree $\leq n$ at $2n$ distinct points $\boldsymbol{f}(s_i)$, $s_i \in [0,h_1]$. The asymptotic approximation order is optimal, i.e., equal to $2n$.*

Now one is left to show the existence of the solution of (6.8) and to prove the nonsingularity of the Jacobian. The expansion

$$1-\alpha\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right) = \sqrt{1-\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)^2} =: 1+\sum_{\ell=1}^{\infty} b_\ell\, t^\ell \tag{6.9}$$

yields

$$\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)^2 + \left(1+\sum_{\ell=1}^{\infty} b_\ell\, t^\ell\right)^2 = 1. \tag{6.10}$$

Since the equations (6.8) for $h \to 0$ are equivalent to the fact that the expansion (6.9) does not contain the powers $n+1, n+2, \ldots, 2n-1$, the relation (6.10) implies

$$\left(t+\sum_{\ell=1}^{n-1} a_\ell\, t^{\ell+1}\right)^2 + \left(1+\sum_{\ell=1}^{n} b_\ell\, t^\ell\right)^2 = 1 + (a_{n-1}^2 + b_n^2)\, t^{2n}. \tag{6.11}$$

One is now left with $2n-1$ equations for $2n-1$ unknowns $\boldsymbol{a}$ and $\boldsymbol{b} := (b_\ell)_{\ell=1}^n$. Once $\boldsymbol{a}$ is obtained, linear relations determine $\boldsymbol{b}$ and vice-versa.

Let

$$\alpha_1 := \frac{1}{\sqrt[2n]{a_{n-1}^2 + b_n^2}}.$$

A regular reparameterization $t \to \alpha_1 \cdot t$, and new variables

$$\alpha_j := (\alpha_1)^j \, a_{j-1}, \quad j = 2, 3, \ldots, n,$$
$$\beta_0 := 1, \quad \beta_j := (\alpha_1)^j \, b_j, \quad j = 1, 2, \ldots, n,$$

simplify (6.11) to the problem of finding two nonconstant polynomials $x_n,\ y_n \in \mathbb{R}[t]$ of degree $\leq n$ such that

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n}, \quad x_n(0) = 0. \tag{6.12}$$

Now, let

$$z_n(t) := x_n^2(t) + y_n^2(t) - \left(1 + t^{2n}\right). \tag{6.13}$$

The relation (6.12) can also be considered as a system of nonlinear equations for the coefficients of the polynomials

$$x_n(t) = \sum_{j=1}^{n} \alpha_j \, t^j, \quad y_n(t) = \sum_{j=0}^{n} \beta_j \, t^j, \tag{6.14}$$

i.e.,

$$\frac{d^j}{dt^j} z_n(t)\Big|_{t=0} = 0, \quad j = 0, 1, \ldots, 2n. \tag{6.15}$$

Although the solutions of the system of nonlinear equations given by (6.12) can be obtained numerically for particular values of n, finding a closed form solution is a much more complicated problem. The importance of equation (6.12) has already been noted in [34] considering a slightly different approximation problem that will be described in Subsection 6.5, and the existence of a solution has been established for odd n. The authors proposed a very nice approach to solve this problem. They have used a particular rational parameterization of the unit circle to obtain the coefficients of the polynomials x_n and y_n. Indeed, if

$$x_0(t) := \frac{2\,t}{1+t^2}, \quad y_0(t) := \frac{1-t^2}{1+t^2}, \quad t \in (-\infty, \infty), \tag{6.16}$$

is a parameterization of a unit circle, then the functions

$$x_n(t) := x_0(t) - (-1)^{(n-1)/2} \, t^n \, y_0(t),$$
$$y_n(t) := y_0(t) + (-1)^{(n-1)/2} \, t^n \, x_0(t),$$

are actually polynomials of degree $\leq n$ for which (6.12) holds. It is also easy to find their explicit form, but unfortunately if n is even, their coefficients are no more real numbers. However, this idea can be applied for even n too, but a slightly different rational parameterization of the unit circle has to be considered. Namely, let x_0, y_0 be redefined as

$$x_0(t) := \frac{2\sqrt{1-c^2}\,t\,(1-c\,t)}{1-2\,c\,t+t^2}, \quad y_0(t) := \frac{1-2\,c\,t+(2\,c^2-1)\,t^2}{1-2\,c\,t+t^2}, \tag{6.17}$$

where $c \in [0,1)$. It is straightforward to see that $x_0^2(t) + y_0^2(t) = 1$. The polynomials (6.14) depend heavily on the degree n of the interpolating curve, and throughout this chapter it will be assumed that the integers n, k, and r are related as

$$n = 2^k\,(2\,r-1), \quad k \geq 0, \quad r \geq 1. \tag{6.18}$$

Each k determines a family of polynomials that satisfy (6.12). In addition, the coefficients of the polynomials x_n and y_n can be given in a closed form with the help of Chebyshev polynomials of the first and the second kind, T_n and U_n.

THEOREM 6.3. *Suppose that n, k, and r satisfy (6.18), and let the constants c_k, s_k be given as*

$$c_k := \cos\left(\frac{\pi}{2^{k+1}}\right), \quad s_k := \sin\left(\frac{\pi}{2^{k+1}}\right). \tag{6.19}$$

Further, suppose that q_i are polynomials of degree ≤ 2, defined as

$$\begin{aligned} q_0(t) &:= q_0(t;k) := 1-2\,c_k\,t+t^2,\\ q_1(t) &:= q_1(t;k) := 2\,s_k\,t\,(1-c_k\,t),\\ q_2(t) &:= q_2(t;k) := 1-2\,c_k\,t+(2\,c_k^2-1)\,t^2. \end{aligned} \tag{6.20}$$

Then, the functions x_n and y_n, defined by

$$\begin{pmatrix} x_n(t)\\ y_n(t)\end{pmatrix} := \frac{1}{q_0(t)} \begin{pmatrix} 1 & (-1)^r\,t^n\\ -(-1)^r\,t^n & 1\end{pmatrix} \begin{pmatrix} q_1(t)\\ q_2(t)\end{pmatrix}, \tag{6.21}$$

are polynomials of degree $\leq n$ that satisfy (6.12)*. Furthermore, their coefficients are given as*

$$\alpha_j = 2\,s_k\,\cos\left((j-1)\frac{\pi}{2^{k+1}}\right) = 2\,s_k\,T_{j-1}(c_k), \quad j = 1,2,\ldots,n-1, \tag{6.22}$$

$$\alpha_n = 2\,s_k\,\cos\left((n-1)\frac{\pi}{2^{k+1}}\right) + (-1)^r = 2\,s_k\,T_{n-1}(c_k) + (-1)^r, \tag{6.23}$$

and

$$\beta_0 = 1,\ \beta_1 = 0, \tag{6.24}$$

$$\beta_j = -2\,s_k\,\sin\left((j-1)\frac{\pi}{2^{k+1}}\right) = -2\,s_k^2\,U_{j-2}(c_k), \quad j = 2,3,\ldots,n. \tag{6.25}$$

Theorem 6.3 actually proves the optimal approximation order for the circular arcs as studied in [34]. But Theorem 6.2 extends the conclusion to Lagrange interpolation of circular arcs and of the circle-like curves of degree n. All that is left to prove is that the Jacobian of the system of equations (6.15) with respect to the variables α_j, β_j at values provided by Theorem 6.3 is nonsingular. A surprisingly simple closed form of the determinant of the Jacobian that confirms this fact is given in the next theorem.

THEOREM 6.4. *With n, k, and r as in* (6.18)*, and α_j, β_j given by Theorem 6.3, the determinant of the Jacobian of the system* (6.15) *is*

$$\det J = (-1)^{nr+1}\, 2^{2n+1}\, n^2\, s_k^2.$$

The asymptotic conclusion of Theorem 6.2 seems to be rather pessimistic since the parameter interval is supposed to be small. As an impetus, consider the approximation of a complete circle. A quick numerical test in Table 6.2 shows that the circle

$$\begin{pmatrix} \sin s \\ \cos s \end{pmatrix}, \quad s \in [-\pi, \pi], \tag{6.26}$$

can be geometrically interpolated at points corresponding to the parameter values

$$-\pi + \frac{2\pi}{2n-1}\ell, \quad \ell = 0, 1, \ldots, 2n-1,$$

by a polynomial curve of small degree n quite accurately. The error seems to decrease

n	Approximation error	τ	n	Approximation error	τ
3	2.85951×10^{-1}	—	10	7.28389×10^{-11}	-1.21
4	2.32476×10^{-2}	-1.11	11	1.14441×10^{-12}	-1.24
5	2.08441×10^{-3}	-0.96	12	1.49890×10^{-14}	-1.26
6	1.22589×10^{-4}	-1.05	13	1.66223×10^{-16}	-1.28
7	5.09328×10^{-6}	-1.11	14	1.58128×10^{-18}	-1.29
8	1.57805×10^{-7}	-1.15	15	1.30483×10^{-20}	-1.30
9	3.79252×10^{-9}	-1.19	16	9.42975×10^{-23}	-1.31

Table 6.2: The error of geometric interpolation of the circle measured as radial distance.

exponentially with n, like $\mathcal{O}(n^{\tau n}), \tau \approx -1.30$. Figure 6.2 shows that the curvatures of geometric interpolants are close to 1 (the curvature of the circle). As can be seen from Table 6.3, they approach 1 with growing n.

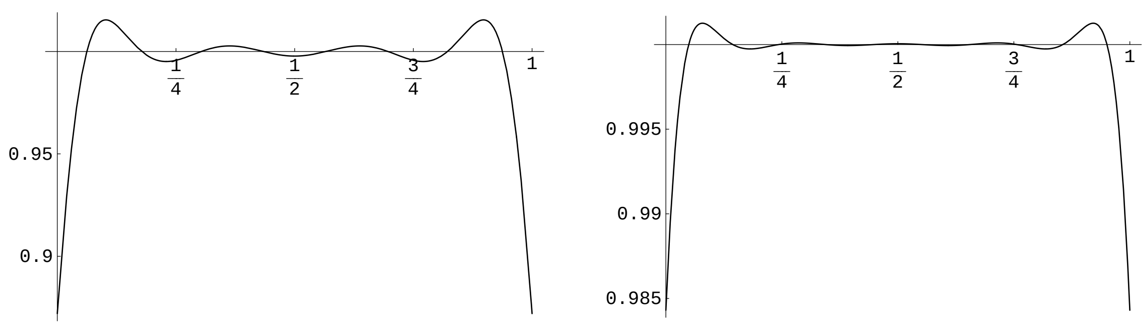


Figure 6.2: The curvatures of geometric interpolants of the complete circle for $n = 5, 6$.

n	3	4	5	6	7	8
$\Vert 1-\kappa_n\Vert_{\infty,[0,1]}$	1.59739	0.48393	0.12798	0.01569	0.00110	0.00005

Table 6.3: Maximal deviation from 1 of the curvatures κ_n of degree n geometric interpolants of the complete circle.

6.4. Proofs

From the previous discussion it is obvious that it suffices to prove Theorem 6.3 and Theorem 6.4 only. Theorem 6.2 then follows as a corollary. Consider the proof of Theorem 6.3 first. Equation (6.21) yields

$$x_n(t) = \frac{q_1(t) + (-1)^r\, t^n\, q_2(t)}{q_0(t)}, \quad y_n(t) = \frac{q_2(t) - (-1)^r\, t^n\, q_1(t)}{q_0(t)}, \tag{6.27}$$

where q_i, $i = 0, 1, 2$, are defined by (6.20). In order to verify that the function x_n is actually a polynomial of the form (6.14), by (6.27) it is sufficient to check that

$$\begin{aligned} q_0(t) \sum_{j=1}^{n} \alpha_j\, t^j = \alpha_1\, t + (\alpha_2 - 2\, c_k\, \alpha_1)\, t^2 + \sum_{j=3}^{n} (\alpha_j - 2\, c_k\, \alpha_{j-1} + \alpha_{j-2})\, t^j \\ + (-2\, c_k\, \alpha_n + \alpha_{n-1})\, t^{n+1} + \alpha_n\, t^{n+2} = q_1(t) + (-1)^r\, t^n\, q_2(t). \end{aligned}$$

A comparison of the coefficients implies the linear recurrence

$$\alpha_1 = 2s_k,\ \alpha_2 = c_k\, \alpha_1,\ \alpha_j - 2\, c_k\, \alpha_{j-1} + \alpha_{j-2} = 0, \quad j = 3, 4, \ldots, n-1, \tag{6.28}$$

with additional conditions

$$\begin{aligned} \alpha_n - 2\, c_k\, \alpha_{n-1} + \alpha_{n-2} &= (-1)^r, \\ 2\, c_k\, \alpha_n - \alpha_{n-1} &= (-1)^r\, 2\, c_k, \\ \alpha_n &= (-1)^r\, (2\, c_k^2 - 1). \end{aligned} \tag{6.29}$$

A straightforward calculation confirms that (6.22) and (6.23) give a solution of (6.28) and (6.29). The proof for the function y_n is similar and will be omitted.

Since by Theorem 6.3 the function (6.13) vanishes identically, the limit solution of the system of equations (6.7) is obtained. The existence of a real solution is verified if the Jacobian at the limit solution is nonsingular. From

$$\begin{aligned} \frac{\partial}{\partial \alpha_j}\, z_n(t) &= \frac{\partial}{\partial \alpha_j}\, x_n^2(t) = 2\, t^j\, x_n(t), \quad j = 1, 2, \ldots, n, \\ \frac{\partial}{\partial \beta_j}\, z_n(t) &= \frac{\partial}{\partial \beta_j}\, y_n^2(t) = 2\, t^j\, y_n(t), \quad j = 0, 1, \ldots, n, \end{aligned}$$

it is straightforward to compute the Jacobian $J := 2\,D$, where

$$D := \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & \beta_0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & \beta_1 & \beta_0 & 0 & & \vdots & 0 \\
\alpha_1 & 0 & \ddots & & \vdots & \vdots & \beta_1 & \beta_0 & \ddots & \vdots & \vdots \\
\alpha_2 & \alpha_1 & \ddots & \ddots & \vdots & \vdots & \vdots & \beta_1 & \ddots & 0 & \vdots \\
\vdots & \alpha_2 & \ddots & 0 & 0 & \vdots & \vdots & \vdots & \ddots & \beta_0 & 0 \\
\vdots & \vdots & \ddots & \alpha_1 & 0 & \beta_n & \vdots & \vdots & \ddots & \beta_1 & \beta_0 \\
\alpha_n & \vdots & & \alpha_2 & \alpha_1 & 0 & \beta_n & \vdots & & \vdots & \beta_1 \\
0 & \alpha_n & & \vdots & \alpha_2 & \vdots & 0 & \beta_n & & \vdots & \vdots \\
\vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \alpha_n & \vdots & \vdots & \vdots & \vdots & \ddots & \beta_n & \vdots \\
0 & 0 & \cdots & 0 & \alpha_n & 0 & 0 & 0 & \cdots & 0 & \beta_n
\end{pmatrix}.$$

Unfortunately, obtaining the explicit formula for $\det D$ is not an easy task, since its entries are given by (6.22)–(6.25). But the columns of D are simply the shifts of the coefficients of x_n and y_n, which leads to the following observation. If

$$u_0 := 0, u_1, u_2, \ldots, u_{2n} \in \mathbb{C}$$

are $2n+1$ pairwise distinct values, and

$$V(u_0, u_1, \ldots, u_{2n}) := \left(u_{j-1}^{\ell-1}\right)_{j,\ell=1}^{2n+1}$$

is the corresponding Vandermonde matrix, then the rows of the product VD are given by

$$u_j\,x_n(u_j),\ u_j^2\,x_n(u_j), \ldots,\ u_j^n\,x_n(u_j),\ y_n(u_j),\ u_j\,y_n(u_j), \ldots,\ u_j^n\,y_n(u_j), \tag{6.30}$$

where $j = 0, 1, \ldots, 2\,n$. Now (6.12) suggests how to choose u_j, i.e., to define u_j as $2\,n$ different solutions of the equation

$$t^{2\,n} + 1 = 0. \tag{6.31}$$

Then (6.12) implies $x_n(u_j) = \pm i\,y_n(u_j)$. Here i will denote the imaginary unit, i.e., $i^2 = -1$. If

$$\begin{aligned}
u_j &:= \exp\left((-1)^r \frac{i\,\pi}{2\,n}(4\,j-3)\right), \quad j = 1, 2 \ldots, n, \quad j \neq j_0, \\
u_{j_0} &:= \exp\left(\frac{i\,\pi}{2\,n}(2\,r-1)\right), \\
u_{n+j} &:= u_j^{-1}, \quad j = 1, 2, \ldots, n,
\end{aligned}$$

where

$$j_0 = \begin{cases} \frac{2n-r+2}{2}, & r \text{ even}, \\ \frac{r+1}{2}, & r \text{ odd}, \end{cases}$$

then

$$y_n(u_j) = -i\, x_n(u_j), \quad j = 1, 2, \ldots, n,$$
$$y_n(u_j) = i\, x_n(u_j), \quad j = n+1, n+2, \ldots, 2\,n, \tag{6.32}$$

as can easily be verified. Recall that $u_0 = 0$, $y_n(0) = 1$, and use (6.30) and (6.32) to see that

$$\det(V\,D) = i^n \prod_{j=1}^{2\,n} x_n(u_j) \det C,$$

where

$$C := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ u_1 & u_1^2 & \cdots & u_1^n & -1 & -u_1 & \cdots & -u_1^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ u_n & u_n^2 & \cdots & u_n^n & -1 & -u_n & \cdots & -u_n^n \\ u_{n+1} & u_{n+1}^2 & \cdots & u_{n+1}^n & 1 & u_{n+1} & \cdots & u_{n+1}^n \\ u_{n+2} & u_{n+2}^2 & \cdots & u_{n+2}^n & 1 & u_{n+2} & \cdots & u_{n+2}^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ u_{2\,n} & u_{2\,n}^2 & \cdots & u_{2\,n}^n & 1 & u_{2\,n} & \cdots & u_{2\,n}^n \end{pmatrix}.$$

Since $J = 2D$,

$$\det J = 2^{2\,n+1}\, i^n \prod_{j=1}^{2\,n} x_n(u_j) \frac{\det C}{\det V}. \tag{6.33}$$

LEMMA 6.5. *If s_k is given by (6.19), then*

$$\prod_{j=1}^{2\,n} x_n(u_j) = n^4\, s_k^4.$$

Proof. Some straightforward computation and (6.27) yield

$$x_n(u_j) x_n(u_{n+j}) = \frac{\sin^2\left(\frac{\pi}{4\,n}((-1)^r(4\,j-3) + 2\,r - 1)\right)}{\sin^2\left(\frac{\pi}{4\,n}((-1)^r(4\,j-3) - 2\,r + 1)\right)}, \; j = 1, 2, \ldots, n, \; j \neq j_0,$$

and, by the L'Hôspital rule,

$$x_n(u_{j_0})\, x_n(u_{n+j_0}) = n^2\, s_k^2. \tag{6.34}$$

The formulae (see [20], e.g.)

$$\prod_{j=1}^{n} \sin\left(\frac{\pi}{2n}(2j - 2m - 1)\right) = \frac{(-1)^m}{2^{n-1}}, \quad m \in \mathbb{Z},$$
$$\prod_{\substack{j=1 \\ j\neq m}}^{n} \sin\left(\frac{\pi}{2n}(2j - 2m)\right) = \frac{(-1)^{m+1} n}{2^{n-1}}, \quad 1 \leq m \leq n,$$

imply

$$\prod_{\substack{j=1 \\ j\neq j_0,\, n+j_0}}^{2n} x_n(u_j) = n^2\, s_k^2,$$

which, together with (6.34), completes the proof of the lemma. □

LEMMA 6.6. *The quotient of determinants in (6.33) is*

$$\frac{\det C}{\det V} = \frac{(-1)^{n\,r+1}}{i^n\, n^2\, s_k^2}.$$

Proof. The determinant of C can be reduced to

$$\det C = (-1)^n \prod_{j=1}^{2\,n} u_j \,\det \begin{pmatrix} V_1 & -V_1 \\ V_2 & V_2 \end{pmatrix},$$

where $V_1 := V(u_1, u_2, \ldots, u_n)$ and $V_2 := V(u_{n+1}, u_{n+2}, \ldots, u_{2\,n})$ are the corresponding Vandermonde matrices. Since u_j are the roots of (6.31), $\prod_{j=1}^{2\,n} u_j = 1$. Further, a simple columnwise reduction implies that

$$\det C = (-1)^n\, 2^n \det V_1 \det V_2,$$

which finally gives

$$\frac{\det C}{\det V} = (-1)^n\, 2^n \frac{1}{\prod_{\ell=1}^{n} \prod_{j=1}^{n} (u_{n+\ell} - u_j)}.$$

If

$$p_1 := \prod_{\substack{\ell=1 \\ \ell\neq j_0}}^{n} \prod_{\substack{j=1 \\ j\neq j_0}}^{n} (u_{n+\ell} - u_j),$$

$$p_2 := (u_{n+j_0} - u_{j_0}) \prod_{\substack{j=1 \\ j\neq j_0}}^{n} (u_{n+j_0} - u_j) \prod_{\substack{\ell=1 \\ \ell\neq j_0}}^{n} (u_{n+\ell} - u_{j_0}),$$

then obviously

$$\prod_{\ell=1}^{n} \prod_{j=1}^{n} (u_{n+\ell} - u_j) = p_1\, p_2.$$

A straightforward computation yields

$$p_1 = (-1)^{n\,r}\, i^{n+1}\, 2^{n-1}\, s_k,$$
$$p_2 = (-1)^n\, n^2\, 2\, i\, s_k,$$

thus

$$\frac{\det C}{\det V} = \frac{(-1)^{n\,r+1}}{i^n\, n^2\, s_k^2}.$$

□

Lemma 6.6 and (6.33) confirm the result of Theorem 6.4 which concludes this subsection.

6.5. Approximation of circular arcs

In this subsection a generalization of the results obtained by Lyche and Mørken in [34] on the approximation of circular arcs by parametric polynomial curves of odd degrees is given. The results of the previous discussion are used to show that for the circular arc of the angular length h, a parametric polynomial curve of arbitrary degree $n \in \mathbb{N}$, which interpolates a given arc at a particular point, can be constructed with a radial distance bounded by h^{2n}.

The approximation problem considered is the following. Let

$$\boldsymbol{A}(\varphi) := \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix}, \quad 0 \le \varphi < L \le 2\pi, \tag{6.35}$$

be a particular parameterization of a circular arc of an angular length L. Find a parametric polynomial curve

$$\boldsymbol{q}_n := \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{6.36}$$

of degree $\le n$ with nonconstant scalar polynomials $x_n, y_n \in \mathbb{R}[t]$ defined in (6.14), which provides "the best approximation" of (6.35). The only prescribed interpolation point is $\boldsymbol{A}(0) := (x_n(0), y_n(0))^T := (0,1)^T$. Note that it is enough to consider arcs of the unit circle only, since any other arc of the same angular length can be obtained by affine transformations.

The approximation error between a circular arc and a parametric polynomial curve is measured as a *"radial distance"* (see Figure 6.3), i.e.,

$$d_r(\boldsymbol{A}, \boldsymbol{q}_n) := \max_{t\in I} \left\{ \left| \sqrt{x_n^2(t) + y_n^2(t)} - 1 \right| \right\}, \tag{6.37}$$

where I is some interval of observation. If $\boldsymbol{q}_n$ is a good approximation of $\boldsymbol{A}$ on I then (6.37) is small and $\sqrt{x_n^2(t) + y_n^2(t)} \approx 1$, thus

$$\left| \sqrt{x_n^2(t) + y_n^2(t)} - 1 \right| = \frac{|x_n^2(t) + y_n^2(t) - 1|}{\sqrt{x_n^2(t) + y_n^2(t)} + 1} \approx \frac{1}{2} \left| x_n^2(t) + y_n^2(t) - 1 \right|,$$

and, for computational purposes, it is enough to consider only the "error"

$$e(t) := \left| x_n^2(t) + y_n^2(t) - 1 \right|. \tag{6.38}$$

Ideally, e would be zero if a polynomial parameterization of a circular arc would exist. But if at least one of x_n or y_n is of degree n, then

$$x_n^2(t) + y_n^2(t) = (\alpha_n^2 + \beta_n^2)\, t^{2n} + \cdots \ne 1. \tag{6.39}$$

Now it follows from (6.38) that e will be small (at least for small t), if coefficients at the lower degree terms in (6.39) will vanish. This implies that e will be as small as possible if

$$x_n^2(t) + y_n^2(t) = 1 + \text{const} \cdot t^{2n}. \tag{6.40}$$

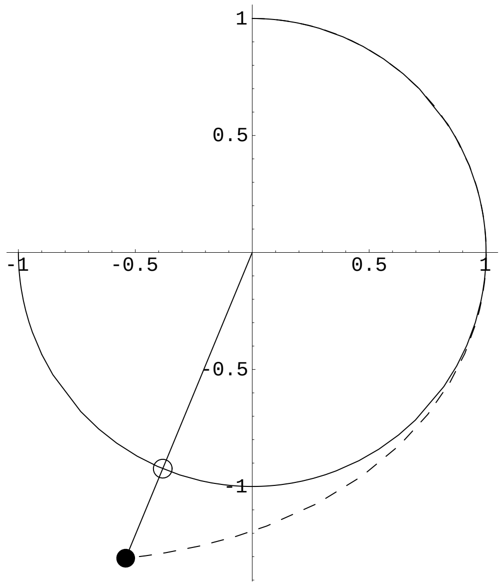


Figure 6.3: Radial distance between a circular arc (solid) and a parametric curve approximation (dashed).

Now, a proper reparameterization

$$t \to \frac{t}{\sqrt[2n]{\alpha_n^2 + \beta_n^2}}$$

transforms (6.40) to relation (6.12), and Theorem 6.3 (eq. (6.22)–(6.25)) gives the closed-form solution for the coefficients of polynomials x_n and y_n.

The approximation order is stated in the following lemma.

LEMMA 6.7. *Let the circular arc $\boldsymbol{A}$ be defined by* (6.35) *and its parametric approximation $\boldsymbol{q}_n$ by* (6.36). *Let the coefficients of x_n and y_n be given by* (6.22)–(6.25). *If $\boldsymbol{q}_n : [0, h] \to \mathbb{R}^2$, where h is sufficiently small, then*

$$d_H(\boldsymbol{A}, \boldsymbol{q}_n) \le d_P(\boldsymbol{A}, \boldsymbol{q}_n) \le d_r(\boldsymbol{A}, \boldsymbol{q}_n) \le h^{2n},$$

where d_r is defined by (6.37).

Proof. It is clear that for a particular θ, which is a regular reparameterization of $\boldsymbol{A}$ on $[0, h]$,

$$d_P(\boldsymbol{A}, \boldsymbol{q}_n) \le \max_{t \in [0,h]} ||\boldsymbol{A}(\theta(t)) - \boldsymbol{q}_n(t)||_2.$$

Thus, it is enough to find a regular reparameterization θ of $\boldsymbol{A}$ for which

$$\max_{t \in [0,h]} ||\boldsymbol{A}(\theta(t)) - \boldsymbol{q}_n(t)||_2 \le h^{2n}.$$

Let $\theta : [0, h] \to I$ be defined as

$$\theta(t) := \arctan\left(\frac{x_n(t)}{y_n(t)}\right). \tag{6.41}$$

Since $x_n(0) = 0$, $y_n(0) = 1$ and by (6.22) $x_n'(0) = 2\,s_k$,

$$\theta'(0) = \frac{x_n'(0)\,y_n(0) - x_n(0)\,y_n'(0)}{x_n^2(0) + y_n^2(0)} = 2\,s_k > 0,$$

and there exists $h_0 > 0$, such that θ is a regular reparameterization on $[0, h]$ for $0 < h < h_0$. But a point $(\boldsymbol{A} \circ \theta)(t)$ lies on the circular arc defined by $\boldsymbol{A}$ and on the ray from the origin to $\boldsymbol{q}_n(t)$. This implies

$$||(\boldsymbol{A} \circ \theta)(t) - \boldsymbol{q}_n(t)||_2 = |\sqrt{x_n^2(t) + y_n^2(t)} - 1| \leq |x_n^2(t) + y_n^2(t) - 1| = t^{2\,n},$$

where the last equality follows from (6.12). Finally,

$$\max_{t\in[0,h]} ||(\boldsymbol{A} \circ \theta)(t) - \boldsymbol{q}_n(t)||_2 \leq h^{2\,n}$$

and the proof of the lemma is complete. □

An interesting question is, how large can be the angular length of the circular arc, which can be approximated by the previous method. First of all, the regularity of θ has to be assured, i.e., h has to be small enough. Then the angular length of the reparameterized circular arc $\boldsymbol{A} \circ \theta$ can be derived at least asymptotically.

LEMMA 6.8. *If θ is a regular reparameterization on $[0, h]$ defined by (6.41), then the length of the circular arc $\boldsymbol{A} \circ \theta : [0, h] \to \mathbb{R}^2$ is $2\,s_k\,h + \mathcal{O}(h^2)$.*

Proof. The proof is straightforward. The regularity of θ, (6.12), (6.22)–(6.25) and the fact that $(1 + t^{2\,n})^{-1} = 1 + \mathcal{O}(t^{2\,n})$, simplify the arc-length to

$$\begin{aligned} s &= \int_0^h \|(\boldsymbol{A} \circ \theta)'(t)\|_2 \, dt = \int_0^h \frac{|x_n'(t)\,y_n(t) - x_n(t)\,y_n'(t)|}{x_n^2(t) + y_n^2(t)} \, dt \\ &= \int_0^h \frac{x_n'(t)\,y_n(t) - x_n(t)\,y_n'(t)}{1 + t^{2\,n}} \, dt \\ &= \int_0^h (x_n'(t)\,y_n(t) - x_n(t)\,y_n'(t))(1 + \mathcal{O}(t^{2\,n})) \, dt = 2\,s_k\,h + \mathcal{O}(h^2). \end{aligned}$$

□

Since we know that the best local approximation at a particular point in the functional case is the Taylor expansion, the natural question arises how good the approximation can be, if x_n and y_n are taken as Taylor polynomials for sine and cosine at $t = 0$. The result is summarized in the following lemma.

LEMMA 6.9. *Let x_n and y_n be the degree n Taylor polynomials of sine and cosine, respectively. Then*

$$x_n^2(t) + y_n^2(t) = 1 + \frac{1}{w_n}\,t^m + \mathcal{O}(t^{m+1}),$$

where

$$m := \begin{cases} n + 1, & n \textit{ is odd}, \\ n + 2, & n \textit{ is even}, \end{cases}$$

and

$$w_n = \begin{cases} \frac{m}{2}\, n!, & \text{if } n \bmod 4 = 1, 2, \\ -\frac{m}{2}\, n!, & \text{otherwise.} \end{cases}$$

Proof. Let

$$R_s(t) = \sin t - x_n(t),$$
$$R_c(t) = \cos t - y_n(t),$$

be Lagrange remainders in Taylor expansions. Since

$$x_n^2(t) + y_n^2(t) = 1 - 2(R_s(t)\, \sin t + R_c(t)\, \cos t) + R_s^2(t) + R_c^2(t) \tag{6.42}$$

and R_s, R_c are of order $\mathcal{O}(t^{n+1})$, it is enough to consider $S(t) := -2(R_s(t)\, \sin t + R_c(t)\, \cos t)$ only.

First, suppose that n is odd, i.e., $n = 2\,\ell - 1$. In this case the expansions of R_s and R_c are

$$R_s(t) = \frac{(-1)^\ell}{(n+2)!} t^{n+2} + \mathcal{O}(t^{n+4}), \quad R_c(t) = \frac{(-1)^\ell}{(n+1)!} t^{n+1} + \mathcal{O}(t^{n+3}),$$

therefore

$$S(t) = -2\, \frac{(-1)^\ell}{(n+1)!} t^{n+1} + \mathcal{O}(t^{n+3}).$$

By (6.42), $m = n + 1$ and

$$w_n = \frac{(-1)^{\ell+1}\,(n+1)!}{2}.$$

If n is even, $n = 2\,\ell$,

$$R_s(t) = \frac{(-1)^\ell}{(n+1)!} t^{n+1} + \mathcal{O}(t^{n+3}), \quad R_c(t) = \frac{(-1)^{\ell+1}}{(n+2)!} t^{n+2} + \mathcal{O}(t^{n+4}),$$

thus

$$S(t) = -2 \left(\frac{(-1)^\ell}{(n+1)!} t^{n+2} + \frac{(-1)^{\ell+1}}{(n+2)!} t^{n+2} \right) + \mathcal{O}(t^{n+4}).$$

Again by (6.42), $m = n + 2$ and

$$w_n = \frac{(-1)^{\ell+1}\,(n+2)\, n!}{2}.$$

□

The last lemma confirms that the Taylor polynomials are not an optimal choice if the radial distance is used as a measure of the approximation order.

Chapter 7

Resultants, Gröbner basis and Brouwer's degree

In this chapter the definition and some of the main properties of resultants, Gröbner basis and Brouwer's mapping degree are given. Further results can be found in [6], [1], [3], [5].

7.1. Resultants

A resultant is defined in the following way.

DEFINITION 7.1. *Given a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

of degree n with zeros α_i, $i = 1, \ldots, n$, and a polynomial

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

of degree m with zeros β_j, $j = 1, \ldots, m$, the resultant is defined by

$$\text{Res}(p, q; x) := a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j).$$

The resultant is also given by the determinant of the corresponding $(n+m) \times (n+m)$

Sylvester matrix. Namely,

$$\operatorname{Res}(p,q;x) = \det \begin{pmatrix} a_0 & & & & b_0 & & & \\ a_1 & a_0 & & & b_1 & b_0 & & \\ & a_1 & \ddots & & & b_1 & \ddots & \\ \vdots & & \ddots & a_0 & \vdots & & \ddots & b_0 \\ & \vdots & & a_1 & & \vdots & \ddots & b_1 \\ a_n & & & & b_m & & & \\ & a_n & & \vdots & & b_m & & \vdots \\ & & \ddots & & & & \ddots & \\ & & & a_n & & & & b_m \end{pmatrix}.$$

It is clear that $\operatorname{Res}(p,q;x) = 0$ if and only if $p(x)$ and $q(x)$ have a common factor in $\mathbb{R}[x]$.

If p and q are polynomials of the same degree $n = m$, then there is another way to compute their resultant. From the following polynomial in two variables

$$\mathcal{B}(x,y) = \frac{p(x)q(y) - p(y)q(x)}{x - y} = \sum_{i,j=0}^{n-1} c_{i,j} x^i y^j,$$

form a symmetric $n \times n$ matrix $C := (c_{i,j})_{i,j=0}^{n-1}$ (Bézout matrix). The determinant of C (Bézout resultant) equals $\pm\operatorname{Res}(p,q;x)$. Practical experience shows that the Bézout matrix is superior to the Sylvester matrix with regard to the numerical stability. So, if one has two polynomials of the same large degree with numerical coefficients, then it is better to use the Bézout matrix.

Computing the resultant of two polynomials with respect to a specific variable is a standard feature in many computer algebra systems. These implementations make use of either the Sylvester matrix or the Bézout matrix, and they work quite fast.

There are many applications of the resultants. Resultants can be used to eliminate the variables from the system of polynomial equations. Furthermore, one can use them to find intersections of two algebraic curves in the real plane, for implicitization of rational curves in the plane, etc.

7.2. Gröbner basis

Let $K := k[x_1, x_2, \dots, x_n]$ denote a ring of multivariate polynomials over k in variables $x_1, x_2, \dots, x_n$. Further, let

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n), \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n), \quad \boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}.$$

Every $f \in K$ can be written as

$$f = a_1 \boldsymbol{x}^{\boldsymbol{\alpha}_1} + a_2 \boldsymbol{x}^{\boldsymbol{\alpha}_2} + \dots + a_r \boldsymbol{x}^{\boldsymbol{\alpha}_r},$$

where $0 \neq a_i \in k$, and $\boldsymbol{x}^{\boldsymbol{\alpha}_1} > \boldsymbol{x}^{\boldsymbol{\alpha}_2} > \dots > \boldsymbol{x}^{\boldsymbol{\alpha}_r}$ with respect to some chosen monomial order. Further let us denote the leading power product by $\operatorname{lp}(f) := \boldsymbol{x}^{\boldsymbol{\alpha}_1}$. A Gröbner basis is defined in the following way.

DEFINITION 7.2. *A set of nonzero polynomials $G = \{g_1, g_2, \ldots, g_\ell\}$ contained in some ideal $I \subseteq K$ is a Gröbner basis for I if for every nonzero $f \in I$ there exists $i \in \{1, 2, \ldots, \ell\}$ such that the* $\mathrm{lp}(g_i)$ *divides* $\mathrm{lp}(f)$.

A Gröbner basis G can also be characterized by one of the properties given in the next theorem.

THEOREM 7.3. *Let $I \subseteq K$ be a nonzero ideal. The following statements are equivalent for a set of nonzero polynomials $G = \{g_1, g_2, \ldots, g_\ell\} \subseteq I$.*

(i) *G is a Gröbner basis for I.*

(ii) *Multivariate division (see [1]) of any polynomial in the ideal I by G gives* 0.

(iii) *$f \in I$ if and only if $f = \sum_{i=1}^{\ell} h_i g_i$, where*

$$\mathrm{lp}(f) = \max_{1 \le i \le \ell} \{\mathrm{lp}(h_i)\, \mathrm{lp}(g_i)\}.$$

(iv) *The ideal given by the leading terms of polynomials in the ideal I is itself generated by the leading terms of the basis G.*

If $G = \{g_1, g_2, \ldots, g_\ell\}$ is a Gröbner basis of the ideal I, then I is generated by its elements, i.e., $I = \langle g_1, g_2, \ldots, g_\ell \rangle$. Furthermore, for every nonzero ideal $I \subseteq K$, generated by a finite set of polynomials, a Gröbner basis exists. One method for computing it is known as Buchberger's algorithm.

Since multivariate division requires a monomial ordering, the basis depends on the chosen monomial ordering, and different orderings can give rise to radically different Gröbner bases. Two of the most commonly used orderings are a *lexicographic* order, and a *degree reverse lexicographic* order. A lexicographic order eliminates variables, but resulting Gröbner bases are often very large and expensive to compute. Degree reverse lexicographic order typically provides the fastest Gröbner basis computations.

A Gröbner basis is called *reduced* if the leading coefficient of each element of the basis is 1 and no monomial in any element of the basis is in the ideal generated by the leading terms of the other elements of the basis. The reduced Gröbner basis is unique.

In the worst case, computation of the Gröbner basis may require time that is exponential or even doubly-exponential in the number of solutions of the polynomial system. Most computer algebra systems contain routines to compute the Gröbner basis.

Gröbner bases with respect to the lexicographic order are very useful for solving polynomial equations and for elimination of variables. The set of polynomials in the Gröbner basis have the same collection of roots as the original polynomials.

7.3. Brouwer's mapping degree

Let U denote a bounded open subset of $\mathbb{R}^n$. For a $\mathcal{C}^1$ mapping $\boldsymbol{f} : U \to \mathbb{R}^n$, $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$, the Jacobian matrix at $\boldsymbol{x} \in U$ is

$$\boldsymbol{f}'(\boldsymbol{x}) = \left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{x}) \right)_{i,j=1}^{n},$$

and the Jacobian determinant is

$$J_{\boldsymbol{f}}(\boldsymbol{x}) = \det \boldsymbol{f}'(\boldsymbol{x}).$$

Further,

$$\mathrm{RV}(\boldsymbol{f}) = \left\{\boldsymbol{y} \in \mathbb{R}^n \middle| \quad \forall \boldsymbol{x} \in \boldsymbol{f}^{-1}(\boldsymbol{y}) : J_{\boldsymbol{f}}(\boldsymbol{x}) \neq 0\right\}$$

is the set of regular values. Its complement $\mathrm{CV}(\boldsymbol{f}) = \mathbb{R}^n \setminus \mathrm{RV}(\boldsymbol{f})$ is called the set of critical values.

Brouwer's degree or simply a *degree* of a continuous mapping $\boldsymbol{f}$ on U is an integer, that gives some evidence on the number of zeros of $\boldsymbol{f}$ in U provided that $\boldsymbol{f}(\boldsymbol{x}) \neq \mathbf{0}$ on ∂U. The precise definition is given in three parts.

(i) Suppose that $\boldsymbol{f} : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ mapping and $\boldsymbol{y}$ is a regular value. Then the degree of $\boldsymbol{f}$ at $\boldsymbol{y}$ relative to U is

$$\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \sum_{\boldsymbol{x} \in \boldsymbol{f}^{-1}(\boldsymbol{y})} \operatorname{sign} J_{\boldsymbol{f}}(\boldsymbol{x}),$$

where the sum is finite and $\sum_{\boldsymbol{x} \in \emptyset} = 0$.

(ii) Suppose that $\boldsymbol{f} : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ mapping. Then one can find a sequence of regular points $(\boldsymbol{y}_i)_i$ that converge to $\boldsymbol{y}$. The degree of $\boldsymbol{f}$ at $\boldsymbol{y}$ is then defined as

$$\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \lim_{i \to \infty} \deg(\boldsymbol{f}, U, \boldsymbol{y}_i).$$

(iii) If $\boldsymbol{f}$ is continuous in U, then there exists a sequence of $\mathcal{C}^1$ mappings $(\boldsymbol{f}_i)_i$ that converge to $\boldsymbol{f}$ uniformly on $\overline{U}$, and

$$\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \lim_{i \to \infty} \deg(\boldsymbol{f}_i, U, \boldsymbol{y}).$$

It can be shown that the function $\deg(\boldsymbol{f}, U, \boldsymbol{y})$ is well defined in (ii) and (iii), i.e., the limits exist and are independent of the approximating sequences.

The basic properties of the degree function are the following:

1. $\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \deg(\boldsymbol{f} - \boldsymbol{y}, U, \mathbf{0})$ (translation invariance).
2. Let $\boldsymbol{H}(\boldsymbol{x}, t)$ be a continuous function of $\boldsymbol{x} \in U$ and $t \in [0, 1]$, and suppose that $\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{y}$ has no solution $\boldsymbol{x} \in \partial U$ for any $t \in [0, 1]$. Then $\deg(\boldsymbol{H}(\cdot, t), U, \boldsymbol{y})$ is constant, independent of $t \in [0, 1]$ (homotopy invariance).
3. $\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \deg(\boldsymbol{f}, U, \boldsymbol{y}')$ for $\boldsymbol{y}$ and $\boldsymbol{y}'$ in the same component of $\mathbb{R}^n \setminus \boldsymbol{f}(\partial U)$.
4. If U_1 and U_2 are open, disjoint subsets of U such that $\boldsymbol{y} \notin \boldsymbol{f}(\overline{U} \setminus (U_1 \cup U_2))$, then $\deg(\boldsymbol{f}, U, \boldsymbol{y}) = \deg(\boldsymbol{f}, U_1, \boldsymbol{y}) + \deg(\boldsymbol{f}, U_2, \boldsymbol{y})$ (additivity).
5. If U is a symmetric domain about the origin, $\boldsymbol{f}(-\boldsymbol{x}) = -\boldsymbol{f}(\boldsymbol{x})$ on ∂U, and $\boldsymbol{f}(\boldsymbol{x}) \neq \mathbf{0}$ on ∂U. Then $\deg(\boldsymbol{f}, U, \mathbf{0})$ is an odd integer.
6. If $\deg(\boldsymbol{f}, U, \boldsymbol{y}) \neq \mathbf{0}$, then the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$ has solutions in U.

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Razširjeni povzetek

Geometrijska interpolacija s parametričnimi polinomskimi krivuljami in zlepki je bila prvič predstavljena v [4], kjer je obravnavana Hermitova kubična interpolacija dveh točk, smeri tangent in ukrivljenosti. Dokazano je, da je red pri aproksimaciji ravninske konveksne krivulje šest, za razliko od funkcijskega primera, kjer je red le štiri. Visok red aproksimacije je le eden od razlogov, ki je k študiju geometrijskih aproksimacijskih shem privabil mnogo avtorjev. Drug pomemben razlog je ta, da je oblika geometrijskih interpolantov v večini primerov boljša od oblike običajnih funkcijskih interpolantov. Osnovni princip geometrijske interpolacije je, da krivulja ni odvisna od dejanske parametrizacije, ampak le od geometrijskih količin, kot so točke, smeri tangent, ukrivljenosti itd. Ker ne predpišemo nobenih dodatnih pogojev, kot so na primer parametri, pri katerih točke interpoliramo, velikosti tangent ali ukrivljenosti, je red aproksimacije višji kot pri običajni funkcijski interpolaciji. Geometrijske sheme so zato pomembno orodje v računalniško podprtem geometrijskem oblikovanju (CAGD). Glavni problem teh shem pa je, da vključujejo nelinearne enačbe, zato so vprašanja o obstoju rešitve, redu aproksimacije in učinkoviti implementaciji zelo težka.

Običajno krivulj ne aproksimiramo z eno samo polinomsko krivuljo, ampak s končnim številom krivulj, ki se v stičnih točkah ujemajo z določenim redom geometrijske zveznosti. Pravimo, da sta dve krivulji G^0 zvezni, če se ujemata v skupni točki, G^1 zvezni, če imata v skupni točki isto smer tangente, G^2 zvezni, če imata v skupni točki isto smer tangente in isto ukrivljenost, itd. V splošnem geometrijsko zveznost definiramo takole.

DEFINICIJA 1. *Dve parametrični krivulji*

$$\boldsymbol{f}_1 : [t_0, t_1] \to \mathbb{R}^d, \quad \boldsymbol{f}_2 : [s_0, s_1] \to \mathbb{R}^d,$$

se ujemata z geometrijsko zveznostjo reda k oziroma sta G^k-zvezni, če se končna točka $\boldsymbol{f}_1$ ujema z začetno točko $\boldsymbol{f}_2$, $\boldsymbol{f}_1(t_1) = \boldsymbol{f}_2(s_0)$, in če obstaja regularna reparametrizacija $\phi : [t_0, t_1] \to [s_0, s_1]$, da velja

$$\left.\frac{d^j \boldsymbol{f}_1}{dt^j}(t)\right|_{t=t_1} = \left.\frac{d^j (\boldsymbol{f}_2 \circ \phi)}{dt^j}(t)\right|_{t=t_1}, \quad j = 0, 1, \ldots, k.$$

Za določanje reda aproksimacije moramo znati meriti razdalje med parametričnimi objekti. Ker so ti največkrat predstavljeni kot množice točk, je ena od možnih metrik zelo znana *Hausdorfova razdalja*.

DEFINICIJA 2. *Naj bosta X in Y podmnožici metričnega prostora M. Hausdorfova razdalja $d_H(X,Y)$ je definirana kot*

$$d_H(X,Y) := \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\},$$

kjer je $d(x,y)$ razdalja v metričnem prostoru M.

Na žalost pa je Hausdorfovo razdaljo v praksi zelo težko računati. Za njeno zgornjo mejo sta T. Lyche and K. Mørken ([34]) predlagala tako imenovano *parametrično razdaljo.*

DEFINICIJA 3. *Naj bosta $\boldsymbol{f}_1$ in $\boldsymbol{f}_2$ dve parametrični krivulji, definirani na intervalih I_1 in I_2. Parametrična razdalja med $\boldsymbol{f}_1$ in $\boldsymbol{f}_2$ je definirana kot*

$$\operatorname{dist}(\boldsymbol{f}_1,\boldsymbol{f}_2) := \inf_{\phi} \|\boldsymbol{f}_1\circ\phi - \boldsymbol{f}_2\| = \inf_{\phi}\max_{t\in I_2}\|\boldsymbol{f}_1(\phi(t)) - \boldsymbol{f}_2(t)\|,$$

kjer je $\phi : I_2 \to I_1$ regularna reparametrizacija.

V [22] sta K. Höllig in J. Koch postavila naslednjo domnevo, ki v splošnem še ni dokazana.

DOMNEVA 4. *Pod ustreznimi generičnimi predpostavkami lahko polinomska krivulja stopnje n interpolira*

$$m = n + 1 + \left\lfloor\frac{n-1}{d-1}\right\rfloor$$

točk na gladki krivulji $\boldsymbol{f}\in\mathbb{R}^d$. Ta interpolacijska krivulja aproksimira $\boldsymbol{f}$ z redom aproksimacije m, ko gredo razlike med interpoliranimi točkami proti nič.

Večina objavljenih rezultatov o geometrijski interpolaciji je dobljenih s pomočjo asimptotične analize. Glavni avtorji in njihova dela so C. de Boor, K. Höllig, M. Sabin, J. Koch ([4], [22], [21]), T. Lyche, K. Mørken, K. Scherer ([37], [45], [46], [34]), W. L. F. Degen ([7], [8], [9]), A. Rababah ([38], [39]), Y. Y. Feng, J. Kozak, E. Žagar ([16], [15], [14], [26], [27], [28], [47]) in R. Schaback ([40], [41], [44], [42], [43]). V praktičnih primerih pa rezultati z asimptotično analizo niso zadostni, saj morajo robustni algoritmi temeljiti na pogojih, ki vnaprej zagotavljajo obstoj interpolantov. Takih rezultatov pa je zaenkrat zelo malo. Razen posebnih primerov, kot so interpolacija krožnih lokov ([34] [10], [19] [35], [17], [18], [11], [12]), je le nekaj rezultatov o geometrijskih pogojih, ki zagotavljajo obstoj rešitve. Pogoji za obstoj parametrične parabole, ki interpolira štiri točke, so podani v [33]. V [36] so ti pogoji razširjeni na vse možne primere (Taylorjev, Hermitov, Lagrangeev). Interpolacija poljubnega števila točk v ravnini s paraboličnimi G^2 zlepki je obravnavana v [41], [14], [26], kjer so izpeljani zadostni pogoji za obstoj in enoličnost rešitve. Najbolj splošen rezultat sta dokazala J. Kozak in E. Žagar v [27], kjer sta izpeljala potrebne in zadostne pogoje za obstoj interpolantov pri interpolaciji $d+2$ različnih točk v $\mathbb{R}^d$ s polinomsko krivuljo stopnje $\leq d$.

V disertaciji obravnavam ravninske geometrijske sheme. Te so v praksi najbolj uporabne, poleg tega pa je v ravnini razmik med parametričnim in funkcijskim primerom največji. Domneva 4 namreč pravi, da lahko $2n$ ravninskih točk interpoliramo s polinomsko krivuljo stopnje n in dosežemo red aproksimacije $2n$, medtem ko lahko v funkcijskem primeru polinomska krivulja stopnje n interpolira največ $n+1$ točk z redom aproksimacije $n+1$.

Lagrangeev interpolacijski problem, na katerega se osredotočim, je naslednji: za danih $2n$ ravninskih točk

$$\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2n-1} \in \mathbb{R}^2, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \tag{1}$$

poišči parametrično polinomsko krivuljo $\boldsymbol{P}_n : [t_0, t_{2n-1}] \to \mathbb{R}^2$ stopnje $\leq n$, ki interpolira dane točke pri naraščajočih vrednostih parametrov t_i,

$$t_0 < t_1 < \cdots < t_{2n-2} < t_{2n-1}. \tag{2}$$

Ker linearna transformacija domene parametrov ohranja stopnjo polinomske krivulje, lahko predpostavimo, da je $t_0 := 0$ in $t_{2n-1} := 1$. Ostali parametri $\boldsymbol{t} := (t_i)_{i=1}^{2n-2}$ pa so še neznani. Parametrom, ki zadoščajo (2), pravimo dopustni parametri. Nanje lahko gledamo kot na komponente točke v odprtem simpleksu

$$\mathcal{D}_n := \left\{ (t_i)_{i=1}^{2n-2}; \quad 0 = t_0 < t_1 < \cdots < t_{2n-2} < t_{2n-1} = 1 \right\},$$

z robom $\partial\mathcal{D}_n$, kjer vsaj dva različna t_i sovpadata. Sistem enačb

$$\boldsymbol{P}_n(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 2n-1, \tag{3}$$

določa tako krivuljo $\boldsymbol{P}_n$ kot tudi neznane parametre $\boldsymbol{t}$. Nelinearni del je določiti parametre. Ko jih izračunamo, dobimo koeficiente polinomske krivulje $\boldsymbol{P}_n$ enostavno tako, da vzamemo katerihkoli $n+1$ različnih interpolacijskih pogojev v (3) in na komponentah $\boldsymbol{P}_n$ uporabimo eno izmed standardnih interpolacijskih shem (Newtonova shema, Lagrangeeva shema).

Ker je sistem enačb (3) nelinearen, je dokaz eksistence dopustne rešitve zelo težek problem. Rešitev tudi ne obstaja nujno. Kot preprost primer vzemimo štiri točke ($n = 2$) z nekonveksnim kontrolnim poligonom. Ker je vsaka komponenta interpolacijske krivulje $\boldsymbol{P}_2$ parabola, ki ne more imeti več kot dve ničli, je jasno, da interpolant ne more obstajati. Zelo lepi, potrebni in zadostni pogoji za obstoj rešitve za $n = 2$ so dani v [27]. Interpolant $\boldsymbol{P}_2$ obstaja natanko takrat, ko imajo determinante

$$\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1), \quad \det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_2), \quad \det(\Delta\boldsymbol{T}_1, \Delta\boldsymbol{T}_2)$$

enak predznak. Tu je $\Delta\boldsymbol{T}_i := \boldsymbol{T}_{i+1} - \boldsymbol{T}_i$ standardna oznaka za premo diferenco.

Za motivacijo si oglejmo dva numerična primera. Najprej primerjajmo kubično geometrijsko shemo s standardno interpolacijo s polinomi stopnje pet, kjer si za parametrizacijo izberemo enakomerno ali tetivno parametrizacijo. Na slikah 1.1 se lepo vidi, da se kubična krivulja (črna krivulja) dosti lepše prilega podatkom kot ostali dve. Računska zahtevnost za izračun teh kubičnih interpolantov je zelo majhna. Newtonova metoda z ekvidistantnimi začetnimi približki $t_i = \frac{i}{5}$ konvergira v povprečju v osmih iteracijah. Kot naslednji primer aproksimirajmo logaritmično spiralo

$$\boldsymbol{f}(t) = \log(t+\pi) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{4}$$

z geometrijskimi interpolanti stopenj $n = 3, 4, 5, 6$. Podatke dobimo iz (4) z enakomerno delitvijo domene parametrov. Tabela 1.1 numerično potrjuje, da je red aproksimacije $2n$. Napako merimo kot parametrično razdaljo med krivuljo in njenim interpolantom.

Geometrijska interpolacija s kubičnimi polinomskimi krivuljami

V tem poglavju obravnavam Lagrangeevo interpolacijo šestih ravninskih točk s kubično polinomsko krivuljo. Podani so zadostni geometrijski pogoji, ki zagotavljajo obstoj interpolanta $\boldsymbol{P}_3$. Pogoji so precej preprosti in odvisni le od določenih determinant, izpeljanih iz danih točk.

Problem je naslednji: poiskati moramo pogoje, ki zagotavljajo obstoj vsaj ene dopustne rešitve nelinearnega sistema

$$\boldsymbol{P}_3(t_i) = \boldsymbol{T}_i, \quad i = 0, 1, \ldots, 5. \tag{5}$$

Rešitev je dopustna, če parametri zadoščajo

$$0 =: t_0 < t_1 < \cdots < t_5 := 1. \tag{6}$$

Pomembno vlogo igra matrika diferenc točk $\left(\Delta \boldsymbol{T}_i\right)_{i=0}^4 \in \mathbb{R}^{2\times 5}$ in predznaki ter razmerja med njenimi minorji $D_{i,j} := \det\left(\Delta \boldsymbol{T}_i, \Delta \boldsymbol{T}_j\right)$. To so predznačene ploščine paralelogramov, napetih na vektorje $\Delta \boldsymbol{T}_i, \Delta \boldsymbol{T}_j$. Definirajmo še

$$\lambda_1 := \frac{D_{0,1}}{D_{1,2}}, \quad \lambda_2 := \frac{D_{0,2}}{D_{1,2}}, \quad \lambda_3 := \frac{D_{2,4}}{D_{2,3}}, \quad \lambda_4 := \frac{D_{3,4}}{D_{2,3}}, \quad \delta := \frac{D_{1,3}}{D_{1,2}}, \quad \mu := \frac{D_{2,3}}{D_{1,2}},$$

$$\gamma_1 := \frac{\lambda_2(1+\lambda_2)}{\lambda_1(1+\lambda_2) + \sqrt{\lambda_1(1+\lambda_2)(\lambda_1+\lambda_2)}},$$

$$\gamma_2 := \frac{\lambda_3(1+\lambda_3)}{\lambda_4(1+\lambda_3) + \sqrt{\lambda_4(1+\lambda_3)(\lambda_3+\lambda_4)}}.$$

Pri podatkih s konveksnim kontrolnim poligonom, kot na prvih treh slikah 1.1, je $\mu > 0$ in $\lambda_i > 0$, $i = 1, 2, 3, 4$. Kontrolni poligoni točk na zadnjih treh slikah 1.1 imajo prevojno točko in zanje je $\mu < 0$ in $\lambda_i > 0$, $i = 1, 2, 3, 4$. Omejimo se na študij takih podatkov. Geometrijska interpretacija λ_i, δ in μ je prikazana na sliki 2.1. Definirajmo še $\boldsymbol{\lambda} := (\lambda_i)_{i=1}^4$ in funkcije

$$\vartheta_1(\boldsymbol{\lambda}, \mu) := \frac{2\mu - \gamma_1 + \sqrt{\gamma_1^2 + 4\mu(1+\gamma_1)}}{2\gamma_1},$$

$$\vartheta_2(\boldsymbol{\lambda}, \mu) := \frac{2 - \mu\gamma_2 + \sqrt{\mu^2\gamma_2^2 + 4\mu(1+\gamma_2)}}{2\gamma_2},$$

$$\vartheta_3(\boldsymbol{\lambda}, \mu) := \frac{\lambda_1\mu}{\lambda_2} + \frac{\lambda_4}{\lambda_3} + \frac{\mu}{\lambda_2}\sqrt{\frac{\lambda_1(\lambda_1+\lambda_2)}{1+\lambda_2}},$$

$$\vartheta_4(\boldsymbol{\lambda}, \mu) := \frac{\lambda_1\mu}{\lambda_2} + \frac{\lambda_4}{\lambda_3} + \frac{1}{\lambda_3}\sqrt{\frac{\lambda_4(\lambda_3+\lambda_4)}{1+\lambda_3}},$$

ki bodo uporabljene v relacijah, ki zagotavljajo obstoj rešitve. Glavna izreka sta naslednja:

IZREK 5. *Naj bo $D_{1,2}D_{2,3} \neq 0$ in naj bodo podatki konveksni, to je $\mu > 0$ in $\lambda_i > 0$, $i = 1, 2, 3, 4$. Če sta bodisi ϑ_ℓ enaka, $\vartheta_1(\boldsymbol{\lambda}, \mu) = \vartheta_2(\boldsymbol{\lambda}, \mu)$, bodisi nista enaka in velja eden od pogojev*

$$\delta < \min_{\ell=1,2} \{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\} \quad \textit{ali} \quad \delta > \max_{\ell=1,2} \{\vartheta_\ell(\boldsymbol{\lambda}, \mu)\},$$

potem kubična interpolacijska krivulja $\boldsymbol{P}_3$, ki zadošča (5), obstaja.

IZREK 6. *Naj bo $D_{1,2}D_{2,3} \neq 0$ in naj ima kontrolni poligon točk prevoj, to je $\mu < 0$ in $\lambda_i > 0$, $i = 1, 2, 3, 4$. Če je*

$$\delta \in (\vartheta_3(\boldsymbol{\lambda}, \mu), \vartheta_4(\boldsymbol{\lambda}, \mu)),$$

potem kubična interpolacijska krivulja $\boldsymbol{P}_3$, ki zadošča (5), obstaja.

Izrek 5 in izrek 6 podata zadostne pogoje za obstoj kubičnega geometrijskega interpolanta. Naslednji izrek pa izključi večino podatkov, ki jih ta dva izreka ne pokrijeta.

IZREK 7. *Pogoji na točke, pri katerih rešitev interpolacijskega problema (5) ne obstaja, so podani v tabeli 1.*

$D_{1,2}D_{2,3} \neq 0$		$D_{1,2}D_{2,3} = 0$
$\mu > 0$	$\mu < 0$	
$\lambda_2 \leq 0,\ \lambda_3 \leq 0$	$\lambda_2 \leq 0$	$D_{1,2} = 0,\ D_{2,3} = 0$
$\delta \leq 0,\ \lambda_1 \leq 0$	$\lambda_3 \leq 0$	$D_{1,2} = 0,\ \lambda_3 \leq 0$
$\delta \leq 0,\ \lambda_4 \leq 0$	$\lambda_1 \leq 0,\ \delta \leq 0$	$D_{2,3} = 0,\ \lambda_2 \leq 0$
$\lambda_1 \leq 0,\ \lambda_3 \leq 0\ ,\ \lambda_4 \geq 0$	$\lambda_4 \leq 0,\ \delta \geq 0$	$D_{1,2} = 0,\ D_{0,1}D_{2,3} \geq 0$
$\lambda_2 \leq 0,\ \lambda_4 \leq 0,\ \lambda_1 \geq 0$		$D_{2,3} = 0,\ D_{1,2}D_{3,4} \geq 0$

Table 1: Pogoji na točke, pri katerih rešitev interpolacijskega problema (5) ne obstaja.

Še vedno pa ostane nekaj primerov, ki z izreki 5, 6 in 7 niso pokriti. Kot primer vzemimo točke

$$\boldsymbol{T}_0 = \begin{pmatrix} -20-\zeta \\ 3 \end{pmatrix}, \qquad \boldsymbol{T}_1 = \begin{pmatrix} -10 \\ 1 \end{pmatrix}, \qquad \boldsymbol{T}_2 = \begin{pmatrix} -5 \\ 0 \end{pmatrix}, \tag{7}$$
$$\boldsymbol{T}_3 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \qquad \boldsymbol{T}_4 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}, \qquad \boldsymbol{T}_5 = \begin{pmatrix} 20+\zeta \\ 3 \end{pmatrix}, \quad \zeta > 0,$$

za katere je $\lambda_1 = \lambda_4 = -\frac{\zeta}{10}$, $\lambda_2 = \lambda_3 = 2$, $\delta = \mu = 1$. Očitno niso izpolnjeni pogoji nobenega od zgornjih izrekov. Ti podatki imajo za $\zeta \in (0, \zeta_0]$, kjer je $\zeta_0 := 2.95373852$, dve dopustni rešitvi (slika 2.2). Pri $\zeta = \zeta_0$ ti dve rešitvi sovpadeta in krivulja ima špico, za $\zeta > \zeta_0$ pa dopustna rešitev ne obstaja več.

Primeri na slikah 1.1 vsi zadoščajo pogojem izreka 5 ali izreka 6. Poglejmo si dva izmed njih bolj natančno. Pri prvem primeru je $\delta < \vartheta_1(\boldsymbol{\lambda}, \mu) < \vartheta_2(\boldsymbol{\lambda}, \mu)$. Slika 2.3 (levo) prikazuje, kako se spreminja položaj točk, ko se δ približuje $\vartheta_1(\boldsymbol{\lambda}, \mu)$. Za $\delta \in [\vartheta_1(\boldsymbol{\lambda}, \mu), \vartheta_2(\boldsymbol{\lambda}, \mu)]$ dopustna rešitev ne obstaja. Podobno nam slika 2.3 (desno) prikazuje spreminjanje položajev točk, ko gre δ od $\vartheta_3(\boldsymbol{\lambda}, \mu)$ do $\vartheta_4(\boldsymbol{\lambda}, \mu)$ za zadnji primer s slike 1.1.

Za $\delta < \vartheta_3(\boldsymbol{\lambda}, \mu)$ ali $\delta > \vartheta_4(\boldsymbol{\lambda}, \mu)$ lahko obstajata dve dopustni rešitvi, zgodi pa se podoben problem kot pri zgornjem primeru.

Pogoji izrekov 5 in 6 so zelo preprosti, v dokazu pa je potrebnih kar nekaj korakov. Poglejmo si idejo in skico dokaza. Sistem (5) najprej pretvorimo v obliko, ki je bolj primerna za nadaljno analizo. Ločimo enačbe za neznane koeficiente od enačb za neznane parametre. To naredimo tako, da na sistemu (5) uporabimo deljene diference $[t_\ell, t_{\ell+1}, \ldots, t_{\ell+4}]$, ki vsak kubični polinom $\boldsymbol{P}_3$ preslikajo v nič. Dobimo enačbe

$$\sum_{i=\ell}^{\ell+4} \frac{1}{\dot{\omega}_{\ell,\ell+4}(t_i)} \boldsymbol{T}_i = \mathbf{0}, \quad \ell = 0, 1, \tag{8}$$

kjer je

$$\omega_{i,j}(t) := (t - t_i)(t - t_{i+1}) \cdots (t - t_{j-1})(t - t_j), \quad \dot{\omega}_{i,j}(t) = \frac{d}{dt}\omega_{i,j}(t), \quad i < j.$$

Nekaj enostavnih transformacij sistema (8) nam da končni sistem enačb za neznane parametre, ki se glasi

$$\frac{1}{\dot{\omega}_{0,4}(t_0)}(1 + \lambda_2) + \frac{1}{\dot{\omega}_{0,4}(t_1)} + \frac{1}{\dot{\omega}_{0,4}(t_4)}\mu = 0, \tag{9}$$

$$\frac{1}{\dot{\omega}_{0,4}(t_0)}\lambda_1 + \frac{1}{\dot{\omega}_{0,4}(t_3)} + \frac{1}{\dot{\omega}_{0,4}(t_4)}(1 + \delta) = 0, \tag{10}$$

$$\frac{1}{\dot{\omega}_{1,5}(t_1)}\left(1 + \frac{\delta}{\mu}\right) + \frac{1}{\dot{\omega}_{1,5}(t_2)} + \frac{1}{\dot{\omega}_{1,5}(t_5)}\lambda_4 = 0, \tag{11}$$

$$\frac{1}{\dot{\omega}_{1,5}(t_1)}\frac{1}{\mu} + \frac{1}{\dot{\omega}_{1,5}(t_4)} + \frac{1}{\dot{\omega}_{1,5}(t_5)}(1 + \lambda_3) = 0. \tag{12}$$

V naslednjem koraku dokažemo, da parametri t_i ne morejo biti pri nobeni rešitvi, ki zadošča (6), poljubno blizu roba $\partial\mathcal{D}_3$. To pove naslednji izrek, katerega dokaz je tehnično precej zapleten, je pa to najpomembnejši del v dokazu glavnih izrekov.

IZREK 8. *Naj bodo izpolnjeni pogoji izreka 5 ali izreka 6. Potem sistem* (9) - (12) *ne more imeti nobene rešitve poljubno blizu roba območja* $\partial\mathcal{D}_3$.

Sledi dokaz, da ima nelinearni sistem za posebne podatke liho število dopustnih rešitev. To dejstvo nato razširimo na splošni sistem s pomočjo homotopije in Brouwerjevega izreka. In sicer definiramo homotopijo $\boldsymbol{H}(\boldsymbol{t}; \zeta)$, ki nam da pri $\zeta = 0$ sistem za posebne, pri $\zeta = 1$ pa sistem za splošne podatke. Dokažemo, da $\boldsymbol{H}(\boldsymbol{t}; \zeta) = \mathbf{0}$ zadošča pogojem izreka 8 za vsak $\zeta \in [0, 1]$. Zato obstaja kompaktna množica $K \subset \mathcal{D}_3$, da velja

$$V := \{\boldsymbol{t} \in \mathcal{D}_3; \quad \boldsymbol{H}(\boldsymbol{t}, \zeta) = \mathbf{0}\} \subset K \subset \mathcal{D}_3, \quad V \cap \partial K = \emptyset.$$

Ker torej $\boldsymbol{H}$ ni nič na robu K za noben $\zeta \in [0, 1]$, je Brouwerjeva stopnja invariantna za $\zeta \in [0, 1]$. Ker pa je liha za posebne podatke, mora imeti sistem vsaj eno dopustno rešitev.

Geometrijska interpolacija s kubičnimi G^1 zlepki

V tem poglavju je obravnavana geometrijska interpolacija štirih točk in dveh smeri tangent ter interpolacija s kubičnimi G^1 zlepki. Določen je širok razred zadostnih pogojev, ki zagotavljajo obstoj interpolantov in pokrijejo tako konveksne kot nekonveksne podatke. Eksistenčni pogoji so odvisni le od geometrije podatkov in so enostavno preverljivi. Dodan je tudi algoritem, s katerim določimo območja za smeri tangent, če le te niso vnaprej predpisane.

Interpolacijski problem je sledeč. Za dano zaporedje točk

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad i = 0, 1, 2, \ldots, 3m, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1},$$

poišči kubično G^1 krivuljo zlepkov $\boldsymbol{S} : [a, b] \to \mathbb{R}^2$ s stičnimi točkami

$$a := u_0 < u_1 < \cdots < u_m := b,$$

ki interpolira dane točke $\boldsymbol{T}_i$, tako da

$$\boldsymbol{S}(u_\ell) = \boldsymbol{T}_{3\ell}, \quad \ell = 0, 1, \ldots, m.$$

Z $\boldsymbol{d}_{3\ell}$, $\|\boldsymbol{d}_{3\ell}\|_2 = 1$, označimo smeri tangent krivulje $\boldsymbol{S}$ v u_ℓ. Z odsekoma polinomsko predstavitvijo

$$\boldsymbol{P}^\ell\left(t^\ell\right) := \boldsymbol{S}(u)\big|_{[u_{\ell-1}, u_\ell]}, \quad t^\ell := \frac{u - u_{\ell-1}}{\Delta u_{\ell-1}} \in [0, 1], \quad \ell = 1, 2, \ldots, m,$$

se interpolacijski problem glasi takole: poišči kubične polinomske krivulje $\boldsymbol{P}^\ell$, za katere velja

$$\begin{array}{l} \boldsymbol{P}^\ell\left(t_i^\ell\right) = \boldsymbol{T}_{3(\ell-1)+i}, \quad i = 0, 1, \ldots, 3, \\ \dfrac{d}{dt^\ell}\boldsymbol{P}^\ell(0) = \alpha_0^\ell \boldsymbol{d}_{3(\ell-1)}, \quad \dfrac{d}{dt^\ell}\boldsymbol{P}^\ell(1) = \alpha_3^\ell \boldsymbol{d}_{3\ell}, \end{array} \quad \ell = 1, 2, \ldots, m, \tag{13}$$

kjer parametri t_1^ℓ, t_2^ℓ, $\alpha_0^\ell, \alpha_3^\ell$ zadoščajo

$$0 =: t_0^\ell < t_1^\ell < t_2^\ell < t_3^\ell := 1, \quad \alpha_0^\ell > 0, \quad \alpha_3^\ell > 0, \quad \ell = 1, 2, \ldots, m.$$

Smeri tangent $\boldsymbol{d}_{3\ell}$, $\ell = 1, 2, \ldots, m-1$, zaenkrat še niso predpisane, zato nas zanima tudi, kako določiti območja za $(\boldsymbol{d}_\ell)_{\ell=1}^m$, ki zagotavljajo obstoj rešitve sistema enačb (13).

Poglejmo si najprej analizo enega odseka, torej $m = 1$ in $\boldsymbol{P}^1 =: \boldsymbol{P}$. Naj velja

$$\boldsymbol{d}_0 := \boldsymbol{d}_0^1, \quad \boldsymbol{d}_3 := \boldsymbol{d}_3^1, \quad t_1 := t_1^1, \quad t_2 := t_2^1, \quad \alpha_0 := \alpha_0^1, \quad \alpha_3 := \alpha_3^1.$$

Nelinearen del interpolacijskega problema (13) je izračunati dopustne parametre $(t_1, t_2, \alpha_0, \alpha_3) \in \mathcal{U}$, kjer je

$$\mathcal{U} := \{(t_1, t_2);\ 0 =: t_0 < t_1 < t_2 < t_3 := 1\} \times \{(\alpha_0, \alpha_3);\ \alpha_0 > 0, \alpha_3 > 0\}.$$

Če na enačbah (13) uporabimo deljene diference, ki preslikajo kubične polinome v nič, dobimo naslednji nelinearen sistem za $(t_1, t_2, \alpha_0, \alpha_3)$:

$$[t_0, t_0, t_1, t_2, t_3]\boldsymbol{P} = \boldsymbol{0} = \frac{\alpha_0}{\dot{\omega}_{0,3}(t_0)}\boldsymbol{d}_0 + \sum_{j=1}^{3}\left(\sum_{i=j}^{3}\frac{1}{\dot{\omega}_{0,3}(t_i)}\frac{1}{t_i - t_0}\right)\Delta\boldsymbol{T}_{j-1},$$

$$[t_0, t_1, t_2, t_3, t_3]\boldsymbol{P} = \boldsymbol{0} = \frac{\alpha_3}{\dot{\omega}_{0,3}(t_3)}\boldsymbol{d}_3 + \sum_{j=0}^{2}\left(\sum_{i=0}^{j}\frac{1}{\dot{\omega}_{0,3}(t_i)}\frac{1}{t_3 - t_i}\right)\Delta\boldsymbol{T}_{j}.$$

Z nekaj preprostimi transformacijami se ga da prepisati v

$$\begin{aligned}\frac{1}{t_1^2(1-t_1)} - \frac{1}{t_2^2(1-t_2)}(1+\mu_1) + \frac{t_2-t_1}{(1-t_1)(1-t_2)}\left(1+\mu_1(1+\lambda_1) - \frac{\lambda_1}{\lambda_2}\right) &= 0,\\ \frac{1}{t_2(1-t_2)^2} - \frac{1}{t_1(1-t_1)^2}(1+\mu_2) + \frac{t_2-t_1}{t_1t_2}\left(1+\mu_2(1+\lambda_2) - \frac{\lambda_2}{\lambda_1}\right) &= 0, \qquad (14)\end{aligned}$$

in

$$\alpha_0 = \delta_1\frac{t_1t_2}{t_2-t_1}\left(\frac{1}{t_2^2(1-t_2)} - \frac{t_2-t_1}{(1-t_1)(1-t_2)}(1+\lambda_1)\right), \qquad (15)$$

$$\alpha_3 = \delta_2\frac{(1-t_1)(1-t_2)}{t_2-t_1}\left(\frac{1}{t_1(1-t_1)^2} - \frac{t_2-t_1}{t_1t_2}(1+\lambda_2)\right), \qquad (16)$$

kjer so nove konstante definirane z

$$\begin{aligned}\lambda_{2\ell-1} &:= \frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-3,3\ell-2}}, & \lambda_{2\ell} &:= \frac{D_{3\ell-3,3\ell-1}}{D_{3\ell-2,3\ell-1}},\\ \mu_{2\ell-1} &:= \frac{\det(\boldsymbol{d}_{3\ell-3}, \Delta\boldsymbol{T}_{3\ell-2})}{\det(\boldsymbol{d}_{3\ell-3}, \Delta\boldsymbol{T}_{3\ell-3})}, & \mu_{2\ell} &:= \frac{\det(\Delta\boldsymbol{T}_{3\ell-2}, \boldsymbol{d}_{3\ell})}{\det(\Delta\boldsymbol{T}_{3\ell-1}, \boldsymbol{d}_{3\ell})},\\ \delta_{2\ell-1} &:= \frac{D_{3\ell-3,3\ell-2}}{\det(\boldsymbol{d}_{3\ell-3}, \Delta\boldsymbol{T}_{3\ell-3})}, & \delta_{2\ell} &:= \frac{D_{3\ell-2,3\ell-1}}{\det(\Delta\boldsymbol{T}_{3\ell-1}, \boldsymbol{d}_{3\ell})}.\end{aligned}$$

Da poenostavimo analizo, predpostavimo $\lambda_k > 0$, $\mu_k > 0$ in $\delta_k > 0$, $k = 1, 2$, za konveksne podatke in $\lambda_1 \cdot \lambda_2 < 0$, $\delta_k > 0$ za nekonveksne, to je podatke s prevojno točko. Naslednji dve lemi povesta, kdaj je odvod $\boldsymbol{P}'$ v krajiščih intervala enak nič.

LEMA 9. *Naj bo $\lambda_1 > 0$. Tedaj obstaja enolična rešitev sistema* (14) *in* (15)*, ki zadošča $0 < t_1 < t_2 < 1$ in pri kateri je $\alpha_0 = 0$, natanko tedaj, ko je $\lambda_2 > 0$ in $\mu_2 = \phi_2(\lambda_1, \lambda_2)$, kjer je*

$$\phi_2(\lambda_1, \lambda_2) := \frac{\lambda_2\dfrac{1-\tilde{t}_1}{\tilde{t}_2^2} - \lambda_1\dfrac{\tilde{t}_1}{(1-\tilde{t}_2)^2}}{\lambda_2\dfrac{1-\tilde{t}_2}{\tilde{t}_1^2} - \lambda_1\dfrac{\tilde{t}_2}{(1-\tilde{t}_1)^2}} - 1,$$

in je $\left(\tilde{t}_1, \tilde{t}_2\right)$ enolična rešitev sistema

$$\frac{1-t_1}{t_2^2(t_2-t_1)} = 1 + \lambda_1, \quad \frac{1-t_2}{t_1^2(t_2-t_1)} = \frac{\lambda_1}{\lambda_2}(1+\lambda_2), \quad 0 < t_1 < t_2 < 1.$$

LEMA 10. *Naj bo* $\lambda_1 > 0$. *Tedaj obstaja enolična rešitev sistema* (14) *in* (16), *ki zadošča* $0 < t_1 < t_2 < 1$ *in pri kateri je* $\alpha_3 = 0$, *natanko tedaj, ko je* $\lambda_2 > 0$ *in* $\mu_1 = \phi_1(\lambda_1, \lambda_2) := \phi_2(\lambda_2, \lambda_1)$.

Definirajmo še funkciji

$$\begin{aligned}\phi_3(\lambda_1, \lambda_2, \mu_1) &:= \frac{\lambda_2\mu_1}{\lambda_1(\lambda_2\mu_1 - 1 - \sqrt{1+\mu_1})},\\ \phi_4(\lambda_1, \lambda_2, \mu_1) &:= \frac{\lambda_2\mu_1(\lambda_2\mu_1(1+2\lambda_1) - 2\lambda_1)}{\lambda_1^2(\lambda_2\mu_1 - 1)^2}.\end{aligned}$$

Naslednja dva izreka podata zadostne pogoje na podatke, ki implicirajo obstoj interpolanta $\boldsymbol{P}$. Prvi pokrije konveksne, drugi pa nekonveksne podatke.

IZREK 11. *Naj podatki* $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ *zadoščajo*

$$\lambda_k > 0, \quad \delta_k > 0, \quad \mu_k > 0, \quad k = 1, 2.$$

Če je

$$0 < \mu_1 < \phi_1(\lambda_1, \lambda_2) \quad \text{in} \quad 0 < \mu_2 < \phi_2(\lambda_1, \lambda_2),$$

ali

$$\mu_1 > \phi_1(\lambda_1, \lambda_2) \quad \text{in} \quad \mu_2 > \phi_2(\lambda_1, \lambda_2),$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$, *ki zadošča* (13), *obstaja.*

IZREK 12. *Naj podatki* $\boldsymbol{d}_0, \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{d}_3$ *zadoščajo*

$$\lambda_1 > 0, \quad \lambda_2 < 0, \quad \delta_1 > 0 \quad \text{in} \quad \delta_2 > 0.$$

Če je $\mu_1 > 0$ *in*

$$\phi_3(\lambda_1, \lambda_2, \mu_1) < \mu_2 < \phi_4(\lambda_1, \lambda_2, \mu_1),$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$, *ki zadošča* (13), *obstaja.*

OPOMBA 13. *Iz simetrije v enačbah* (14)-(16) *sledi, da izrek 12 velja tudi, če zamenjamo vlogi* λ_1 *in* λ_2 *ter vlogi* μ_1 *in* μ_2.

Geometrijska interpretacija izrekov 11 in 12 je prikazana na slikah 3.4 in 3.5. Ideja dokaza je podobna kot v prejšnjem poglavju. Najprej dokažemo, da pod pogoji izrekov sistem (14)–(16) ne more imeti nobene rešitve poljubno blizu roba $\partial\mathcal{U}$. V naslednjem koraku pokažemo, da imajo posebni podatki, ki zadoščajo pogojem izrekov, liho število rešitev. Nato to dejstvo prenesemo na splošne podatke s pomočjo homotopije in Brouwerjeve stopnje.

Poglejmo si sedaj interpolacijo z G^1 kubičnimi zlepki ob predpostavki, da so smeri tangent $\boldsymbol{d}_{3\ell}$ neznane. Vsaka smer tangente je odvisna le od enega parametra. Če sta vektorja $\Delta\boldsymbol{T}_{3\ell-1}$ in $\Delta\boldsymbol{T}_{3\ell}$ nekolinearna, smeri tangent izrazimo kot

$$\begin{aligned}\boldsymbol{d}_0 &:= \boldsymbol{d}_0(\xi_0) := (\xi_0 - 1)\Delta\boldsymbol{T}_1 + \xi_0\Delta\boldsymbol{T}_0,\\ \boldsymbol{d}_{3\ell} &:= \boldsymbol{d}_{3\ell}(\xi_\ell) := \sigma_{3\ell}(1-\xi_\ell)\Delta\boldsymbol{T}_{3\ell-1} + \sigma_{3\ell-1}\xi_\ell\Delta\boldsymbol{T}_{3\ell}, \quad \ell = 1, \ldots, m-1,\\ \boldsymbol{d}_{3m} &:= \boldsymbol{d}_m(\xi_m) := (1-\xi_m)\Delta\boldsymbol{T}_{3m-1} - \xi_m\Delta\boldsymbol{T}_{3m-2},\end{aligned}$$

kjer je $\sigma_k := \operatorname{sign}\left(\frac{D_{k-1,k}}{D_{k,k+1}}\right)$. Čeprav tako predstavljene tangente niso normalizirane, to ne spremeni eksistenčnih pogojev. Po predpostavkah v eksistenčnih izrekih morajo biti konstante $\delta_{2\ell}(\xi_\ell) > 0$ in $\delta_{2\ell+1}(\xi_\ell) > 0$ pozitivne. To je res, če ξ_ℓ leži na intervalu $(0,1)$. Preprosto se da določiti tudi podinterval v $(0,1)$, kjer sta pozitivna tudi $\mu_{2\ell-1}(\xi)$ in $\mu_{2\ell}(\xi)$. Rekli bomo, da je neka izbira parametrov $(\xi_\ell)_{\ell=0}^m$ dopustna, če so na vsakem polinomskem odseku izpolnjeni pogoji izreka 11 ali izreka 12. Algoritem *ForwardSweep*, ki je predstavljen v disertaciji, vrne množico vektorjev intervalov $\Xi = \{\Xi_0, \Xi_1, \ldots, \Xi_m\}$, za katere velja naslednje: za vsak r, $0 \le r \le m$, in poljubno izbran $\xi_r \in \Xi_r$ obstaja vsaj ena dopustna izbira

$$(\xi_0, \ldots, \xi_{r-1}, \xi_r, \xi_{r+1}, \ldots, \xi_m), \quad \xi_\ell \in \Xi_\ell.$$

Kadar so podatki na vsakem odseku konveksni, to je

$$\lambda_{2\ell-1} > 0,\ \lambda_{2\ell} > 0, \quad \ell = 1, 2, \ldots, m,$$

pa velja mnogo močnejši rezultat. Če je Ξ vektor intervalov, ki ga vrne algoritem, potem je poljubna izbira parametrov

$$(\xi_0, \xi_1, \ldots, \xi_m), \quad \xi_\ell \in \Xi_\ell,$$

dopustna. Algoritem je preprost *algoritem sestopanja*, ki pregleda podatke najprej v eno smer $\boldsymbol{T}_0 \to \boldsymbol{T}_{3m}$ in določi vmesni rezultat Ξ_ℓ, $\ell = 0, 1, \ldots, m$, ki je tak, da za poljuben $\xi_\ell \in \Xi_\ell$ obstaja izbira $\xi_i \in \Xi_i$, $i = 0, 1, \ldots, \ell-1$, da je $(\xi_0, \xi_1, \ldots, \xi_\ell)$ dopustna izbira za točke $\boldsymbol{T}_i$, $i = 0, 1, \ldots, 3\ell$. Korak v nasprotno smer $\boldsymbol{T}_{3m} \to \boldsymbol{T}_0$ skrči trenutne Ξ_ℓ, $\ell = m-1, m-2, \ldots, 0$, tako da za vsak $\xi_\ell \in \Xi_\ell$ obstaja izbira

$$\xi_i \in \Xi_i, \quad i = 0, 1, \ldots, \ell-1, \ell+1, \ldots, m,$$

da je $(\xi_0, \xi_1, \ldots, \xi_m)$ dopustna za vse podatke. Ko enkrat imamo določene meje Ξ, moramo izbrati dejanske smeri tangent, to je izbrati $\xi_\ell \in \Xi_\ell$. Predstavljen je preprost algoritem *deli in vladaj*, s katerim iz neprazne Ξ vedno dobimo dopustno izbiro. S tem algoritmom pa lahko tudi preverimo ali ima interpolacijski problem (13) rešitev, kadar so smeri tangent predpisane vnaprej.

Hermitova geometrijska interpolacija s kubičnimi G^1 zlepki

V tem poglavju obravnavam Hermitovo geometrijsko interpolacijo z ravninskimi kubičnimi G^1 zlepki. Na vsakem polinomskem odseku interpoliram tri točke in tri smeri tangent. Podani so zadostni geometrijski pogoji za obstoj rešitve, ki pokrijejo večino primerov.

Problem je sledeč. Za danih $2m+1$ točk in smeri tangent

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad \boldsymbol{d}_i \in \mathbb{R}^2, \qquad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \quad \|\boldsymbol{d}_i\|_2 = 1, \quad i = 0, 1, \ldots, 2m,$$

poišči kubično G^1 krivuljo zlepkov $\boldsymbol{S} : [0,1] \to \mathbb{R}^2$ s predpisanimi stičnimi točkami $(t_{2i})_{i=0}^m$,

$$0 =: t_0 < t_1 < \cdots < t_{2m-1} < t_{2m} := 1,$$

ki interpolira točke $\boldsymbol{T}_i$ in smeri tangent $\boldsymbol{d}_i$ pri parametrih t_i,

$$\boldsymbol{S}(t_i) = \boldsymbol{T}_i, \quad \frac{1}{\|\boldsymbol{S}'(t_i)\|_2}\boldsymbol{S}'(t_i) = \boldsymbol{d}_i, \quad i = 0, 1, \ldots, 2m, \tag{17}$$

kjer so $(t_{2i-1})_{i=1}^m$ neznanke. Interpolacijska shema je očitno lokalna, saj sprememba ene točke ali smeri tangente vpliva le na odsek, ki podatek vsebuje. Analizo in vse izračune lahko zato naredimo lokalno. Za $m = 1$ je zlepek $\boldsymbol{S}$ kar polinom, ki ga označimo s $\boldsymbol{P}$. Enačbe za polinomski problem se dajo prepisati v

$$\boldsymbol{P}(t_i) = \boldsymbol{T}_i, \qquad \boldsymbol{P}'(t_i) = \alpha_i \boldsymbol{d}_i, \quad \alpha_i > 0, \qquad i = 0, 1, 2, \tag{18}$$

kjer mora veljati $0 := t_0 < t_1 < t_2 := 1$. Preostane nam torej dvanajst enačb za osem neznanih koeficientov krivulje $\boldsymbol{P}$ in štiri neznane parametre t_1, α_0, α_1 in α_2. Da lahko podamo glavne eksistenčne izreke, potrebujemo še nekaj oznak. Definirajmo konstante

$$\begin{aligned}
\lambda_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_0)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, & \lambda_2 &:= \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \\
\lambda_3 &:= \frac{\det(\boldsymbol{d}_1, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, & \lambda_4 &:= \frac{\det(\Delta\boldsymbol{T}_1, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, \\
\mu_1 &:= \frac{\det(\boldsymbol{d}_0, \Delta\boldsymbol{T}_1)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}, & \mu_2 &:= \frac{\det(\Delta\boldsymbol{T}_0, \boldsymbol{d}_2)}{\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1)}
\end{aligned}$$

in funkcije

$$\begin{aligned}
&\varphi_1(\boldsymbol{\lambda}, \mu_2) := \frac{2\lambda_1\lambda_3\lambda_4}{\lambda_3\mu_2 - \lambda_2\lambda_4}, \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) := \frac{2\lambda_1\lambda_2\lambda_4}{\lambda_2\mu_1 - \lambda_1\lambda_3} \\
&\psi_1(\boldsymbol{\lambda}) := \lambda_1 \frac{\tau(\lambda_3, \lambda_2)^2(3 - 2\tau(\lambda_3, \lambda_2))}{(1 - \tau(\lambda_3, \lambda_2))^2(1 + 2\tau(\lambda_3, \lambda_2))}, \\
&\psi_2(\boldsymbol{\lambda}) := \lambda_4 \frac{\tau(\lambda_2, \lambda_3)^2(3 - 2\tau(\lambda_2, \lambda_3))}{(1 - \tau(\lambda_2, \lambda_3))^2(1 + 2\tau(\lambda_2, \lambda_3))}, \\
&\psi_3(\lambda_1, \lambda_4, \mu_1) := -\lambda_4 \frac{(3 - \tau(-\lambda_1, \mu_1))\,\tau(-\lambda_1, \mu_1)^2}{(1 - \tau(-\lambda_1, \mu_1))^3}, \\
&\psi_4(\lambda_1, \lambda_4, \mu) := \psi_3(\lambda_4, \lambda_1, \mu),
\end{aligned}$$

kjer je $\tau(\lambda_2, \lambda_3)$ definiran kot enolična rešitev problema

$$g(t_1; \lambda_2, \lambda_3) := \frac{1}{\lambda_2}\frac{t_1^2}{1 - t_1} - \frac{1}{\lambda_3}\frac{(1 - t_1)(2 + t_1)}{t_1} = 0, \quad 0 < t_1 < 1.$$

Zadostni pogoji za obstoj interpolacijske polinomske krivulje so naslednji:

IZREK 14. *Naj podatki* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0, 1, 2$, *zadoščajo* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$ *in* $\boldsymbol{\lambda} > 0$. *Če velja katerakoli od naslednjih možnosti*

1. $\mu_1 > \psi_1(\boldsymbol{\lambda})$ *in* $\mu_2 > \psi_2(\boldsymbol{\lambda})$,
2. $\frac{\lambda_1\lambda_3}{\lambda_2} \le \mu_1 < \psi_1(\boldsymbol{\lambda})$ *in* $\mu_2 < \psi_2(\boldsymbol{\lambda})$,
3. $\frac{\lambda_2\lambda_4}{\lambda_3} \le \mu_2 < \psi_2(\boldsymbol{\lambda})$ *in* $\mu_1 < \psi_1(\boldsymbol{\lambda})$,
4. $-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2}$ *in* $\varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \psi_2(\boldsymbol{\lambda})$,
5. $-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3}$ *in* $\varphi_1(\boldsymbol{\lambda}, \mu_2) < \mu_1 < \psi_1(\boldsymbol{\lambda})$,

potem kubična interpolacijska krivulja $\boldsymbol{P}$*, ki zadošča* (18)*, obstaja.*

IZREK 15. *Naj podatki* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, *zadoščajo* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$, $\lambda_i > 0$, $i = 1,2,3$, *in* $\lambda_4 < 0$. *Če velja*

$$\begin{aligned} \mu_1 > \psi_1(\boldsymbol{\lambda}) \quad &\text{in} \quad \mu_2 > \varphi_2(\boldsymbol{\lambda}, \mu_1) \quad \text{ali} \\ \frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 < \psi_1(\boldsymbol{\lambda}) \quad &\text{in} \quad \mu_2 < \varphi_2(\boldsymbol{\lambda}, \mu_1), \end{aligned}$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$*, ki zadošča* (18)*, obstaja.*

OPOMBA 16. *Primer, ko je* $\lambda_1 < 0$ *in* $\lambda_i > 0$, $i = 2,3,4$, *je simetričen primeru iz izreka 15. Velja naslednje: če je*

$$\begin{aligned} \mu_2 > \psi_2(\boldsymbol{\lambda}) \quad &\text{in} \quad \mu_1 > \varphi_1(\boldsymbol{\lambda}, \mu_2) \quad \text{ali} \\ \frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 < \psi_2(\boldsymbol{\lambda}) \quad &\text{in} \quad \mu_1 < \varphi_1(\boldsymbol{\lambda}, \mu_2), \end{aligned}$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$*, ki zadošča* (18)*, obstaja.*

IZREK 17. *Naj podatki* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, *zadoščajo* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$ *in* $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\lambda_4 < 0$. *Če je*

$$-\frac{\lambda_1\lambda_3}{\lambda_2} < \mu_1 \quad \text{in} \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \frac{\lambda_2\lambda_4}{\lambda_3} + \frac{2\lambda_1\lambda_4}{\mu_1},$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$*, ki zadošča* (18)*, obstaja.*

OPOMBA 18. *Primer, ko je* $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, *je simetričen primeru iz izreka 17. Rezultat je sledeč. Če je*

$$-\frac{\lambda_2\lambda_4}{\lambda_3} < \mu_2 \quad \text{in} \quad \varphi_1(\boldsymbol{\lambda}, \mu_2) < \mu_1 < \frac{\lambda_1\lambda_3}{\lambda_2} + \frac{2\lambda_1\lambda_4}{\mu_2},$$

potem kubična interpolacijska krivulja $\boldsymbol{P}$*, ki zadošča* (18)*, obstaja.*

IZREK 19. *Naj podatki* $\boldsymbol{T}_i$, $\boldsymbol{d}_i$, $i = 0,1,2$, *zadoščajo* $\det(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1) \neq 0$ *in* $\lambda_1 < 0$, $\lambda_4 < 0$, $\mu_1 > 0$. *Če velja katerakoli od naslednjih možnosti*

1. $\lambda_2 > 0, \quad \lambda_3 > 0, \quad \mu_2 > \psi_3(\lambda_1, \lambda_4, \mu_1)$,

2. $\lambda_2 < 0, \quad \lambda_3 > 0, \quad 0 < \mu_1 \leq \frac{\lambda_1\lambda_3}{\lambda_2}, \quad \mu_2 > \psi_3(\lambda_1, \lambda_4, \mu_1),$

3. $\lambda_2 > 0, \quad \lambda_3 < 0, \quad 0 < \mu_2 \leq \frac{\lambda_2\lambda_4}{\lambda_3}, \quad \mu_1 > \psi_4(\lambda_1, \lambda_4, \mu_2),$

4. $\lambda_2 < 0, \quad \lambda_3 < 0, \quad \mu_1 > -\frac{\lambda_1\lambda_3}{\lambda_2}, \quad \varphi_2(\boldsymbol{\lambda}, \mu_1) < \mu_2 < \psi_3(\lambda_1, \lambda_4, \mu_1),$

5. $\lambda_2 < 0, \quad \lambda_3 < 0, \quad 0 < \mu_1 \leq -\frac{\lambda_1\lambda_3}{\lambda_2}, \quad \frac{2\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_4\mu_1}{\lambda_3\mu_1} < \mu_2 < \psi_3(\lambda_1, \lambda_4, \mu_1),$

potem kubična interpolacijska krivulja $\boldsymbol{P}$, ki zadošča (18)*, obstaja.*

Dokazi teh izrekov so tehnično precej zapleteni, potekajo pa podobno kot v prejšnjih poglavjih z uporabo homotopije in Brouwerjeve stopnje. Enostavno pa se da poiskati pogoje, ko rešitev ne obstaja.

LEMA 20. *Kubična interpolacijska krivulja $\boldsymbol{P}$, ki zadošča* (18)*, ne obstaja v nobenem od naslednjih primerov:*

1. $\lambda_2 \leq 0$ *in* $\lambda_1 \geq 0$, 2. $\lambda_3 \leq 0$ *in* $\lambda_4 \geq 0$,
3. $\lambda_3 \leq 0$ *in* $\mu_1 \leq 0$, 4. $\lambda_2 \leq 0$ *in* $\mu_2 \leq 0$.

V primeru zlepkov velja naslednji rezultat.

IZREK 21. *Če na vsakem odseku $[t_{2\ell-2}, t_{2\ell}]$, $\ell = 1, 2, \ldots, m$, velja eden od pogojev iz izrekov 14–19, potem kubična G^1 krivulja zlepkov $\boldsymbol{S}$, ki zadošča* (17)*, obstaja.*

Red aproksimacije je optimalen, kar pove naslednji izrek.

IZREK 22. *Naj bodo podatki vzeti iz gladke konveksne ravninske parametrične krivulje $\boldsymbol{f} : [a, b] \to \mathbb{R}^2$ pri parametrih s_i, $a = s_0 < s_1 < \cdots < s_{2m} = b$, $i = 0, 1, \ldots, 2m$,*

$$\boldsymbol{T}_i = \boldsymbol{f}(s_i), \quad \boldsymbol{d}_i = \frac{1}{\|\boldsymbol{f}'(s_i)\|_2}\boldsymbol{f}'(s_i),$$

in naj bo $h := \max\{\Delta s_i : i = 1, 2, \ldots, 2m-1\}$. Potem obstaja taka konstanta $h_0 > 0$, da za vse h, $0 < h \leq h_0$, kubična G^1 krivulja zlepkov $\boldsymbol{S}$, ki zadošča (17)*, obstaja in aproksimira $\boldsymbol{f}$ z optimalnim redom aproksimacije šest.*

Asimptotična analiza

Iz analize kubičnega primera se vidi, da je nelinearni sistem (3) za poljubno stopnjo n nemogoče obravnavati brez kakšnih dodatnih predpostavk. V tem poglavju je uporabljen asimptotični pristop, kar pomeni, da vzamemo točke $\boldsymbol{T}_\ell$ iz gladke regularne konveksne ravninske parametrične krivulje $\boldsymbol{f} : [0, h] \to \mathbb{R}^2$, kjer je h majhen. Ker afine transformacije na točkah $\boldsymbol{T}_\ell$ ne spremenijo rešitve $\boldsymbol{t}$ sistema (3), lahko predpostavimo, da je $\boldsymbol{f}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\boldsymbol{f}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, in reparametriziramo $\boldsymbol{f}$ po prvi komponenti

$$\boldsymbol{f}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix},$$

kjer je razvoj y enak

$$y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{3!}y^{(3)}(0)x^3 + \cdots + \frac{1}{(2\,n-1)!}y^{(2\,n-1)}(0)x^{2\,n-1} + \mathcal{O}(x^{2\,n}),\ y''(0) > 0.$$

Ker nas zanimajo vrednosti $\boldsymbol{f}$ za majhne h, koordinatni sistem skaliramo z matriko

$$D_h := \operatorname{diag}\left(\frac{1}{h}, \frac{2}{y''(0)h^2}\right).$$

Za poljubno izbrane η_ℓ,

$$\eta_0 := 0 < \eta_1 < \cdots < \eta_{2n-2} < \eta_{2n-1} := 1,$$

so točke dane s $\boldsymbol{T}_\ell = D_h \boldsymbol{f}(\eta_\ell h)$, njihov razvoj pa je enak

$$\boldsymbol{T}_\ell = \begin{pmatrix} \eta_\ell \\ \sum_{k=2}^{\infty} c_k h^{k-2}\eta_\ell^k \end{pmatrix}, \quad \ell = 0, 1, \ldots, 2n-1,$$

kjer so konstante

$$c_k \;=\; \frac{2}{k!}\frac{y^{(k)}(0)}{y''(0)}, \quad k = 2, 3, \ldots$$

odvisne le od krivulje, ne pa od h ali η_ℓ. Z uporabo deljenih diferenc $[t_0, t_1, \ldots, t_{n+j}]$, $j = 1, 2, \ldots, n-1$, dobimo iz sistema (3) nelinearen sistem

$$\sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)}\boldsymbol{T}_\ell = 0, \quad j = 1, 2, \ldots, n-1, \tag{19}$$

za neznane parametre $\boldsymbol{t}$, kar je tudi edini nelinearni del celotnega problema. Rešitev v limiti, ko je $h = 0$, je enaka $\boldsymbol{t} = \boldsymbol{\eta}$. Na žalost pa je Jakobijan v rešitvi izrojen, zato z izrekom o implicitnih funkcijah ne moremo sklepati na obstoj rešitve v okolici $h = 0$. Lahko pa zamenjamo vlogo neznank t_ℓ in parametrov η_ℓ. Uvedemo še nove neznanke $\boldsymbol{\xi} := (\xi_\ell)_{\ell=1}^{2n-2}$ z reparametrizacijo η krivulje $\boldsymbol{f}$,

$$\eta \;\to\; \eta(t) \;:=\; \eta(t;\boldsymbol{\xi}), \tag{20}$$

ki je določena pri t_ℓ z enačbami

$$\eta_\ell = \eta(t_\ell;\boldsymbol{\xi}) = t_\ell + u(t_\ell;\boldsymbol{\xi}) + \xi_{n-2+\ell}h^{n-1}p(t_\ell), \quad \ell = 1, 2, \ldots, 2n-2,$$

kjer definiramo $\xi_\ell := 0,\ \ell > 2n-2$, in

$$p(t) := (t-t_0)\prod_{\ell=n+1}^{2n-1}(t-t_\ell), \quad u(t;\boldsymbol{\xi}) := (t-t_0)(t-t_{2n-1})\sum_{j=1}^{n-2}\xi_j h^j t^{j-1}.$$

Zamenjava spremenljivk $\boldsymbol{\eta} \to \boldsymbol{\xi}$ je bijektivna. Sistem enačb (19) za neznane $\boldsymbol{\xi}$ pa se zdaj glasi

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{\xi}; h) &:= (F_j(\boldsymbol{\xi}; h))_{j=1}^{n-1} = \mathbf{0}, \\ \boldsymbol{G}(\boldsymbol{\xi}; h) &:= (G_j(\boldsymbol{\xi}; h))_{j=1}^{n-1} = \mathbf{0}, \end{aligned} \tag{21}$$

kjer je

$$F_j(\boldsymbol{\xi}; h) := \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(t_\ell + u(t_\ell; \boldsymbol{\xi}) + \xi_{n-2+\ell} h^{n-1} p(t_\ell)\right)$$

in

$$G_j(\boldsymbol{\xi}; h) := \sum_{\ell=0}^{n+j} \frac{1}{\dot{\omega}_{0,n+j}(t_\ell)} \left(\sum_{k=2}^{\infty} c_k h^{k-2} \left(t_\ell + u(t_\ell; \boldsymbol{\xi}) + \xi_{n-2+\ell} h^{n-1} p(t_\ell)\right)^k\right).$$

Velja naslednji izrek.

IZREK 23. *Neznanke* $\boldsymbol{\xi}$ *so rešitev sistema* (21) *natanko takrat, ko velja*

$$\xi_{n-1} = \xi_n = \xi_{n+1} = \cdots = \xi_{2n-2}. \tag{22}$$

Iz izreka 23 sledi, da mora veljati $\xi_j = \xi_{n-1}$, $j = n, n+1, \ldots, 2n-2$. Preostale neznanke $(\xi_\ell)_{\ell=1}^{n-1}$ pa moramo določiti iz drugega dela enačb (21). Toda (22) poenostavi reparametrizacijo (20) v polinom

$$\eta(t; \boldsymbol{\xi}) = t + u(t; \boldsymbol{\xi}) + \xi_{n-1} h^{n-1} p(t)$$

in dalje funkcije $G_j(\boldsymbol{\xi}; h)$ v

$$G_j(\boldsymbol{\xi}; h) = [t_0, t_1, \ldots, t_{n+j}] \sum_{k=2}^{\infty} c_k h^{k-2} \eta(\cdot\,; \boldsymbol{\xi})^k. \tag{23}$$

Če definiramo $q(t; \boldsymbol{\xi}) := t + \sum_{\ell=1}^{n-1} \xi_\ell h^\ell t^{\ell+1}$, potem je

$$[t_0, t_1, \ldots, t_{n+j}] \eta(\cdot\,; \boldsymbol{\xi})^k = [t_0, t_1, \ldots, t_{n+j}] q(\cdot\,; \boldsymbol{\xi})^k + \mathcal{O}(h^{n+j+1-k})$$

in $[t_0, t_1, \ldots, t_{n+j}] q(\cdot\,; \boldsymbol{\xi})^k = \mathcal{O}(h^{n+j-k})$. Funkcije (23) postanejo

$$G_j(\boldsymbol{\xi}; h) = [t_0, t_1, \ldots, t_{n+j}] \sum_{k=2}^{\infty} c_k h^{k-2} q(\cdot\,; \boldsymbol{\xi})^k + \mathcal{O}(h^{n+j-1}), \tag{24}$$

naslednji izrek pa poda končno obliko nelinearnega sistema (21).

IZREK 24. *Razvoj* (24) *je enak*

$$\sum_{k=2}^{\infty} c_k h^{k-2} q(t; \boldsymbol{\xi})^k = \sum_{k=2}^{\infty} C_k(\boldsymbol{\xi}) h^{k-2} t^k,$$

kjer so

$$C_k(\boldsymbol{\xi}) := \frac{2}{k!\, h^k y''(0)} \left(\frac{d^k}{dx^k}\, y\left(h\, q(x;\boldsymbol{\xi})\right)\right)\Big|_{x=0}.$$

Polinomi $C_k(\boldsymbol{\xi})$ so odvisni le od neznank $\boldsymbol{\xi}$, ne pa tudi od h ali od parametrov $\boldsymbol{t}$. Končni nelinearni sistem (21) *za majhne h je enak*

$$C_{n+j}(\boldsymbol{\xi}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \ldots, n-1. \tag{25}$$

Glavna rezultata sta podana v naslednjih dveh izrekih.

IZREK 25. *Če obstaja tak $h_0 > 0$, da ima nelinearen sistem* (25) *vsaj eno realno rešitev za vse h, $0 \le h \le h_0$, potem interpolacijska polinomska parametrična krivulja $\boldsymbol{P}_n$ obstaja in aproksimira $\boldsymbol{f}$ z optimalnim redom aproksimacije $2n$.*

IZREK 26. *Nelinearen sistem enačb* (25) *ima realno rešitev za $n \le 5$ in dovolj majhen h.*

Dokaz zadnjega izreka je preprost za $n = 3$, za $n = 4$ in $n = 5$ pa je dokaz obstoja rešitve precej tehnično zapleten.

Krivulje blizu krožnice

Domnevo, da lahko parametrična polinomska krivulja stopnje n interpolira $2n$ točk v $\mathbb{R}^2$ v asimptotičnem smislu, smo v prejšnjem poglavju dokazali za stopnje $n \le 5$. Toda obstoj za splošen n je odprt problem že kar precej časa. V tem poglavju je dokazan obstoj interpolacijske krivulje za splošen n pod pogojem, da so točke vzete iz posebnih krivulj, tako imenovanih *krivulj blizu krožnice*.

Naj bo $\boldsymbol{\mathcal{A}} : [0, h] \mapsto \mathbb{R}^2$ krožni lok, parametriziran po naravnem parametru. Krivulja $\boldsymbol{f} \approx \boldsymbol{\mathcal{A}}$ je *krivulja blizu krožnice*, če se ujema z $\boldsymbol{\mathcal{A}}$ v točki 0 dvakratno, ima v 0 enako predznačeno ukrivljenost,

$$\boldsymbol{f}(0) = \boldsymbol{\mathcal{A}}(0), \quad \boldsymbol{f}'(0) = \boldsymbol{\mathcal{A}}'(0), \quad \det\left(\boldsymbol{f}'(0), \boldsymbol{f}''(0)\right) \det\left(\boldsymbol{\mathcal{A}}'(0), \boldsymbol{\mathcal{A}}''(0)\right) > 0,$$

in je razvoj gladkega popravka $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{\mathcal{A}}$ enak

$$\boldsymbol{g}(s) = \frac{1}{2!}\, \boldsymbol{g}''(0)\, s^2 + \frac{1}{3!}\, \boldsymbol{g}^{(3)}(0)\, s^3 + \ldots$$

Da bi razlikovali med krivuljami blizu krožnice, uvedemo konstanto M,

$$\max_{2 \le r \le 2n-1} \left\| \boldsymbol{g}^{(r)}(0) \right\|_\infty \le M, \tag{26}$$

ki meri velikosti odvodov v točki 0. Množico krivulj blizu krožnice, ki ustreza konstanti M, označimo s $\mathbb{F}_M$.

Interpolacijski problem je problem (3), pri čemer vzamemo točke $\boldsymbol{T}_j$ iz gladke krivulje blizu krožnice $\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} : [0, h] \to \mathbb{R}^2$, kjer je $\boldsymbol{\mathcal{A}}(t) := (\sin t, 1 - \cos t)^T$. Za dovolj majhen h lahko $\boldsymbol{f}$ reparametriziramo po prvi komponenti

$$\boldsymbol{f}(s) := \begin{pmatrix} s \\ u(s) \end{pmatrix} := \begin{pmatrix} s \\ \alpha(s) + \gamma(s) \end{pmatrix},$$

kjer je $\alpha(s) := 1 - \sqrt{1 - s^2}$ krožni lok, parametriziran po prvi komponenti, in

$$\gamma(s) = f_2\left(f_1^{-1}(s)\right) - \alpha(s) = \frac{\gamma''(0)}{2!} s^2 + \frac{\gamma^{(3)}(0)}{3!} s^3 + \dots.$$

Da se pokazati, da obstaja konstanta $c(M)$, odvisna le od M, da velja

$$|\gamma^{(i)}(0)| \leq c(M), \quad i = 2, 3, \dots, 2n - 1.$$

Z ustrezno izbiro konstante M ima torej krivulja blizu krožnice $\boldsymbol{f} \in \mathbb{F}_M$ poljubno majhen popravek γ, prav tako pa so poljubno majhni njegovi odvodi.

Iz prejšnega poglavja sledi, da je asimptotičen obstoj rešitve interpolacijskega problema (3) ekvivalenten dejstvu, da ima nelinearni sistem

$$C_{n+j}(\boldsymbol{a}) + \mathcal{O}(h) = 0, \quad j = 1, 2, \dots, n - 1, \tag{27}$$

realno rešitev $\boldsymbol{a} := (a_\ell)_{\ell=1}^{n-1}$ za vse dovolj majhne h. Za krivulje blizu krožnice se da funkcije C_{n+j} prepisati v

$$C_{n+j}(\boldsymbol{a}) := \frac{1}{(n+j)!} \frac{d^{n+j}}{dt^{n+j}} \left(\alpha\left(t + \sum_{\ell=1}^{n-1} a_\ell t^{\ell+1} \right) + \gamma\left(t + \sum_{\ell=1}^{n-1} a_\ell t^{\ell+1} \right) \right) \Bigg|_{t=0}.$$

Dokaže se, da ima sistem

$$\frac{1}{(n+j)!} \frac{d^{n+j}}{dt^{n+j}} \left(\alpha\left(t + \sum_{\ell=1}^{n-1} a_\ell t^{\ell+1} \right) \right) \Bigg|_{t=0} = 0, \quad j = 1, 2, \dots, n - 1, \tag{28}$$

realno rešitev in da je Jakobijan v rešitvi nesingularen. Zato po izreku o implicitni funkciji obstaja taka konstanta $c(M)$ za dovolj majhen M, da imajo enačbe

$$C_{n+j}(\boldsymbol{a}) = 0, \quad j = 1, 2, \dots, n - 1,$$

za krivulje blizu krožnice, ki zadoščajo (26) za ta poseben M, tudi realno rešitev z nesingularnim Jakobijanom. Zato pa ima, še enkrat po izreku o implicitni funkciji, tudi sistem (27) realno rešitev za dovolj majhne h. Rezultat je povzet v naslednjem izreku.

IZREK 27. *Naj bo $\boldsymbol{\mathcal{A}} : [0, h] \mapsto \mathbb{R}^2$ krožni lok, parametriziran z naravnim parametrom. Tedaj obstaja pozitivna konstanta M in tak h_0, $h_0 \leq h$, da lahko za vsak $h_1 \leq h_0$ vsako krivuljo blizu krožnice $\boldsymbol{f} = \boldsymbol{\mathcal{A}} + \boldsymbol{g} \in \mathbb{F}_M$ geometrijsko interpoliramo s polinomsko parametrično krivuljo stopnje $\leq n$ v $2n$ različnih točkah $\boldsymbol{f}(s_i)$, $s_i \in [0, h_1]$. Red aproksimacije je optimalen, to je $2n$.*

Glavni del v dokazu je pokazati, da ima sistem (28) realno rešitev z nesingularnim Jakobijanom. Da se videti, da je vprašanje obstoja rešitve ekvivalentno vprašanju, ali obstajata dva nekonstantna polinoma $x_n,\ y_n \in \mathbb{R}[t]$ stopnje $\leq n$, ki zadoščata relaciji

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n}, \quad x_n(0) = 0. \tag{29}$$

Če definiramo

$$z_n(t) := x_n^2(t) + y_n^2(t) - \left(1 + t^{2n}\right),$$

potem lahko gledamo na relacijo (29) kot na nelinearen sistem enačb

$$\frac{d^j}{dt^j} z_n(t)\Big|_{t=0} = 0, \quad j = 0, 1, \ldots, 2n, \tag{30}$$

za neznane koeficiente polinomov

$$x_n(t) = \sum_{j=1}^{n} \alpha_j\, t^j, \quad y_n(t) = \sum_{j=0}^{n} \beta_j\, t^j.$$

Rešitve sistema (30) se da numerično izračunati za nizke stopnje n, poiskati rešitev v zaključeni obliki pa je precej kompliciran problem. V pomoč nam je posebna racionalna parametrizacija enotske krožnice, in sicer

$$x_0(t) := \frac{2\sqrt{1-c^2}\, t\ (1 - c\, t)}{1 - 2\, c\, t + t^2}, \quad y_0(t) := \frac{1 - 2\, c\, t + (2\, c^2 - 1)\, t^2}{1 - 2\, c\, t + t^2},$$

kjer je $c \in [0, 1)$. Velja $x_0^2(t) + y_0^2(t) = 1$. Koeficienti polinomov x_n in y_n so v naslednjem izreku podani v zaključeni obliki s pomočjo Čebiševih polinomov prve in druge vrste.

IZREK 28. *Naj bodo števila $n,\ k$ in r povezana z enačbo*

$$n = 2^k\, (2\, r - 1), \quad k \geq 0, \quad r \geq 1, \tag{31}$$

in naj bosta konstanti $c_k,\ s_k$ definirani kot

$$c_k := \cos\left(\frac{\pi}{2^{k+1}}\right), \quad s_k := \sin\left(\frac{\pi}{2^{k+1}}\right).$$

Dalje, naj bodo q_i polinomi stopnje $\leq\ 2$ definirani kot

$$\begin{aligned} q_0(t) &:= q_0(t; k) := 1 - 2\, c_k\, t + t^2, \\ q_1(t) &:= q_1(t; k) := 2\, s_k\, t\ (1 - c_k\, t)\,, \\ q_2(t) &:= q_2(t; k) := 1 - 2\, c_k\, t + (2\, c_k^2 - 1)\, t^2. \end{aligned}$$

Tedaj sta funkciji x_n in y_n

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} := \frac{1}{q_0(t)} \begin{pmatrix} 1 & (-1)^r\, t^n \\ -(-1)^r\, t^n & 1 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix},$$

polinoma stopnje $\leq n$*, ki zadoščata* (29)*. Njuni koeficienti so enaki*

$$\begin{aligned}\alpha_j =& 2\, s_k \cos\left((j-1)\frac{\pi}{2^{k+1}}\right) = 2\, s_k\, T_{j-1}(c_k), \quad j = 1, 2, \ldots, n-1,\\ \alpha_n =& 2\, s_k \cos\left((n-1)\frac{\pi}{2^{k+1}}\right) + (-1)^r = 2\, s_k\, T_{n-1}(c_k) + (-1)^r,\end{aligned}$$

in

$$\begin{aligned}&\beta_0 = 1,\ \beta_1 = 0,\\ &\beta_j = -2\, s_k \sin\left((j-1)\frac{\pi}{2^{k+1}}\right) = -2\, s_k^2\, U_{j-2}(c_k), \quad j = 2, 3, \ldots, n.\end{aligned}$$

Iz izreka 28 sledi tudi, da je red pri aproksimaciji krožnih lokov, kot je obravnavana v [34], optimalen. Toda izrek 27 razširi rezultat na Lagrangeevo interpolacijo krožnih lokov in krivulj blizu krožnice. Videti moramo le še, da je Jakobijan sistema enačb (30) glede na neznanke α_j in β_j pri vrednostih iz izreka 28 nesingularen. Zelo lepa formula za determinanto Jakobijana, ki to potrjuje, je podana v naslednjem izreku.

IZREK 29. *Naj bodo števila* n*,* k *in* r *povezana z enačbo* (31) *in naj bodo koeficienti* α_j*,* β_j *podani z izrekom 28. Tedaj je determinanta Jakobijana sistema* (30) *v rešitvi enaka*

$$\det J = (-1)^{nr+1}\, 2^{2n+1}\, n^2\, s_k^2.$$

Izjava

Podpisana Marjetka Krajnc izjavljam, da je disertacija z naslovom *Geometrijska interpolacija z ravninskimi parametričnimi polinomskimi krivuljami* oziroma *Geometric interpolation by planar parametric polynomial curves* plod lastnega raziskovalnega dela.

Ljubljana, april 2008 Marjetka Krajnc