Triangularizability of Polynomially Compact Operators

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Abstract

An operator on a complex Banach space is polynomially compact if a non-zero polynomial of the operator is compact, and power compact if a power of the operator is compact. Theorems on triangularizability of algebras (resp. semigroups) of compact operators are shown to be valid also for algebras (resp. semigroups) of polynomially (resp. power) compact operators, provided that pairs of operators have compact commutators.

1 Basic definitions and properties

An operator T on an infinite-dimensional complex Banach space $\mathcal{X}$ is called *polynomially compact* if there exists a non-zero complex polynomial p such that the operator $p(T)$ is compact. If T^k is compact for some k we say that T is a *power compact* operator. Trivial examples of polynomially compact operators are compact and algebraic operators; the sum of a compact and an algebraic operator is also a polynomially compact operator: if $p(A) = 0$ and K is compact, then $p(A + K)$ is obviously compact. A polynomially compact operator on a Hilbert space is a compact perturbation of an algebraic operator, see Section 4 and [3, Theorem 2.4].

The monic polynomial p of the smallest degree for which the operator $p(T)$ is compact is called the *minimal polynomial* of the polynomially compact operator T. Note that if A is algebraic with minimal polynomial p then p is in general only divisible by (and not necessarily equal to) the minimal polynomial of A as a polynomially compact operator; for example, a non-zero finite-rank projection is algebraic with a minimal polynomial of degree 2 and polynomially compact with a minimal polynomial of degree 1. Obviously, the minimal polynomial of a polynomially compact operator T is the minimal polynomial of the algebraic element $\pi(T)$ of the Calkin algebra $\mathcal{B}(\mathcal{X})/\mathcal{K}(\mathcal{X})$, where $\mathcal{B}(\mathcal{X})$ denotes the algebra of all bounded linear operators on $\mathcal{X}$, and $\mathcal{K}(\mathcal{X})$ the ideal of compact operators.

A semigroup $\mathcal{S}$ on $\mathcal{X}$ is a subset of $\mathcal{B}(\mathcal{X})$ which is closed under multiplication of operators, and $\mathcal{I} \subseteq \mathcal{S}$ is an ideal if the implication

$$A \in \mathcal{I},\ B, C \in \mathcal{S} \cup \{I\} \Longrightarrow BAC \in \mathcal{I}$$

holds. An algebra $\mathcal{A}$ on $\mathcal{X}$ is a linear subset of $\mathcal{B}(\mathcal{X})$ which is also a semigroup.

A closed subspace $\mathcal{M}$ of $\mathcal{X}$ is said to be *invariant* (respectively, *hyperinvariant*) for a family of operators $\mathcal{F}$ if $A(\mathcal{M}) \subseteq \mathcal{M}$ for every $A \in \mathcal{F}$ (respectively, $B(\mathcal{M}) \subseteq \mathcal{M}$ for every B which commutes with all $A \in \mathcal{F}$).

Propositions 1.1 and 1.3 show that polynomially compact operators have many of the well-known spectral properties of compact operators. We include the proofs for the sake of completeness.

Proposition 1.1: *Let T be a polynomially compact operator on $\mathcal{X}$ and let $p(z) = (z - \lambda_1)\cdots(z - \lambda_k)$ be its minimal polynomial. Then the essential spectrum $\sigma_e(T)$ of T is the set $\{\lambda_1, \ldots, \lambda_k\}$, the spectrum $\sigma(T)$ of T consists of at most countably many points, and each point in $\sigma(T) \setminus \sigma_e(T)$ is an isolated point of the spectrum and an eigenvalue of T.*

Proof. The first statement follows from the facts that the essential spectrum of an operator T is the spectrum of the element $\pi(T)$ of the Calkin algebra and that the spectrum of an algebraic element of a Banach algebra is the set of zeros of its minimal polynomial. The countability of the spectrum is justified by the inclusion

$$\sigma(T) \subseteq \bigcup_{\mu \in \sigma(p(T))} p^{-1}(\mu),$$

where the set on the right-hand side is a countable union of finite sets and is hence countable. Finally, let $\lambda \in \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_k\}$. Since $p(\lambda)$ is a non-zero element of the spectrum of a compact operator and is therefore an isolated point of $\sigma(p(T))$, λ is an isolated point of $\sigma(T)$ by continuity of p and the fact that $p(z) = p(\lambda)$ has only finitely many solutions. We would like to prove that λ is an eigenvalue. Isolated points of the spectrum are in the boundary of the spectrum, and since it is well known that $\partial\sigma(T)$ is a subset of the approximative spectrum $\sigma_a(T)$, we infer that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors such that the sequence $\{(\lambda I - T)x_n\}_{n=1}^{\infty}$ converges to 0. Let q be the polynomial satisfying $p(\lambda) - p(z) = q(z)(\lambda - z)$; then the sequence of $(p(\lambda)I - p(T))x_n = q(T)(\lambda I - T)x_n$ also converges to 0. By definition of compactness of an operator, we can choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ for which $\{p(T)x_{n_k}\}_{k=1}^{\infty}$ converges. Without loss of generality we may assume that this is already satisfied by the original sequence. Hence, $\{p(\lambda)x_n\}_{n=1}^{\infty}$ (and, since $p(\lambda)$ is non-zero, $\{x_n\}_{n=1}^{\infty}$) also converges. Let x be the (non-zero) limit of $\{x_n\}_{n=1}^{\infty}$. Then $(\lambda I - T)x = \lim(\lambda I - T)x_n = 0$, and we can conclude that x is an eigenvector for λ. □

Remark 1.2: If $p(T)$ is compact (and p is not necessarily the minimal polynomial of T), then $\sigma_e(T)$ is a subset of the set of zeros of p.

Proposition 1.3 (Riesz Decomposition for Polynomially Compact Operators)**:** *Let T be a polynomially compact operator on $\mathcal{X}$ and let $p(z) = (z - \lambda_1)\cdots(z - \lambda_k)$ be its minimal polynomial. Let $\lambda \in \sigma(T) \setminus \sigma_e(T)$ be an eigenvalue. The following statements hold:*

1. *The kernel $\ker(\lambda I - T)^n$ is finite-dimensional for every n, and the image $\mathrm{im}(\lambda I - T)^n$ is closed and of finite codimension.*
2. *There exists $N \in \mathbb{N}$ with $\ker(\lambda I - T)^n = \ker(\lambda I - T)^N$ and $\mathrm{im}(\lambda I - T)^n = \mathrm{im}(\lambda I - T)^N$ for all $n \geq N$. Moreover, the subspaces $\ker(\lambda I - T)^N =: \mathcal{N}_\lambda$ and $\mathrm{im}(\lambda I - T)^N =: \mathcal{R}_\lambda$ are complementary hyperinvariant subspaces for T, $\sigma(T|_{\mathcal{N}_\lambda}) = \{\lambda\}$, $\sigma(T|_{\mathcal{R}_\lambda}) = \sigma(T) \setminus \{\lambda\}$.*
3. *Any subspace $\mathcal{M}$ that is invariant for T can be decomposed as $\mathcal{M} = \mathcal{N} \oplus \mathcal{R}$, where $\mathcal{N} \subseteq \mathcal{N}_\lambda$ and $\mathcal{R} \subseteq \mathcal{R}_\lambda$.*

Proof. We have already proved 1 in Proposition 1.1. Since the corresponding proposition holds for compact operators and $\ker(\lambda I - T)^n \subseteq \ker(p(\lambda)I - p(T))^n$, $\{\ker(\lambda I - T)^n\}_{n=1}^{\infty}$ is an increasing chain of subspaces contained in a finite-dimensional space and attains its supremum. Similarly, $\mathrm{im}(\lambda I - T)^n \supseteq \mathrm{im}(p(\lambda)I - p(T))^n$ is a decreasing chain of subspaces containing a subspace of finite codimension, and attains its infimum. The subspaces $\mathcal{N}_\lambda$ and

$\mathcal{R}_\lambda$ are obviously hyperinvariant for T and are complementary by a standard argument. The restriction of $\lambda I - T$ to $\mathcal{N}_\lambda$ is nilpotent, so $\sigma(T|_{\mathcal{N}_\lambda}) = \{\lambda\}$. On the other hand, λ cannot be an eigenvalue of the polynomially compact operator $T|_{\mathcal{R}_\lambda}$ (since $Tx = \lambda x$ implies $x \in \mathcal{N}_\lambda$), and the spectrum of a direct sum is the union of the spectra, so $\sigma(T|_{\mathcal{R}_\lambda})$ must be $\sigma(T) \setminus \{\lambda\}$. This proves 2. The claim 3 follows from 2 if we substitute $T|_{\mathcal{M}}$ for T. □

The set of compact operators on a Banach space is an ideal of the algebra $\mathcal{B}(\mathcal{X})$. The next example shows that the set of polynomially compact operators is not closed under either addition or multiplication.

EXAMPLE 1.4: The operators T and S on $\mathcal{X} = \ell^2$ defined by

$$T\colon (x_1, x_2, x_3, \ldots) \mapsto (x_2, 0, x_4, 0, x_6, 0, \ldots)$$

and

$$S\colon (x_1, x_2, x_3, \ldots) \mapsto (0, x_3, 0, x_5, 0, x_7, \ldots)$$

are polynomially compact ($T^2 = S^2 = 0$) but the point spectra of

$$T + S\colon (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$$

and

$$TS\colon (x_1, x_2, x_3, \ldots) \mapsto (x_3, 0, x_5, 0, x_7, 0, \ldots)$$

are the uncountably infinite set $\{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$. By Proposition 1.1, $T + S$ and TS are not polynomially compact.

The following theorem provides a sufficient condition for the polynomial compactness of a non-commutative polynomial of a given set of polynomially compact operators.
We say that the operators A and B are *essentially commuting* if the commutator $AB - BA$ is compact, or equivalently, if the images $\pi(A)$ and $\pi(B)$ in the Calkin algebra commute.

Theorem 1.5: *Let $\{T_1, \ldots, T_n\}$ be a set of essentially commuting polynomially compact operators and let r be a complex non-commutative polynomial in n variables. Then $r(T_1, \ldots, T_n)$ is a polynomially compact operator and*

$$\sigma_e(r(T_1, \ldots, T_n)) \subseteq r(\sigma_e(T_1), \ldots, \sigma_e(T_n)).$$

Here $r(\Lambda_1, \ldots, \Lambda_n)$ is the set $\{r(\lambda_1, \ldots, \lambda_n)\colon \lambda_j \in \Lambda_j \text{ for } j = 1, \ldots, n\}$. Theorem 1.5 is a corollary of the following lemma.

Lemma 1.6: *Let $t_1, \ldots, t_n$ be commuting algebraic elements of an arbitrary unital algebra $\mathcal{A}$ and let r be a complex (non-commutative) polynomial in n variables. Then $r(t_1, \ldots, t_n)$ is an algebraic element of $\mathcal{A}$ and the set of zeros of the minimal polynomial of $r(t_1, \ldots, t_n)$ is a subset of $r(\Lambda_1, \ldots, \Lambda_n)$, where Λ_j is the set of zeros of the minimal polynomial of t_j.*

Sketch of proof. First, let us assume that $n = 2$ and $r(t, s) = t + s$ (where $t = t_1$ and $s = t_2$). Let $(z - \lambda_1) \cdots (z - \lambda_k)$ and $(z - \mu_1) \cdots (z - \mu_l)$ be the minimal polynomials of t and s respectively. We claim that $q(t + s)$ is zero, where

$$q(z) := \prod_{1 \le i \le k,\, 1 \le j \le l} (z - \lambda_i - \mu_j).$$

In the product

$$q(t+s) = \prod(t+s-\lambda_i-\mu_j) = \prod((t-\lambda_i)+(s-\mu_j))$$

distributivity can be used: we get a sum of 2^{kl} terms, each of which is a product of kl elements of the forms $t-\lambda_i$ and $s-\mu_j$. Commutativity of t and s ensures that we can write the elements in these products in arbitrary order.
If a term lacks any of $t-\lambda_i$ for $1 \leq i \leq k$, that means that it includes *all* $s-\mu_j$ (for $1 \leq j \leq l$) and is hence equal to zero. If a term includes all $t-\lambda_i$ for $1 \leq i \leq k$ then it obviously equals 0.
The proof in the general case is more cumbersome, but the reasoning is essentially the same. Because $t+s-(\lambda+\mu) = (t-\lambda)+(s-\mu)$ and $ts-\lambda\mu = (t-\lambda)s+\lambda(s-\mu)$, we can write the polynomial $r(t_1,\ldots,t_n)-r(\lambda_1,\ldots,\lambda_n)$ as the sum

$$(t_1-\lambda_1)r_1(t_1,\ldots,t_n)+\ldots+(t_n-\lambda_n)r_n(t_1,\ldots,t_n)$$

for some non-commutative polynomials $r_1,\ldots,r_n$ (we omit the rather tedious proof by induction on the degree of the non-commutative polynomial r). In the product of all $r(t_1,\ldots,t_n)-r(\lambda_1,\ldots,\lambda_n)$ for $\lambda_j \in \Lambda_j$ we use distributivity and the same conclusion as in the special case in the first part of the proof. $\square$

2 Algebras of polynomially compact operators

Throughout this paper, the term "triangularizability" will mean "simultaneous triangularizability" as defined by Wojtyński ([7]) and Laurie-Nordgren-Radjavi-Rosenthal ([2]): a family $\mathcal{F}$ of operators on a Banach space $\mathcal{X}$ is triangularizable if there is a chain $\mathcal{C}$ of closed subspaces that is maximal as a chain of subspaces of $\mathcal{X}$ and has the property that every subspace in $\mathcal{C}$ is invariant under operators in $\mathcal{F}$. In [4], a wide range of results on triangularizability (of algebras and semigroups) of (especially compact and algebraic) operators is presented.
A family of operators is *reducible* if there exists a common non-trivial (non-zero and proper) invariant subspace. A chain of subspaces is *complete* if it is closed under intersections and closed linear spans. For a complete chain of subspaces $\mathcal{C}$ and $\mathcal{M} \in \mathcal{C}$, the *predecessor* $\mathcal{M}_-$ of $\mathcal{M}$ is the closed linear span of

$$\{\mathcal{N} \in \mathcal{C} \colon \mathcal{N} \subseteq \mathcal{M}, \mathcal{N} \neq \mathcal{M}\}.$$

A chain of subspaces is maximal if and only if it is complete, it includes $\{0\}$ and $\mathcal{X}$, and the quotient spaces $\mathcal{M}/\mathcal{M}_-$ are at most one-dimensional. A property of families of operators is *inherited by quotients* if for an arbitrary family of operators $\mathcal{F}$ satisfying the property, the family of all quotient operators $\widetilde{F} \colon \mathcal{M}/\mathcal{N} \to \mathcal{M}/\mathcal{N}$ for all $F \in \mathcal{F}$ also satisfies this property for every pair $\{\mathcal{M},\mathcal{N}\}$ of common invariant subspaces, $\mathcal{M} \supseteq \mathcal{N}$. The following lemma reduces the concept of triangularizability of a family satisfying a certain property inherited by quotients to reducibility, i.e. to the existence of a common invariant subspace. See [4] for details and proofs.

Lemma 2.1 (The Triangularization Lemma)**:** *If every family of operators satisfying a property inherited by quotients has a common invariant subspace, then every family of operators satisfying this property is triangularizable.*

To prove that a property is inherited by quotients, we need not consider all pairs $\{\mathcal{M},\mathcal{N}\}$ of invariant subspaces. It suffices to do so for all pairs $\{0,\mathcal{M}\}$ and $\{\mathcal{M},\mathcal{X}\}$, where $\mathcal{M}$ is a common invariant subspace of the family. The following lemma states that in a number of important cases the proof of the latter is redundant, and in order to deduce triangularizability from reducibility it is enough to check that a property is *inherited by subspaces*, i.e. that if $\mathcal{F}$ is a family satisfying $\mathcal{P}$ and $\mathcal{M}$ is a common invariant subspace, then $\mathcal{F}|_{\mathcal{M}} = \{F|_{\mathcal{M}} \colon F \in \mathcal{F}\}$ satisfies $\mathcal{P}$. The proof is based on the proof of [4, Corollary 8.4.2].

Lemma 2.2: *Let us assume that the property $\mathcal{P}$ fulfills the following two conditions:*

1. *A family of operators $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$ satisfies $\mathcal{P}$ if and only if the family of adjoints $\mathcal{F}^* = \{F^* \colon F \in \mathcal{F}\} \subseteq \mathcal{B}(\mathcal{X}^*)$ satisfies $\mathcal{P}$.*

2. *If $\Phi : \mathcal{X} \to \mathcal{Y}$ is an isometric isomorphism of Banach spaces and the family $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$ satisfies $\mathcal{P}$, then $\Phi\mathcal{F}\Phi^{-1} = \{\Phi F \Phi^{-1} \colon F \in \mathcal{F}\} \subseteq \mathcal{B}(\mathcal{Y})$ satisfies $\mathcal{P}$.*

Then the property $\mathcal{P}$ is inherited by quotients if and only if it is inherited by subspaces.

Proof. Let us assume that $\mathcal{P}$ is inherited by subspaces, and let $\mathcal{F}$ be a family of operators on $\mathcal{X}$ satisfying $\mathcal{P}$. In view of the paragraph preceding this lemma, it is enough to show that the family $\widetilde{\mathcal{F}} = \{\widetilde{F} \in \mathcal{B}(\mathcal{X}/\mathcal{M}) \colon F \in \mathcal{F}\}$ satisfies $\mathcal{P}$, where $\mathcal{M}$ is a common invariant subspace of the family $\mathcal{F}$. Let $\mathcal{M}^\perp \subseteq \mathcal{X}^*$ denote the annihilator of $\mathcal{M}$, the space of all bounded linear functionals on $\mathcal{X}$ which are zero on $\mathcal{M}$. By [5, Theorem 4.9 (b)]

$$\Phi \colon (\mathcal{X}/\mathcal{M})^* \to M^\perp,$$

defined by

$$\varphi \mapsto \varphi \circ \pi$$

(here $\pi \colon \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is the quotient map), is an isometric isomorphism. By condition 1, the family $\mathcal{F}^*$ satisfies $\mathcal{P}$, and hence the family $\mathcal{F}^*|_{\mathcal{M}^\perp}$ satisfies $\mathcal{P}$. For an operator F from $\mathcal{F}$, $F^*|_{\mathcal{M}^\perp}$ and $\widetilde{F}^*$ can be identified through the isometric isomorphism Φ:

$$((F^*\Phi(\varphi))(x) = \Phi(\varphi)(Fx) = \varphi(\pi(Fx))$$

and

$$(\Phi\widetilde{F}^*(\varphi))(x) = \widetilde{F}^*(\varphi)(\pi(x)) = \varphi(\widetilde{F}(\pi(x))) = \varphi(\pi(Fx)).$$

By condition 2, the family $\widetilde{\mathcal{F}}^*$ satisfies $\mathcal{P}$. Again by 1, the same holds for the family $\widetilde{\mathcal{F}}$. $\square$

Note that the hypotheses of the lemma are fulfilled by any properties involving compactness (e.g. polynomial compactness or essential commutativity) and conditions on spectrum (such as quasinilpotency or sublinearity, cf. Proposition 3.5).
This section will show that the results on algebras of compact operators can be generalized to algebras of essentially commuting polynomially compact operators. As in the compact case, the key arguments are the following two results due to Lomonosov (see e.g. [1, 4.9 and 4.13]).

Theorem 2.3 (Lomonosov's Lemma)**:** *Let $\mathcal{A}$ be an irreducible subalgebra of $\mathcal{B}(\mathcal{X})$, and let K be an arbitrary non-zero compact operator in $\mathcal{B}(\mathcal{X})$. Then there exists $A \in \mathcal{A}$ such that 1 is in the (point) spectrum of AK.*

Corollary 2.4 (Lomonosov's Theorem)**:** *Let T be a non-scalar operator (i.e., an operator which is not a scalar multiple of identity) which commutes with a non-zero compact operator K. Then T has a non-trivial hyperinvariant subspace.*

Proposition 2.5: *A non-scalar polynomially compact operator T has a non-trivial hyperinvariant subspace.*

Proof. If T is algebraic, then any of the subspaces $\ker(\lambda I - T)$ for $\lambda \in \sigma(T)$ is hyperinvariant for T and non-trivial. Otherwise, let $K = p(T)$ be a non-zero compact operator. Obviously, K and T commute, and by the preceding corollary T has a non-trivial hyperinvariant subspace. □

Corollary 2.6: *A commuting family of polynomially compact operators is triangularizable.*

Proof. By the Triangularization Lemma and the fact that polynomial compactness is inherited by quotients, it suffices to prove that there exists a common hyperinvariant subspace for the family $\mathcal{F}$. If all operators in $\mathcal{F}$ are scalar, any subspace will do. Otherwise, such a subspace is given by the previous proposition. □

The proof of Ringrose's Theorem for compact operators works as well for polynomially compact operators, with only minor modifications, and this theorem will be of paramount importance for the remaining results of this paper.

Definition: Let T be polynomially compact, and let $\mathcal{C}$ be a triangularizing chain for T (i.e. a chain of invariant subspaces for T that is maximal as a chain of subspaces; such a chain exists by the last corollary). Let $\mathcal{C}'$ be the subchain of all $\mathcal{M} \in \mathcal{C}$ with $\mathcal{M}_- \neq \mathcal{M}$. For $\mathcal{M} \in \mathcal{C}'$ the space $\mathcal{M}/\mathcal{M}_-$ is one-dimensional. The *diagonal coefficient of T corresponding to $\mathcal{M}$*, $\lambda_{\mathcal{M}}$, is the (only) point in the spectrum of $\widetilde{T} \in \mathcal{B}(\mathcal{M}/\mathcal{M}_-)$.

Theorem 2.7 (Ringrose's Theorem for Polynomially Compact Operators)**:** *Let T be a polynomially compact operator and let $p(z) = (z - \lambda_1) \cdots (z - \lambda_k)$ be its minimal polynomial. If $\mathcal{C}$ is any triangularizing chain for T, then*

$$\sigma(T) = \sigma_e(T) \cup \{\lambda_{\mathcal{M}} \colon \mathcal{M} \in \mathcal{C}'\},$$

where $\sigma_e(T) = \{\lambda_1, \ldots, \lambda_k\}$.

Proof. If $\mathcal{M} \in \mathcal{C}'$ and $\lambda_{\mathcal{M}} \notin \sigma_e(T)$, then $(\lambda_{\mathcal{M}} I - T)\mathcal{M} \subseteq \mathcal{M}_-$ and $\lambda_{\mathcal{M}}$ is an element of the spectrum of the polynomially compact operator $T|_{\mathcal{M}}$ that is not in the essential spectrum, and is therefore an eigenvalue of this restriction. We conclude that $\lambda_{\mathcal{M}}$ is also an eigenvalue of T, so $\sigma_e(T) \cup \{\lambda_{\mathcal{M}} \colon \mathcal{M} \in \mathcal{C}'\} \subseteq \sigma(T)$.
Let us prove the other inclusion. Let $\lambda \in \sigma(T) \setminus \sigma_e(T)$ be an eigenvalue of T; we want to find $\mathcal{M} \in \mathcal{C}'$ for which $\lambda_{\mathcal{M}} = \lambda$. The set $\mathcal{O} := \{x \in \mathcal{X} \colon Tx = \lambda x, \|x\| = 1\}$ is compact, so the subspace $\mathcal{M} := \cap\{\mathcal{N} \in \mathcal{C} \colon \mathcal{N} \cap \mathcal{O} \neq \emptyset\} \in \mathcal{C}$ has a non-empty intersection with $\mathcal{O}$.
For every proper subspace $\mathcal{L} \in \mathcal{C}$ of $\mathcal{M}$ we have $\ker((\lambda I - T)|_{\mathcal{L}})^n = \{0\}$ for any n, and by Proposition 1.3 (claim 3), $\mathcal{L} \subseteq \operatorname{im}(\lambda I - T)^N$ (for some N). Since $\mathcal{M}_-$ is spanned by proper subspaces of $\mathcal{M}$, we have $\mathcal{M}_- \subseteq \operatorname{im}(\lambda I - T)^N$, and from $\mathcal{M} \cap \ker(\lambda I - T)^N \neq \{0\}$ we conclude that $\mathcal{M}_- \neq \mathcal{M}$.
Choose $x \in \mathcal{M}$, $\|x\| = 1$, $Tx = \lambda x$. Then

$$\widetilde{T}(x + \mathcal{M}_-) = Tx + \mathcal{M}_- = \lambda x + \mathcal{M}_- = \lambda(x + \mathcal{M}_-)$$

and consequently $\lambda = \lambda_{\mathcal{M}}$. $\square$

Theorem 2.8 (Spectral Mapping Theorem)**:** *Let $\{T_1, \ldots, T_n\}$ be a triangularizable set of essentially commuting polynomially compact operators. Then for any complex non-commutative polynomial r in n variables the operator $r(T_1, \ldots, T_n)$ is polynomially compact and*

$$\sigma(r(T_1, \ldots, T_n)) \subseteq r(\sigma(T_1), \ldots, \sigma(T_n)).$$

Proof. Let $\mathcal{C}$ be a triangularizing chain for $\{T_1, \ldots, T_n\}$ (and $r(T_1, \ldots, T_n)$). By Theorem 1.5, $r(T_1, \ldots, T_n)$ is a polynomially compact operator, and we have the inclusion

$$\sigma_e(r(T_1, \ldots, T_n)) \subseteq r(\sigma_e(T_1), \ldots, \sigma_e(T_n)) \subseteq r(\sigma(T_1), \ldots, \sigma(T_n)).$$

To complete the proof, let $\lambda \in \sigma(r(T_1, \ldots, T_n)) \setminus \sigma_e(r(T_1, \ldots, T_n))$. By Theorem 2.7, $\lambda = \lambda_{\mathcal{M}}$ for some $\mathcal{M} \in \mathcal{C}'$. For any $x \in \mathcal{M} \setminus \mathcal{M}_-$, $r(\widetilde{T}_1, \ldots, \widetilde{T}_n)(x + \mathcal{M}_-) = \lambda(x + \mathcal{M}_-)$ and $\widetilde{T}_j(x + \mathcal{M}_-) = \lambda_j(x + \mathcal{M}_-)$ for some $\lambda_j \in \sigma(T_j)$. Obviously, $r(\lambda_1, \ldots, \lambda_n) = \lambda$. $\square$

Proposition 2.9: *If T and S are essentially commuting polynomially compact operators, then the pair $\{T, S\}$ is triangularizable if and only if $r(T, S)(TS - ST)$ is quasinilpotent for every non-commutative polynomial r.*

Proof. By the Spectral Mapping Theorem,

$$\sigma(r(T, S)(TS - ST)) \subseteq \{r(\lambda, \mu)(\lambda\mu - \mu\lambda) \colon \lambda \in \sigma(T), \mu \in \sigma(S)\} = \{0\}$$

if $\{T, S\}$ is triangularizable, so the condition is necessary. For the converse, first note that by the Triangularization Lemma (quasinilpotency is a property inherited by quotients, by Lemma 2.2 or otherwise), it suffices to prove reducibility. If T and S commute, they have a common invariant subspace by Corollary 2.6. Otherwise, assume irreducibility; the unital algebra generated by T and S (the algebra of all non-commutative polynomials in T and S) is also irreducible, and $TS - ST$ is a non-zero compact operator. By Lomonosov's Lemma, there exists a non-commutative polynomial r such that 1 is in the spectrum of $r(T, S)(TS - ST)$, which is therefore not quasinilpotent. $\square$

Theorem 2.10: *An algebra $\mathcal{A}$ of essentially commuting polynomially compact operators is triangularizable if and only if every pair of operators in the algebra is triangularizable.*

Proof. Let $\mathcal{C}$ be a maximal chain of invariant subspaces for the algebra (such a chain exists by the Hausdorff Maximality Principle). We have to prove that $\mathcal{C}$ is maximal as a chain of subspaces in $\mathcal{X}$, or equivalently, that the dimension of $\mathcal{M}/\mathcal{M}_-$ is at most 1 for any $\mathcal{M} \in \mathcal{C}$. Assume $\dim(\mathcal{M}/\mathcal{M}_-) > 1$. The quotient algebra $\widetilde{A}$ is irreducible by maximality of $\mathcal{C}$ and cannot be commutative. Let T and S be such operators in $\mathcal{A}$ that the compact operator $\widetilde{T}\widetilde{S} - \widetilde{S}\widetilde{T}$ is non-zero. By Lomonosov's Lemma, there is an R in $\mathcal{A}$ and $x \in \mathcal{M} \setminus \mathcal{M}_-$ such that

$$\widetilde{R}(\widetilde{T}\widetilde{S} - \widetilde{S}\widetilde{T})(x + \mathcal{M}_-) = x + \mathcal{M}_-,$$

in other words, $R(TS - ST)x \in x + \mathcal{M}_-$. Let $\mathcal{N}$ denote the subspace generated by x and $\mathcal{M}_-$. Obviously we have $\operatorname{im}(R(TS - ST) - I)|_{\mathcal{N}} \subseteq \mathcal{M}_-$; so 1 is an eigenvalue of $R(TS - ST)|_{\mathcal{N}}$ and of $R(TS - ST)$. However, the pairs $\{T, S\}$ and $\{R, TS - ST\}$ are triangularizable and by the Spectral Mapping Theorem, $\sigma(TS - ST) = \{0\}$ and $\sigma(R(TS - ST)) = \{0\}$. The contradiction completes the proof. $\square$

Corollary 2.11: *An algebra $\mathcal{A}$ of essentially commuting polynomially compact operators is triangularizable if and only if for every pair $\{T, S\}$ of operators in $\mathcal{A}$, $r(T,S)(TS - ST)$ is quasinilpotent for every non-commutative polynomial r.*

3 Semigroups of power compact operators

The theorems on triangularizability of semigroups of compact operators depend heavily on the following two results: Turovskii's Theorem, which states that a semigroup of compact quasinilpotents is reducible (triangularizable), and a lemma due to Radjavi ([4, Lemma 7.4.5]) which guarantees the existence of a non-zero finite-rank operator (a nilpotent or an idempotent) in a uniformly closed semigroup of compact operators which is closed under multiplication by non-negative real numbers and which contains a non-quasinilpotent element.
Example 3.1 shows that the latter does not extend to semigroups of arbitrary essentially commuting polynomially compact operators.

EXAMPLE 3.1: Let $\mathcal{X} = \ell^2$ and let T be the projection

$$T\colon (x_1, x_2, x_3, \ldots) \mapsto (0, x_2, x_3, \ldots).$$

Since $T^2 = T$, $\mathcal{S} = \{\lambda T\colon \lambda \geq 0\}$ is a semigroup of (essentially) commuting polynomially compact operators ($\lambda T - \lambda I$ is compact), and there is an element of $\mathcal{S}$ (namely, λT for any $\lambda > 0$) with $\sigma(T) \neq \sigma_e(T)$. The semigroup is uniformly closed and is also closed under multiplication by non-negative real numbers, however, it does not contain any non-zero finite-rank operators.

Therefore, generalizing the results to arbitrary semigroups of (essentially commuting) polynomially compact operators appears to be quite a formidable task. We will restrict our attention to semigroups of *power* compact operators. Lemma 3.2 gives the crucial result of this section. As an application we prove Theorem 3.3, which gives several powerful sufficient conditions for the triangularizability of semigroups of essentially commuting power compact operators.

Lemma 3.2: *Let $\mathcal{P}$ be a property defined for semigroups of essentially commuting power compact operators such that the ideal $\mathcal{S} \cap \mathcal{K}(\mathcal{X})$ of a semigroup $\mathcal{S}$ satisfying $\mathcal{P}$ also satisfies $\mathcal{P}$. If every semigroup of compact operators satisfying $\mathcal{P}$ is reducible, then every semigroup of essentially commuting power compact operators satisfying $\mathcal{P}$ is reducible. If, in addition, $\mathcal{P}$ is inherited by quotients, every semigroup of essentially commuting power compact operators satisfying $\mathcal{P}$ is triangularizable.*

Proof. Let $\mathcal{S}$ be a semigroup of essentially commuting power compact operators satisfying $\mathcal{P}$. If $\mathcal{S}$ contains non-zero compact operators then the ideal $\mathcal{S} \cap \mathcal{K}(\mathcal{X})$ is non-zero; it satisfies $\mathcal{P}$ and is therefore reducible by hypothesis. It is well known that a semigroup with a non-zero reducible ideal is also reducible.
Otherwise, $\mathcal{S}$ consists of essentially commuting nilpotents. Let T be a non-zero element of $\mathcal{S}$, without loss of generality we may assume that it is a nilpotent of order 2. For any S in $\mathcal{S}$, $TST = T(ST - TS)$ is a compact element of $\mathcal{S}$ and hence zero. In other words, $\mathcal{S}$ maps the image of T to the kernel of T. If x is a non-zero vector in $\operatorname{im} T$, then the closed linear span of

$$\{Sx\colon S \in \mathcal{S} \cup \{I\}\}$$

is a non-zero invariant subspace for $\mathcal{S}$ which is contained in the subspace $\ker T$ and is therefore non-trivial. If $\mathcal{P}$ is inherited by quotients, Triangularization Lemma yields triangularizability. □

Theorem 3.3: *Let $\mathcal{S}$ be a semigroup of essentially commuting power compact operators. Then any of the following conditions on $\mathcal{S}$ implies reducibility (and, in cases 1, 4, 5 and 6, triangularizability):*

1. *All elements of $\mathcal{S}$ are quasinilpotents.*
2. *Spectrum is permutable on $\mathcal{S}$, i.e. for any $T_1, \ldots, T_n$ in $\mathcal{F}$ and any permutation π of n elements, $\sigma(T_1 \cdots T_n) = \sigma(T_{\pi(1)} \cdots T_{\pi(n)})$, or equivalently, for any T, S and R in $\mathcal{S}$, $\sigma(TSR) = \sigma(STR)$.*
3. *Spectrum is submultiplicative on $\mathcal{S}$, i.e. $\sigma(TS) \subseteq \sigma(T) \cdot \sigma(S)$ for any T and S in $\mathcal{S}$.*
4. *Spectrum is sublinear on $\mathcal{S}$, i.e. $\sigma(T + \lambda S) \subseteq \sigma(T) + \lambda \cdot \sigma(S)$ for any T, S in $\mathcal{S}$ and $\lambda \in \mathbb{C}$.*
5. *Spectrum is real-sublinear on $\mathcal{S}$, i.e. $\sigma(T + \lambda S) \subseteq \sigma(T) + \lambda \cdot \sigma(S)$ for any T, S in $\mathcal{S}$ and $\lambda \in \mathbb{R}$.*
6. *Every pair $\{T, S\}$ in $\mathcal{S}$ is triangularizable.*

Proof. First, let us prove reducibility in the cases 1–5: by Lemma 3.2, since all the properties are trivially inherited by all subfamilies, it suffices to know that these conditions imply reducibility for semigroups of compact operators. The first is given by the well-known result of Turovskii ([6, Theorem 4]); the rest are Theorems 8.3.1, 8.3.5, 8.4.3, and 8.4.7 from [4] respectively.
What remains is to prove that conditions 1, 4 and 5 are inherited by quotients. We have already mentioned that this is true for 1, and the proof for 5 is almost the same as that for 4, which is given (in a slightly more general form, namely for families of essentially commuting polynomially – not necessarily power – compact operators) in Proposition 3.5.
The condition 6 implies 4 (by the Spectral Mapping Theorem), and we conclude that a semigroup of essentially commuting power compact operators with triangularizable pairs is triangularizable. □

Remark 3.4: Note that the conditions 2 – 6 of the last theorem are necessary for the triangularizability of any family of essentially commuting polynomially compact operators: 3, 4 and 5 by the Spectral Mapping Theorem (Theorem 2.8), 6 trivially, and 2 by a simple argument using Theorem 2.7. Indeed, if a family of essentially commuting polynomially compact operators $\{T_1, \ldots, T_n\}$ is triangularizable and π is a permutation of n elements, then the essential spectra of $T_1 \cdots T_n$ and $T_{\pi(1)} \cdots T_{\pi(n)}$ are the same because one is a compact perturbation of the other; if $\lambda \in \sigma(T_1 \cdots T_n) \setminus \sigma_e(T_1 \cdots T_n)$, there is a subspace $\mathcal{M} \in \mathcal{C}'$ in the triangularizing chain $\mathcal{C}$ such that $\lambda = \lambda_{\mathcal{M}}$; λ must be equal to $\lambda_1 \cdots \lambda_n$ for some $\lambda_j \in \sigma(T_j)$, and $\lambda = \lambda_{\pi(1)} \cdots \lambda_{\pi(n)} \in \sigma(T_{\pi(1)} \cdots T_{\pi(n)})$.

The proof of Proposition 3.5 follows the steps of the proof of [4, Lemma 8.4.1] but avoids the use of the Baire Category Theorem.

Proposition 3.5: *Sublinearity of spectrum for essentially commuting polynomially compact operators is inherited by quotients.*

Proof. In view of Lemma 2.2, we have to prove that the property is inherited by subspaces. Let us suppose that spectrum is sublinear on the pair $\{T, S\}$, where T and S are essentially commuting polynomially compact operators. We want to prove that spectrum is sublinear on the pair $\{T|_{\mathcal{M}}, S|_{\mathcal{M}}\}$ for any common invariant subspace $\mathcal{M}$ of T and S.
Let $\mathcal{D}$ be the set

$$\{\lambda \in \mathbb{C}\colon \sigma(\widetilde{T} + \lambda\widetilde{S}) \subseteq \sigma(\widetilde{T}) + \lambda \cdot \sigma(\widetilde{S})\},$$

where $\widetilde{T} = T|_{\mathcal{M}}$ and $\widetilde{S} = S|_{\mathcal{M}}$. First, let us prove that $\mathcal{D}$ is closed in $\mathbb{C}$. Assume that a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathcal{D}$ converging to $\lambda \in \mathbb{C}$ is given. By Theorem 1.5, $\widetilde{T}+\lambda\widetilde{S}$ is polynomially compact and

$$\sigma_e(\widetilde{T} + \lambda\widetilde{S}) \subseteq \sigma_e(\widetilde{T}) + \lambda \cdot \sigma_e(\widetilde{S}) \subseteq \sigma(\widetilde{T}) + \lambda \cdot \sigma(\widetilde{S}).$$

If $\mu \in \sigma(\widetilde{T} + \lambda\widetilde{S}) \setminus \sigma_e(\widetilde{T} + \lambda\widetilde{S})$, then it is an isolated point of the spectrum (Proposition 1.1). This means (e.g. [4, Theorem 7.2.10]) that there is a sequence $\{\mu_n\}_{n=1}^{\infty}$, $\mu_n \in \sigma(\widetilde{T} + \lambda_n\widetilde{S})$, that converges to μ. Since $\lambda_n \in \mathcal{D}$ for all n, there exist sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$, $\alpha_n \in \sigma(\widetilde{T})$, $\beta_n \in \sigma(\widetilde{S})$, such that $\mu_n = \alpha_n + \lambda_n\beta_n$. Without loss of generality we may assume that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ converge to $\alpha \in \sigma(\widetilde{T})$ and $\beta \in \sigma(\widetilde{S})$, respectively. We conclude that $\mu = \lim \mu_n = \alpha + \lambda\beta$ is in $\sigma(\widetilde{T}) + \lambda \cdot \sigma(\widetilde{S})$, and λ is an element of $\mathcal{D}$.
Now we will prove that $\mathbb{C} \setminus \mathcal{D}$ is countable (and hence empty). For

$$(\alpha, \beta) \in (\sigma(T) \times \sigma(S)) \setminus (\sigma(\widetilde{T}) \times \sigma(\widetilde{S}))$$

(this set is countable by Proposition 1.1), let $\mathcal{E}_{\alpha,\beta}$ denote the set

$$\{\lambda \in \mathbb{C}\colon \alpha + \lambda\beta \in \sigma(\widetilde{T} + \lambda\widetilde{S})\}.$$

For $\lambda \in \mathbb{C} \setminus \mathcal{D}$, the inclusion $\sigma(\widetilde{T}+\lambda\widetilde{S}) \subseteq \sigma(\widetilde{T}) + \lambda \cdot \sigma(\widetilde{S})$ does not hold. In other words, there exists $\mu \in \sigma(\widetilde{T} + \lambda\widetilde{S}) \subseteq \sigma(T + \lambda S) \subseteq \sigma(T) + \lambda \cdot \sigma(S)$, $\mu = \alpha + \lambda\beta$ for $\alpha \in \sigma(T)$, $\beta \in \sigma(S)$, which is not an element of $\sigma(\widetilde{T}) + \lambda \cdot \sigma(\widetilde{S})$. We have proved

$$\mathbb{C} \setminus \mathcal{D} \subseteq \bigcup \mathcal{E}_{\alpha,\beta}$$

and all that remains to be seen is that the sets $\mathcal{E}_{\alpha,\beta}$ are countable.
One of $\widetilde{T}-\alpha I$ and $\widetilde{S}-\beta I$ is invertible, we can assume that it is $\widetilde{T}-\alpha I$. Hence $\widetilde{T}+\lambda\widetilde{S}-(\alpha+\lambda\beta)I$ is non-invertible if and only if $I + \lambda(\widetilde{S} - \beta I)(\widetilde{T} - \alpha I)^{-1}$ is non-invertible. The inverse of an invertible polynomially compact operator is again polynomially compact (if $R^k + a_{k-1}R^{k-1} + \ldots + a_0 I$ is compact, then so is $a_0 R^{-k} + a_1 R^{-k+1} + \ldots + I$), and if polynomially compact R and Q essentially commute, so do R^{-1} and Q (since $R^{-1}Q - QR^{-1} = R^{-1}(QR - RQ)R^{-1}$), so $(\widetilde{S} - \beta I)(\widetilde{T} - \alpha I)^{-1}$ is a polynomially compact operator by Theorem 1.5, and its spectrum is countable by Proposition 1.1. In other words, $I + \lambda(\widetilde{S} - \beta I)(\widetilde{T} - \alpha I)^{-1}$ is non-invertible only for countably many λ, and therefore $\mathcal{E}_{\alpha,\beta}$ is countable. □

4 A remark on the structure of polynomially compact operators on Hilbert spaces

The fundamental theorem on the structure of polynomially compact operators was given by Olsen in [3]: any polynomially compact operator (an operator whose image in the Calkin

algebra is algebraic) on a separable Hilbert space can be written as a sum of a compact and an algebraic operator (whose minimal polynomial is equal to the minimal polynomial of the given polynomially compact operator). This is a complete analog of the well-known West decomposition, which states that a Riesz operator (an operator whose image in the Calkin algebra is quasinilpotent) on a (not necessarily separable) Hilbert space can be written as a sum of a compact and a quasinilpotent operator. A question arises whether the decomposition given by Olsen's theorem holds in the general (non-separable) Hilbert-space setting. Let us offer two proofs that the answer is affirmative; the first is a modification of the proof for separable spaces (a certain sequence is replaced by a net), and the second uses Olsen's result for the restriction of the polynomially compact operator to a separable subspace.
Observe that in the proof of Olsen's theorem the only time separability is used is in the proof of [3, Lemma 2.1] where a sequence of finite-rank projections converging strongly to the given projection is considered. In a non-separable Hilbert space such a sequence does not necessarily exist. However, the net given by Lemma 4.1 suffices for the completion of the proof.

Lemma 4.1: *Let $\mathcal{H}$ be a Hilbert space. Then there exists a net $\{P_i\}_{i\in\mathcal{I}}$ of finite-rank projections that converges strongly to the identity operator.*

Proof. By the Zorn Lemma, there is a maximal orthonormal system $\{e_\lambda\colon \lambda\in\Lambda\}$ in $\mathcal{H}$. Let $\mathcal{I}$ be the set $\{\mathcal{K}\subseteq\Lambda\colon \mathcal{K} \text{ finite}\}$ ordered by inclusion and let $P_\mathcal{K}$ be the projection

$$x\mapsto\sum_{\lambda\in\mathcal{K}}\langle x,e_\lambda\rangle e_\lambda.$$

We claim that the net $\{P_\mathcal{K}\}_{\mathcal{K}\in\mathcal{I}}$ converges strongly to identity.
Choose $x\in\mathcal{H}$. For any $\varepsilon>0$, $|\langle x,e_\lambda\rangle|\geq\varepsilon$ for only finitely many $\lambda\in\Lambda$ by Bessel's inequality. Hence, $\langle x,e_\lambda\rangle=0$ for all $\lambda\in\Lambda\setminus\{\lambda_n\}_{n=1}^\infty$. The series $\sum_{n=1}^\infty\langle x,e_{\lambda_n}\rangle e_{\lambda_n}$ converges to some $y\in\mathcal{H}$, for any n we have $\langle x-y,e_{\lambda_n}\rangle=\langle x,e_{\lambda_n}\rangle-\langle y,e_{\lambda_n}\rangle=0$, and $\langle x,e_\lambda\rangle=\langle y,e_\lambda\rangle=0$ for $\lambda\in\Lambda\setminus\{\lambda_n\}_{n=1}^\infty$. The maximality of $\{e_\lambda\colon\lambda\in\Lambda\}$ guarantees that $x=y$. We conclude that for a positive ε there is a finite set $\mathcal{K}$ ($\mathcal{K}=\{\lambda_1,\dots,\lambda_n\}$ for some $n\in\mathbb{N}$) such that $\|x-P_\mathcal{L}x\|<\varepsilon$ for any set $\mathcal{L}\in\mathcal{I}$ that includes $\mathcal{K}$ as a subset. □

Alternatively, let $K=p(T)$ be compact (possibly zero). The image of K is separable, so $\mathcal{H}_1$, the closure of the subspace

$$\{q(T,T^*)x\colon q \text{ non-commutative polynomial}, x\in\operatorname{im}K\},$$

is also separable, and is invariant for T and T^*. Therefore, with respect to the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ (where $\mathcal{H}_2=\mathcal{H}_1^\perp$), T can be written as $T=T_1\oplus T_2$ for $T_1\in\mathcal{B}(\mathcal{H}_1)$, $T_2\in\mathcal{B}(\mathcal{H}_2)$. The operator T_1 is an operator on a separable Hilbert space, $p(T_1)$ is compact, so by Olsen's theorem $T_1=K_1+A_1$ for K_1 compact and $p(A_1)=0$. The image of $p(T_2)\in\mathcal{B}(\mathcal{H}_2)$ is contained in $\operatorname{im}p(T)=\operatorname{im}K\subseteq\mathcal{H}_1$ and must be zero. We conclude that $T=(K_1\oplus 0)+(A_1\oplus T_2)$, $K_1\oplus 0$ is compact, and $p(A_1\oplus T_2)=0$. If p is minimal for T, p is also minimal for $A_1\oplus T_2$.

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