Divisibility of generalized Catalan numbers

Matjaž Konvalinka

Abstract

We define a q generalization of weighted Catalan numbers studied by Postnikov and Sagan, and prove a result on the divisibility by p of such numbers when p is a prime and q its power.

1 Introduction

The n-th Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is equal to the number of binary trees on n vertices, of lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that stay above the y-axis (Dyck paths), and of many other objects (see [S99]). If $\mathbf{b}\colon \mathbb{N} \to \mathbb{Z}$ is a function and $\{b_i\} = \mathbf{b}(i)$ is the corresponding sequence, we weight a vertex of a binary tree $\mathcal{T}$ by b_i, where i is the number of left edges on the unique path from the vertex to the root of the tree, and we define the weight $w(\mathcal{T})$ of the tree to be the product of the weights of its vertices. Then the *weighted n-th Catalan number* is

$$C_n(\mathbf{b}) = \sum_{\mathcal{T}} w(\mathcal{T}),$$

where the sum is over all binary trees on n points. We weight each step $(1,1)$ of a Dyck path by b_i, where i is the y-coordinate of the starting point, and we weight a Dyck path by the product of weights of its up steps; then

$$C_n(\mathbf{b}) = \sum_{\mathcal{P}} w(\mathcal{P}),$$

where the sum is over all Dyck paths from $(0,0)$ to $(2n,0)$.

The divisibility of Catalan numbers C_n by powers of 2 has been determined both arithmetically and combinatorially (see for example [AK73], [E83], [SU91], [D99], [DS06]); if we denote the maximal ξ for which $q^\xi | m$ by $\xi_q(m)$, and the sum of the digits in the q-ary expansion of m by $s_q(m)$, then

$$\xi_2(C_n) = s_2(n+1) - 1. \tag{1}$$

A natural question arises: under what conditions on $\mathbf{b}$ do we have $\xi_2(C_n(\mathbf{b})) = \xi_2(C_n)$? Postnikov and Sagan ([SP06, Theorem 2.1]) found the following sufficient condition. Here the operator Δ is defined by $\Delta\mathbf{f}(x) = \mathbf{f}(x+1) - \mathbf{f}(x)$.

Theorem 1 *Assume that* $\mathbf{b}$ *satisfies* $\mathbf{b}(0) = 1 \pmod 2$ *and* $2^{n+1} | \Delta^n \mathbf{b}(x)$ *for all* $n \geq 1$ *and* $x \in \mathbb{N}$. *Then* $\xi_2(C_n) = \xi_2(C_n(\mathbf{b}))$. □

In this paper, we will define a generalization of weighted Catalan numbers and prove an analogous theorem.

2 Generalized Catalan numbers

For $q \geq 1$ and $n \geq 0$, define

$$C_n^{(q)} = \frac{1}{(q-1)n+1}\binom{qn}{n}.$$

It is well known that this counts the number of lattice paths P in the plane from $(0,0)$ to $(qn,0)$ using steps $(1,q-1)$ and $(1,-1)$ that never go below the y axis, and the number of q-ary trees on n vertices (recall that a rooted tree is q-ary if every vertex has q distinguishable possibly empty branches). If $F^{(q)}(x)$ is the ordinary generating function for $C_n^{(q)}$, then obviously

$$F^{(q)}(x) = 1 + x\left(F^{(q)}(x)\right)^q,$$

and so the numbers $C_n^{(q)}$ are the coefficients in the Taylor expansion of the continued fraction

$$\cfrac{1}{1 - \cfrac{x}{\left(1 - \cfrac{x}{\left(1 - \frac{x}{(1-\ldots)^{q-1}}\right)^{q-1}}\right)^{q-1}}}.$$

The following is a generalization of (1).

Proposition 2 *Assume that* $q = p^k$ *where* p *is a prime and* $k \geq 1$. *Then we have*

$$\xi_p(C_n^{(q)}) = \frac{s_p((q-1)n+1)-1}{p-1} \tag{2}$$

for any n.

Proof: The exponent of p in the prime factorization of $m!$ is

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \ldots$$

In

$$C_n^{(q)} = \frac{1}{(q-1)n+1}\binom{qn}{n} = \frac{(qn)!}{n!((q-1)n+1)!}$$

the numerator contains $p^{k-1}n+p^{k-2}n+\ldots+n+\lfloor n/p\rfloor+\lfloor n/p^2\rfloor+\lfloor n/p^3\rfloor+\ldots$ p factors, $n!$ contains $\lfloor n/p\rfloor+\lfloor n/p^2\rfloor+\lfloor n/p^3\rfloor+\ldots$ p factors, and $((q-1)n+1)!$ contains

$$(a_1+a_2p+\ldots+a_tp^{t-1})+(a_2+a_3p+\ldots+a_tp^{t-1})+\ldots+(a_{t-1}+a_tp)+a_t=$$

$$=a_1+a_2(1+p)+\ldots+a_t(1+\ldots+p^{t-1})=\frac{a_1(p-1)+\ldots+a_t(p^t-1)}{p-1}=$$

$$=\frac{(q-1)n+1-a_0-a_1-\ldots-a_t}{p-1}$$

p factors, where $(q-1)n+1=a_0+a_1p+\ldots+a_tp^t$ is the expansion of $(q-1)n+1$ in base p. But then $C_n^{(q)}$ contains

$$\frac{(q-1)n}{p-1}-\frac{(q-1)n+1-s_p((q-1)n+1)}{p-1}=\frac{s_p((q-1)n+1)-1}{p-1}$$

p factors. □

Remark 3 It is possible (but cumbersome) to calculate explicitly the residue of $C_n^{(q)}/p^{\xi}$ modulo q (with $\xi=\xi_p(C_n^{(q)})$). For example, if $q=p$ is a prime, then this residue is

$$(-1)^{\frac{\sum_{i=0}^t a_i-1}{p-1}+\sum_{i=0}^t(a_i-1)}(p-a_0-1)!(p-a_1-1)!\cdots(p-a_t-1)!;$$

we get a much more complicated formula for general q. □

For a q-ary tree $\mathcal{T}$ on n vertices, weight the vertex v by $b_i=\mathbf{b}(i)$ where i is the number of non-right edges on the unique path from the root of $\mathcal{T}$ to v, and let $w_{\mathbf{b}}(\mathcal{T})$, the weight of $\mathcal{T}$, be the product of the weights of its vertices (see Figure 1).

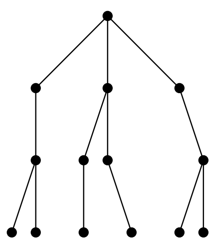

Figure 1: A ternary tree with weight $b_0^3b_1^4b_2^4b_3^3$

Obviously we have $C_n^{(q)}=C_n^{(q)}(\mathbf{b})$ for the constant function $\mathbf{b}(x)=1$.

Define the weighted analogues of $C_n^{(q)}$ by

$$C_n^{(q)}(\mathbf{b})=\sum_{\mathcal{T}}w_{\mathbf{b}}(\mathcal{T}),$$

where the sum is over all q-ary trees on n-vertices. For example,

$$C_0^{(q)}(\mathbf{b}) = 1, \qquad C_1^{(q)} = b_0, \qquad C_2^{(q)} = (q-1)b_0b_1 + b_0^2,$$

$$C_3^{(q)} = (q-1)^2 b_0b_1b_2 + \binom{q-1}{2} b_0b_1^2 + 3(q-1)b_0^2b_1 + b_0^3.$$

The same proof as in the non-weighted case shows that

$$\sum_{n\geq 0} C_n^{(q)}(\mathbf{b})x^n = \cfrac{1}{1 - \cfrac{b_0x}{\left(1 - \cfrac{b_1x}{\left(1-\frac{b_2x}{(1-\ldots)^{q-1}}\right)^{q-1}}\right)^{q-1}}}.$$

Proposition 4 *For each path P from $(0,0)$ to $(qn,0)$ using steps $(1,q-1)$ and $(1,-1)$, weight the step $(x,y) \to (x+1,y-1)$ by 1 and the step $(x,y) \to (x+1,y+q-1)$ by b_i where i is the number of points (x',y') on P satisfying $x' < x$ and $y' < y''$ for any $(x'',y'') \in P$, $x' < x'' \leq x$. Let $w_{\mathbf{b}}(P)$ denote the product of the weights of the steps of P. Then*

$$C_n^{(q)}(\mathbf{b}) = \sum_P w_{\mathbf{b}}(P),$$

Sketch of proof: Consider a depth-first search of a weighted tree $\mathcal{T}$. If a branch is empty, do a $(1,-1)$ step (and backtrack if it is the right-most branch of a vertex); otherwise do a $(1,q-1)$ step. It is easy to see that this gives a bijection

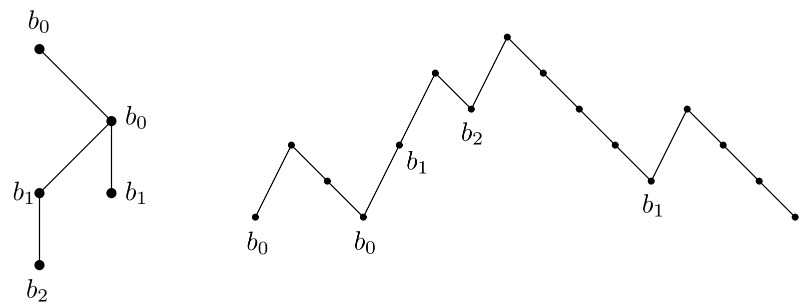

Figure 2: A tree and the corresponding path

between q-ary trees and paths, and that the weights of the paths are as described above. See Figure 2 for an example. □

Our main result is the following generalization of Theorem 1.

Theorem 5 *Let $q = p^k$ for p prime and $k \geq 1$, and let a function $\mathbf{b}\colon \mathbb{N} \to \mathbb{Z}$ satisfy $\mathbf{b}(0) = 1 \pmod q$ and $q^{n+1} | \Delta^n \mathbf{b}(x)$ for all x. Then*

$$C_n^{(q)}(\mathbf{b}) = C_n^{(q)} \pmod{p^{\xi+k}}$$

where

$$\xi = \frac{s_p((q-1)n+1)-1}{p-1},$$

i.e. the same powers of p divide $C_n^{(q)}(\mathbf{b})$ and $C_n^{(q)}$, and $C_n^{(q)}(\mathbf{b})/p^{\xi}$ and $C_n^{(q)}/p^{\xi}$ have the same remainder modulo q.

3 Proof of Theorem 5

For any i, define

$$\mathcal{F}_i^{(q)} = \{\mathbf{f}\colon \mathbb{N} \to \mathbb{Z}\colon \mathbf{f}(0) = i \pmod q,\ \ q^{n+1} | \Delta^n \mathbf{f}(x) \text{ for all } n \geq 1 \text{ and all x}\}.$$

The following generalization of [SP06, Lemma 2.2] is true for *any* q, although we will only need it for q a prime power.

Proposition 6 *We have:*

(1) If $\mathbf{f} \in \mathcal{F}_i^{(q)}$ then $\Delta\mathbf{f}/q \in \mathcal{F}_0^{(q)}$.

(2) If $\mathbf{f} \in \mathcal{F}_i^{(q)}$ and $\mathbf{g} \in \mathcal{F}_j^{(q)}$ then $\mathbf{f} + \mathbf{g} \in \mathcal{F}_{i+j}^{(q)}$.

(3) If $\mathbf{f} \in \mathcal{F}_i^{(q)}$ and $\mathbf{g} \in \mathcal{F}_j^{(q)}$ then $\mathbf{f} \cdot \mathbf{g} \in \mathcal{F}_{ij}^{(q)}$.

(4) If $\mathbf{f}_1, \ldots, \mathbf{f}_q \in \mathcal{F}_1^{(q)}$ then $\left(\sum_i \mathbf{f}_i(x) \prod_{j \neq i} \mathbf{f}_j(x+1)\right)/q \in \mathcal{F}_1^{(q)}$.

Proof: The first two claims are obvious, and (3) follows from

$$\Delta^n(\mathbf{f} \cdot \mathbf{g}) = \sum_{j=0}^{n} \binom{n}{k} \Delta^{n-k}(S^k(\mathbf{f})) \cdot \Delta^k(\mathbf{g})$$

where S is the shift operator, $S\mathbf{f}(x) = \mathbf{f}(x+1)$. For (4), note that the right-hand side can be written as

$$\frac{\sum_i \mathbf{f}_i(x) \prod_{j \neq i}(\mathbf{f}_j(x) + \Delta\mathbf{f}_j(x))}{q} = \mathbf{f}_1(x) \cdots \mathbf{f}_q(x) + \sum_l \mathbf{F}_l,$$

where each $\mathbf{F}_l$ is a product of some elements of $\mathcal{F}_1^{(q)}$ and (by (1)) at least one element of $\mathcal{F}_0^{(q)}$. By (3), $\mathbf{F}_l \in \mathcal{F}_0^{(q)}$ and $\mathbf{f}_1 \cdots \mathbf{f}_q \in \mathcal{F}_1^{(q)}$, so (4) holds by (2). □

As in [DS06] and [SP06], we will need to study the orbits of the action of $\mathcal{G}_n^{(q)}$ on the set $\mathcal{T}_n^{(q)}$ of q-ary trees on n points, where $\mathcal{G}_n^{(q)}$ is the group of symmetries of the complete q-ary tree of depth n.

Proposition 7 *Let $q = p^k$, and let $\mathcal{O}$ be an orbit of $\mathcal{G}_n$ acting on $\mathcal{T}_n$. Then p^ξ divides $|\mathcal{O}|$ where*

$$\xi = \frac{s_p((q-1)n+1)-1}{p-1}.$$

Let us postpone the proof.

Note that the proposition is a combinatorial proof of only a part of Proposition 2: it shows that p^ξ divides $C_n^{(q)}$, but not that it is the highest power of p that divides it. However, this is enough to prove the main result.

Proof (of Theorem 5)*:* As in [SP06, Lemma 2.4] let $\mathcal{O}$ denote an orbit of $\mathcal{G}_n^{(q)}$ acting on $\mathcal{T}_n^{(q)}$, and define

$$\mathbf{r_b}(\mathcal{O})(x) = \frac{\mathbf{w_b}(\mathcal{O})(x)}{|\mathcal{O}|}$$

where $\mathbf{w_b}(\mathcal{O})(x) = \sum_{\mathcal{T}\in\mathcal{O}} \mathbf{w_b}(\mathcal{T})(x)$ and $\mathbf{w_b}(\mathcal{T})(x) = \prod_{v\in\mathcal{T}} \mathbf{b}(x+i_v)$ with i_v the number of non-right edges on the unique path from the vertex v to the root. In particular, $\mathbf{w_b}(\mathcal{O})(0) = C_n^{(q)}(\mathbf{b})$. We will prove by induction on n that for any orbit $\mathcal{O}$, $\mathbf{r_b}(\mathcal{O}) \in \mathcal{F}_1^{(q)}$.
If $n = 0$, $\mathcal{O}$ is empty, $\mathbf{r_b}(\mathcal{O})(x) = 1$ and $\mathbf{r_b}(\mathcal{O}) \in \mathcal{F}_1^{(q)}$. Suppose $n \geq 1$, pick a tree $\mathcal{T}$ in $\mathcal{O}$, and let $\mathcal{T}_1, \ldots, \mathcal{T}_q$ be the branches of the root of $\mathcal{T}$. Some of the corresponding orbits $\mathcal{O}_i$ can be the same; assume that there are l different orbits $\mathcal{V}_1, \ldots, \mathcal{V}_l$, and that they appear $q_1, \ldots, q_l$ times in $\mathcal{O}_1, \ldots, \mathcal{O}_q$. Then $|\mathcal{O}| = \binom{q}{q_1,\ldots,q_l}|\mathcal{V}_1|^{q_1}\cdots|\mathcal{V}_l|^{q_l}$ and $\mathbf{w_b}(\mathcal{O})(x)$ is equal to

$$b(x)\left(\sum_{i=1}^{l}\binom{q-1}{q_1,\ldots,q_i-1\ldots,q_l}\mathbf{w_b}(\mathcal{V}_i)(x)\left(\mathbf{w_b}(\mathcal{V}_i)(x+1)\right)^{q_i-1}\prod_{j\neq i}\left(\mathbf{w_b}(\mathcal{V}_j)(x+1)\right)^{q_j}\right);$$

hence

$$\mathbf{r_b}(\mathcal{O})(x) = b(x)\cdot\frac{\sum_{i=1}^{l} q_i\mathbf{r_b}(\mathcal{V}_i)(x)\left(\mathbf{r_b}(\mathcal{V}_i)(x+1)\right)^{q_i-1}\prod_{j\neq i}\left(\mathbf{r_b}(\mathcal{V}_j)(x+1)\right)^{q_j}}{q} =$$

$$= b(x)\cdot\frac{\sum_{i=1}^{q}\mathbf{r_b}(\mathcal{O}_i)(x)\prod_{j\neq i} r_{\mathbf{b}}(\mathcal{O}_j)(x+1)}{q}$$

and this function is in $\mathcal{F}_1^{(q)}$ by induction, (4) and (3).
Since

$$C_n^{(q)}(\mathbf{b}) = \sum_{\mathcal{O}} |\mathcal{O}|\cdot\mathbf{r_b}(\mathcal{O})(0),$$

$|\mathcal{O}|$ is divisible by p^ξ by Proposition 7 and $\mathbf{r_b}(\mathcal{O})(0) \in \mathbb{Z}$, we have $p^\xi | C_n^{(q)}(\mathbf{b})$, and, modulo q,

$$\frac{C_n^{(q)}(\mathbf{b})}{p^\xi} = \sum_{\mathcal{O}}\frac{|\mathcal{O}|}{p^\xi}\cdot\mathbf{r_b}(\mathcal{O})(0) = \sum_{\mathcal{O}}\frac{|\mathcal{O}|}{p^\xi} = \frac{C_n^{(q)}}{p^\xi}. \qquad \square$$

In order to prove Proposition 7, we will have to explore the structure of the *minimal* orbits of the action of $\mathcal{G}_n^{(q)}$ on $\mathcal{T}_n^{(q)}$, i.e. the orbits whose cardinalities have the lowest power of p in their prime factorization.

Color a vertex of $\mathcal{T}$ black if all its branches are equivalent, and white otherwise.

The number of trees in the orbit of a q-ary tree $\mathcal{T}$ is the product of

$$P_v = \binom{p^k}{q_1^v, \ldots, q_l^v} = \frac{(p^k)!}{Q_v}$$

over all white vertices v of $\mathcal{T}$, where $q_1^v, \ldots, q_l^v$ are the sizes of equivalence classes of the children of v. Note that $P_v = 1$ for a black vertex and $p|P_v$ for a white vertex.

Lemma 8 *If $\mathcal{T}$ is a tree in a minimal orbit $\mathcal{O}$, then no black vertex of $\mathcal{T}$ can have a white child.*

Proof: If a black vertex v of $\mathcal{T} \in \mathcal{O}$ has white children, and a child of v has branches $\mathcal{T}_1, \ldots, \mathcal{T}_q$, we can form a tree $\mathcal{T}'$ with at least $q-1$ fewer p factors by attaching all q copies of $\mathcal{T}_1$ to the first child of v, all q copies of $\mathcal{T}_2$ to the second child of v, etc.

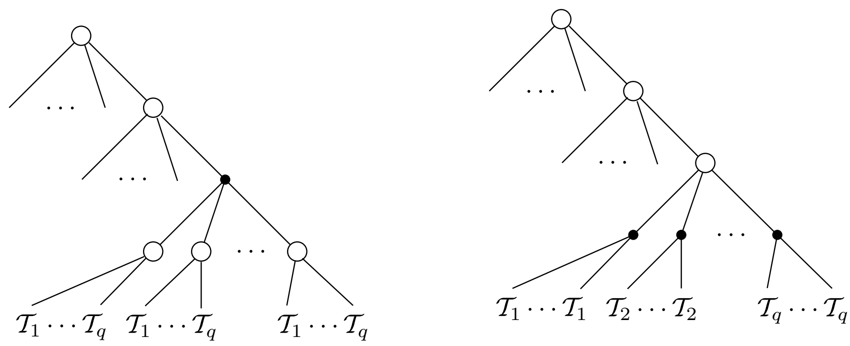

Figure 3: Transforming a tree

See Figure 3 for an example. □

Therefore a minimal orbit must have the following structure: there are some (white) vertices forming a q-ary tree, and there are some complete q-ary trees (containing black vertices) attached – see the left drawing of Figure 4. We can visualize such an orbit as a plane q-ary tree with each endpoint denoted by a non-negative integer (see the right drawing on Figure 4) indicating the depth of the attached complete q-ary tree. If $\mathcal{T}$ has b white vertices and A_i complete q-ary trees of depth i, then $\sum A_i = (q-1)b + 1$, in other words, the tree has $(\sum A_i - 1)/(q-1)$ white vertices. We will call complete q-ary trees of depth i i-trees, and if an i-tree is the child of a white vertex, we will call it an i-child.

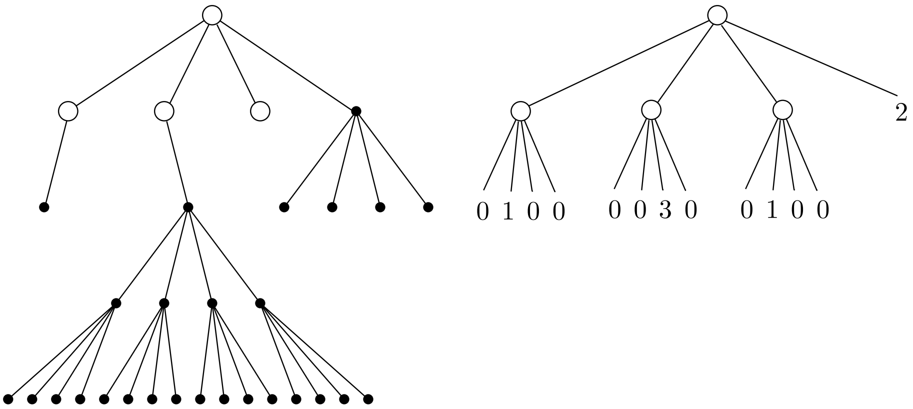

Figure 4: A tree in which no black vertex has a white child

Lemma 9 *If $\mathcal{T}$ is a tree in a minimal orbit $\mathcal{O}$ and A_i is the number of complete q-ary trees of depth i in $\mathcal{T}$, then $A_i < q$.*

Proof: Assume $A_i \geq q = p^k$ for some i and write the number of i-children of a white vertex v as $r^v_{k-1} \cdots r^v_1 r^v_0$ in base p. Pick the vertex v with the highest r^v_0, and assume that there is another vertex v' with $r^{v'}_0 > 0$. Since the number of children of v is divisible by p, there must be a j-child or a white vertex so that the number of equivalent children of v is not divisible by p. Switching this j-tree or white vertex (together with its successors) with one of the i-trees among the children of v' does not decrease the number of p factors in Q_v or $Q_{v'}$ and increases r^v_0. But if the new r^v_0 is p, the number of p factors in Q_v actually increases, which contradicts the minimality of $\mathcal{O}$. Therefore we can assume that there is only one vertex v with a positive r^v_0.
We can repeat the same process for r^v_1: if there are two vertices with non-zero r^v_1, we can exhange p i-trees of one with p j-trees or equivalent white children of the other and repeat this until only one r^v_1 (which must remain smaller than p) is non-zero. Continue with $r^v_2, \ldots, r^v_{k-2}$. When we do the same procedure for r^v_{k-1}, we will get a complete q-ary tree of depth $i+1$ (after at most $p-1$ exchanges), and we will strictly decrease the number of p factors in $|\mathcal{O}|$, which contradicts its minimality. $\square$

Since a complete q-ary tree of depth l has $1 + q + \ldots + q^{l-1} = (q^l - 1)/(q-1)$ vertices and since

$$n = b + A_0 \frac{1-1}{q-1} + A_1 \frac{q-1}{q-1} + A_2 \frac{q^2-1}{q-1} + \ldots$$

implies $(q-1)n + 1 = A_0 + A_1 q + A_2 q^2 + \ldots$, the number of white points in a minimal orbit is

$$\frac{s_q((q-1)n+1) - 1}{q-1}.$$

Lemma 10 *If $\mathcal{T}$ is a tree in a minimal orbit $\mathcal{O}$, then the number of white children of a (white) vertex v with equivalent subtrees is strictly smaller than p.*

Proof: Assume that p^l is the highest power of p that is smaller than or equal to the number of equivalent white children of v.
Assume that the p^l equivalent white children of v have q_i copies of $\mathcal{T}_i$, $q_1+\ldots+q_t=q$. Note that q_i must be smaller than p^{k-l} by Lemma 9, and each white child contributes

$$\sum_i \left(\left\lfloor \frac{q_i}{p} \right\rfloor + \left\lfloor \frac{q_i}{p^2} \right\rfloor + \ldots + \left\lfloor \frac{q_i}{p^{k-l-1}} \right\rfloor \right)$$

p factors to the denominator of $|\mathcal{O}|$.
Write $q_i = c_0^i + c_1^i p + \ldots + c_{k-l-1}^i p^{k-l-1}$ in base p, and split the $q_i p^l$ equivalent children in $c_0^i + c_1^i + \ldots + c_{k-l-1}^i$ groups of size p^l (c_0^i times), p^{l+1} (c_1^i times) etc. and attach each group to one of the chosen p^l children of v. Note that we can do that for all i simultaneously: first attach groups of size p^{k-1} to the first, second etc. child of v. After we run out of groups of size p^{k-1}, we repeat the same process with p^{k-2}, p^{k-3} etc.
We get a new orbit with a different number of p factors. The vertex v has p^l fewer children of some equivalent class; since l is maximal, the new Q_v has at most $1+p+\ldots+p^{l-1}$ fewer p factors.
The new grandchildren of v contribute

$$\sum_i \left(c_0^i(1+\ldots+p^{l-1}) + c_1^i(1+\ldots+p^l) + \ldots + c_{k-l-1}^i(1+\ldots+p^{k-2}) \right) =$$

$$= \sum_i \left((1+p+\ldots+p^{l-1})q_i + \left\lfloor \frac{q_i}{p} \right\rfloor + \left\lfloor \frac{q_i}{p^2} \right\rfloor + \ldots + \left\lfloor \frac{q_i}{p^{k-l-1}} \right\rfloor \right) =$$

$$= (1+p+\ldots+p^{l-1})p^k + \sum_i \left(\left\lfloor \frac{q_i}{p} \right\rfloor + \left\lfloor \frac{q_i}{p^2} \right\rfloor + \ldots + \left\lfloor \frac{q_i}{p^{k-l-1}} \right\rfloor \right)$$

p factors. Since

$$\sum_i \left(\left\lfloor \frac{q_i}{p} \right\rfloor + \ldots + \left\lfloor \frac{q_i}{p^{k-l-1}} \right\rfloor \right) \leq \left\lfloor \frac{\sum q_i}{p} \right\rfloor + \ldots + \left\lfloor \frac{\sum q_i}{p^{k-l-1}} \right\rfloor = p^{l+1} + \ldots + p^{k-1},$$

the difference between the old and the new number of p factors in $|\mathcal{O}|$ is at least

$$(1+\ldots+p^{l-1})p^k - \left(\sum_i \left(\left\lfloor \frac{q_i}{p} \right\rfloor + \ldots + \left\lfloor \frac{q_i}{p^{k-l-1}} \right\rfloor \right) \right)(p^l-1) - (1+\ldots+p^{l-1}) \geq$$

$$\geq (1+\ldots+p^{l-1})p^k - (p^{l+1}+\ldots+p^{k-1})(p^l-1) - (1+\ldots+p^{l-1}) =$$

$$= (p^k+\ldots+p^{k+l-1}) - (p^{2l+1}+\ldots+p^{k+l-1}) + (p^{l+1}+\ldots+p^{k-1}) - (1+\ldots+p^{l-1}) =$$

$$= (p^{l+1}+\ldots+p^{2l}) - (1+\ldots+p^{l-1}) = (p^{l+1}-1)(1+\ldots+p^{l-1}),$$

which is strictly positive and hence contradicts the minimality of $|\mathcal{O}|$ unless $l=0$. $\square$

Now it is easy to prove the proposition. We have determined the number of white points in a minimal orbit; they contribute

$$b \cdot (1 + p + \ldots + p^{k-1}) = \frac{s_q((q-1)n+1)-1}{q-1} \cdot \frac{q-1}{p-1} = \frac{s_q((q-1)n+1)-1}{p-1}$$

p factors to the numerator of $|\mathcal{O}|$. There are $A_i = a_{ik} + a_1 p + \ldots + a_{ik+k-1} p^{k-1}$ complete trees of depth i, and they contribute at most

$$\xi_p(A_i!) = \frac{A_i - a_{ik} - a_{ik+1} - \ldots - a_{ik+k-1}}{p-1}$$

p factors to the denominator of $|\mathcal{O}|$. The white vertices do not contribute any p factors to the denominator of $|\mathcal{O}|$ by Lemma 10. That means that the prime factorization of the cardinality of the minimal orbit has at least (and, by Proposition 2, exactly)

$$\frac{s_q((q-1)n+1)-1}{p-1} - \frac{s_q((q-1)n+1) - s_p((q-1)n+1)}{p-1} =$$

$$= \frac{s_p((q-1)n+1)-1}{p-1}$$

p factors.

Note that in the case $q = p$ both Lemma 9 and Lemma 10 are trivial.

4 Concluding remarks

It is natural to ask whether the results extend to arbitrary q.

Question 1 *Let $q = p^k q'$ for p prime, $k \geq 1$ and $\gcd(p, q') = 1$, and let a function $\mathbf{b}\colon \mathbb{N} \to \mathbb{Z}$ satisfy $\mathbf{b}(0) = 1 \pmod q$ and $q^{n+1} | \Delta^n \mathbf{b}(x)$ for all x. Is it true then that*

$$C_n^{(q)}(\mathbf{b}) = C_n^{(q)} \pmod{p^{\xi+k}}$$

where ξ is the highest power of p dividing $C_n^{(q)}$; i.e. do the same powers of p divide $C_n^{(q)}(\mathbf{b})$ and $C_n^{(q)}$, and do $C_n^{(q)}(\mathbf{b})/p^\xi$ and $C_n^{(q)}/p^\xi$ have the same remainder modulo p^k?

The answer is negative. The author wrote a program in C++ that generated random $\mathbf{b}$'s satisfying the hypothesis, and checked the condition $C_n^{(q)}(\mathbf{b}) = C_n^{(q)} \pmod{p^{\xi+k}}$ for low n's ($n \leq 250$). It appears that the equality fails to be satisfied for sporadic n's whenever q is not a prime power; when $q = 6$ and $p = 2$, the equality is not necessarily satisfied for $n = 22, 43, 86, 107, 150, 171, 214, 235$. For example, we have

$$C_{22}^{(6)} = 5.643274... \cdot 10^{22} = 1011111100...1011000000_{[2]}$$

and

$$C_{22}^{(6)}(\mathbf{b}) = 1.071965... \cdot 10^{71} = 1111100010...0100000000_{[2]}.$$

for $\mathbf{b}(x) = 36x + 1$. What about the following?

Question 2 *Let q be arbitrary, and let a function $\mathbf{b}\colon \mathbb{N} \to \mathbb{Z}$ satisfy $\mathbf{b}(0) = 1 \,(\text{mod } q)$ and $q^{n+1}|\Delta^n \mathbf{b}(x)$ for all x. Is it true that*

$$C_n^{(q)}(\mathbf{b}) = C_n^{(q)} \; (\text{mod } q^{\xi+1})$$

where ξ is the highest power of q dividing $C_n^{(q)}$; i.e. do the same powers of q divide $C_n^{(q)}(\mathbf{b})$ and $C_n^{(q)}$, and do $C_n^{(q)}(\mathbf{b})/q^{\xi}$ and $C_n^{(q)}/q^{\xi}$ have the same remainder modulo q?

The answer, again, is no, although the computation that proves this is considerably harder. A Maple program showed that (again for $\mathbf{b}(x) = 36x + 1$) $C_n^{(6)} = C_n^{(6)}(\mathbf{b})$ (mod $6^{\xi+1}$) (where ξ is the highest power of 6 dividing $C_n^{(6)}$) holds for $n \leq 202$, while

$$C_{203}^{(6)} = 6.506438... \cdot 10^{233} = 2155553502...5211200000_{[6]},$$

and

$$C_{203}^{(6)}(\mathbf{b}) = 9.873449... \cdot 10^{878} = 2521223211...3050000000_{[6]}.$$

It is interesting to explore *necessary* conditions for the conclusion of Theorem 5 to hold for low n. The following is a sample, and it suggests that the conditions of the theorem are too strong.

Proposition 11 *Let $q = p^k$ for p prime, and let $\mathbf{b}\colon \mathbb{N} \to \mathbb{Z}$. Then the following statements are equivalent:*

(a) For $n \leq q + 2$ we have

$$C_n^{(q)}(\mathbf{b}) = C_n^{(q)} \; (\text{mod } p^{\xi+k})$$

where

$$\xi = \frac{s_p((q-1)n + 1) - 1}{p - 1}$$

(b) $\mathbf{b}(0) = 1 \,(\text{mod}\, q)$, $q|\Delta\mathbf{b}(q-1)$ and $q^2|\Delta\mathbf{b}(x)$ for $x = 0, 1, \ldots, q-3, q-2, q$.

Sketch of proof: We will prove that the conditions are necessary, and it will be clear from the proof that they are also sufficient. We have $C_1^{(q)}(\mathbf{b}) = b_0$ and $C_1^{(q)} = 1$, so $\mathbf{b}(0) = 1$ (mod q). It is easy to see either directly or using Proposition 2 that for $n = 2, 3, \ldots, q$ p^k is the highest power of p that divides $C_n^{(q)}$, and it is obvious from the definition of $C_n^{(q)}(\mathbf{b})$ that the only term in $C_n^{(q)}(\mathbf{b})$ that contains b_{n-1} is

$$(q-1)^n b_0 b_1 \cdots b_{n-1}.$$

If we assume by induction that $b_0 = \ldots = b_{n-2} \pmod{q^2}$ and if $n \le q$, then we get (modulo q^2)

$$C_n^{(q)}(\mathbf{b}) - C_n^{(q)} = (q-1)^n b_0^{n-2} b_{n-1} + (C_n^{(q)} - (q-1)^n) b_0^{n-1} - C_n^{(q)} =$$

$$= (q-1)^n b_0^{n-2}(b_{n-1} - b_0) + C_n^{(q)}(b_0^{n-1} - 1).$$

Since $(q-1)^n$, b_0^{n-2} are invertible in $\mathbb{Z}_{q^2}$ and since $q | b_0^{n-1} - 1$ and $q | C_n^{(q)}$, we get that $b_{n-1} = b_0 \pmod{q^2}$.
Since $C_{q+1}^{(q)}$ is not divisible by p and $b_0 = \ldots = b_{q-1} = 1 \pmod{q}$,

$$C_{q+1}^{(q)}(\mathbf{b}) = (q-1)^q b_q + (C_{q+1}^{(q)} - (q-1)^q) = C_{q+1}^{(q)} \pmod{q}$$

implies $b_q = 1 \pmod{q}$.
The Catalan number $C_{q+2}^{(q)}$ is divisible by q and not by q^2. A careful consideration of q-ary trees of depth at least q on $q+2$ vertices gives

$$C_{q+2}^{(q)}(\mathbf{b}) = (q-1)^{q+1} b_0^q b_q b_{q+1} + \frac{1}{2} q(q-1)^q b_0^q b_q^2 + (2-q+q^2)(q-1)^q b_0^{q+1} b_q +$$

$$+ \left(C_{q+2}^{(q)} - (q-1)^{q+1} - \frac{1}{2} q(q-1)^q - (2-q+q^2)(q-1)^q \right) b_0^{q+2},$$

if we set this equal to $C_{q+2}^{(q)}$ and do some elementary arithmetic in $\mathbb{Z}_{q^2}$, we get $b_{q+1} = b_q \pmod{q^2}$. This concludes the proof. □

For example, when $\mathbf{b}$ is a polynomial, this gives $q+1$ conditions on the coefficients. It is interesting that for $q = 2$ these conditions appear to be sufficient as well.

Conjecture *Let $\mathbf{b}(x) = c_0 + c_1 x + \ldots + c_d x^d$. Then $\xi_2(C_n(\mathbf{b})) = \xi_2(C_n)$ for all n if and only if*

(1) $2 | c_0 - 1$,

(2) $4 | c_1 + c_2 + c_3 + \ldots$,

(3) $2 | c_3 + c_5 + c_7 + \ldots$ □

The conjecture was verified with a C++ program for a large number of $\mathbf{b}$'s and for $n \le 250$.

Acknowledgment

I would like to thank A. Postnikov for suggesting the problem.

References

[AK73] R. Alter and K. K. Kubota, Prime and prime power divisibility of Catalan numbers. *J. Combinatorial Theory Ser. A* **15** (1973), 243–256.

[D99] E. Deutsch, An involution on Dyck paths and its consequences. *Discrete Math.* **204** (1999), no. 1-3, 163–166.

[DS06] E. Deutsch, B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences. *J. Number Theory* **117** (2006), no. 1, 191–215.

[E83] Ö. Eğecioğlu, The parity of the Catalan numbers via lattice paths. *Fibonacci Quart.* **21** (1983), no. 1, 65–66.

[SP06] A. Postnikov, B. E. Sagan, What power of two divides a weighted Catalan number? to appear in *J. Combinatorial Theory Ser. A*

[SU91] R. Simion, D. Ullman, On the structure of the lattice of noncrossing partitions. *Discrete Math.* **98** (1991), no. 3, 193–206.

[S99] R. P. Stanley, *Enumerative combinatorics*. Vol. 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.