TABLEAU POSETS AND THE FAKE DEGREES OF COINVARIANT ALGEBRAS

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Abstract. We introduce two new partial orders on the standard Young tableaux of a given partition shape, in analogy with the strong and weak Bruhat orders on permutations. Both posets are ranked by the major index statistic offset by a fixed shift. The existence of such ranked poset structures allows us to classify the realizable major index statistics on standard tableaux of arbitrary straight shape and certain skew shapes. By a theorem of Lusztig–Stanley, this classification can be interpreted as determining which irreducible representations of the symmetric group exist in which homogeneous components of the corresponding coinvariant algebra, strengthening a recent result of the third author for the modular major index. Our approach is to identify patterns in standard tableaux that allow one to mutate descent sets in a controlled manner. By work of Lusztig and Stembridge, the arguments extend to a classification of all nonzero fake degrees of coinvariant algebras for finite complex reflection groups in the infinite family of Shephard–Todd groups.

1. Introduction

Let $\mathrm{SYT}(\lambda)$ denote the set of all standard Young tableaux of partition shape λ. We say i is a *descent* in a standard tableau T if $i+1$ comes before i in the row reading word of T, read from bottom to top along rows in English notation. Equivalently, i is a descent in T if $i+1$ appears in a lower row in T. Let $\mathrm{maj}(T)$ denote the *major index statistic* on $\mathrm{SYT}(\lambda)$, which is defined to be the sum of the descents of T. The major index generating function for $\mathrm{SYT}(\lambda)$ is given by

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T\in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k\geq 0} b_{\lambda,k} q^k. \tag{1}$$

The polynomial $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ has two elegant closed forms, one due to Steinberg based on dimensions of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$, see

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[Ste51], and one due to Stanley [Sta79] generalizing the Hook-Length Formula; see Theorem 2.11.

For fixed λ, consider the *fake degree sequence*

$$b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\} \text{ for } k = 0, 1, 2, \ldots \tag{2}$$

The fake degrees have appeared in a variety of algebraic and representation-theoretic contexts including Green's work on the degree polynomials of unipotent $\mathrm{GL}_n(\mathbb{F}_q)$-representations [Gre55, Lemma 7.4], the irreducible decomposition of type A coinvariant algebras [Sta79, Prop. 4.11], Lusztig's work on the irreducible representations of classical groups [Lus77], and branching rules between symmetric groups and cyclic subgroups [Ste89, Thm. 3.3]. The term "fake degree" was apparently coined by Lusztig [Car89], perhaps because $\#\,\mathrm{SYT}(\lambda) = \sum_{k\geq 0} b_{\lambda,k}$ is the degree of the irreducible S_n-representation indexed by λ, so a q-analog of this number is not itself a degree but related to the degree.

We consider three natural enumerative questions involving the fake degrees:

(I) which $b_{\lambda,k}$ are zero?
(II) are the fake degree sequences unimodal?
(III) are there efficient asymptotic estimates for $b_{\lambda,k}$?

We completely settle (I) with the following result. Denote by λ' the conjugate partition of λ, and let $b(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$.

Theorem 1.1. *For every partition $\lambda \vdash n \geq 1$ and integer k such that $b(\lambda) \leq k \leq \binom{n}{2} - b(\lambda')$, we have $b_{\lambda,k} > 0$ except in the case when λ is a rectangle with at least two rows and columns and k is either $b(\lambda)+1$ or $\binom{n}{2} - b(\lambda') - 1$. Furthermore, $b_{\lambda,k} = 0$ for $k < b(\lambda)$ or $k > \binom{n}{2} - b(\lambda')$.*

As a consequence of the proof of Theorem 1.1, we identify two ranked poset structures on $\mathrm{SYT}(\lambda)$ where the rank function is determined by maj. Furthermore, as a corollary of Theorem 1.1 we have a new proof of a complete classification due to the third author [Swa18, Thm. 1.4] generalizing an earlier result of Klyachko [Kly74] for when the counts

$$a_{\lambda,r} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) \equiv_n r\}$$

for $\lambda \vdash n$ are nonzero.

The easy answer to question (II) is "no". The fake degree sequences are not always unimodal. For example, $\mathrm{SYT}(4,2)^{\mathrm{maj}}(q)$ is not unimodal. See Example 2.13. Nonetheless, certain inversion number generating functions $p_\alpha^{(k)}(q)$ which appear in a generalization of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are in fact unimodal; see Definition 7.7 and Corollary 7.10. Furthermore, computational evidence suggests $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ is typically not far from unimodal.

Questions (II) and (III) are addressed in a separate article [BKS20a]. In particular, we show in that article that the coefficients of $\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q)$ are asymptotically normal for any sequence of partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots$ such that $\mathrm{aft}(\lambda^{(i)})$ approaches infinity where $\mathrm{aft}(\lambda)$ is the number of boxes outside the first row or column, whichever is smaller. The aft statistics on partitions is in FindStat as [RS+18, St001214].

We note that there are polynomial expressions for the fake degrees $b_{\lambda,k}$ in terms of parameters H_i, the number of cells of λ with hook length equal to i. These polynomials are closely related to polynomials that express the number of permutations S_n of a given inversion number $k \le n$ as a function of n by work of Knuth. See Lemma 3.1 and Corollary 3.3. These polynomials are useful in some cases, however, we find that in practice Stanley's formula is the most effective way to compute a given fake degree sequence for partitions up to size 200. See Remark 2.12 for more on efficient computation using cyclotomic polynomials.

Symmetric groups are the finite reflection groups of type A. The classification and invariant theory of both finite irreducible real reflection groups and complex reflection groups developed over the past century builds on our understanding of the type A case [Hum90]. In particular, these groups are classified by Shephard–Todd into an infinite family $G(m, d, n)$ together with 34 exceptions. Using work of Stembridge on generalized exponents for irreducible representations, the analog of (1) can be phrased for all Shephard–Todd groups as

$$g^{\{\underline{\lambda}\}^d}(q) \coloneqq \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d} \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m) = \sum b_{\{\underline{\lambda}\}^d,k} q^k \tag{3}$$

where $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is a sequence of m partitions with n cells total, $\alpha(\underline{\lambda}) = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|) \vDash n$, $d \mid m$, and $\{\underline{\lambda}\}^d$ is the orbit of $\underline{\lambda}$ under the group C_d of (m/d)-fold cyclic rotations; see Corollary 8.2. The polynomials $\begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d}$ are deformations of the usual q-multinomial coefficients which we explore in Section 7. The coefficients $b_{\{\underline{\lambda}\}^d,k}$ are the fake degrees in this case.

We use (3) and Theorem 1.1 to completely classify all nonzero fake degrees for coinvariant algebras for all Shephard–Todd groups $G(m, d, n)$, which includes the finite real reflection groups in types A, B, and D. See Corollary 6.4 and Corollary 8.4 for the type B and D cases, respectively. See Theorem 6.3 and Theorem 8.3 for the general $C_m \wr S_n$ and $G(m, d, n)$ cases, respectively.

The rest of the paper is organized as follows. In Section 2, we give background on tableau combinatorics, Shephard–Todd groups, and their irreducible representations. Section 3 describes the polynomial formulas for fake degrees in type A. Section 4 presents our combinatorial argument proving Theorem 1.1 and giving poset structures on tableaux of a given shape. Section 5 uses the argument in Section 4 to answer in the affirmative a question of Adin–Elizalde–Roichman about internal zeros of $\mathrm{SYT}(\lambda)^{\mathrm{des}}(q)$; see Corollary 5.3. In Section 6, we begin to address the question of characterizing nonzero fake degrees by starting with the wreath products $C_m \wr S_n = G(m,1,n)$; see Theorem 6.3. In Section 7, we define the deformed q-multinomials $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d}$ as rational functions and give a summation formula, Theorem 7.6, which shows they are polynomial. Finally, in Section 8, we complete the classification of nonzero fake degrees for $G(m,d,n)$ and spell out how (3) relates to Stembridge's original generating function for the fake degrees in $G(m,d,n)$; see Theorem 8.3 and Corollary 8.2. We discuss potential algebraic and geometric directions for future work in Section 9.

2. Background

In this section, we review some standard terminology and results on combinatorial statistics and tableaux. Many further details in this area can be found in [Sta12, Sta99]. We also review background on the finite complex reflection groups and their irreducible representations. Further details in this area can be found in [Car89, Sag91].

2.1. **Word and Tableau Combinatorics.**

Here we review standard combinatorial notions related to words and tableaux.

Definition 2.1. Given a word $w = w_1 w_2 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *content* of w is the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ where α_i is the number of times i appears in w. Such a sequence α is called a (weak) *composition* of n, written as $\alpha \vDash n$. Trailing 0's are often omitted when writing compositions, so $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ for some m. Note, a word of content $(1, 1, \ldots, 1) \vDash n$ is a permutation in the symmetric group S_n written in one-line notation. The *inversion number* of w is

$$\mathrm{inv}(w) := \#\{(i,j) : i < j, w_i > w_j\}.$$

The *descent set* of w is

$$\mathrm{Des}(w) := \{0 < i < n : w_i > w_{i+1}\}$$

and the *major index* of w is

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

The study of permutation statistics is a classical topic in enumerative combinatorics. The major index statistic on permutations was introduced by Percy MacMahon in his seminal works [Mac13, Mac17]. At first glance, this function on permutations may be unintuitive, but it has inspired hundreds of papers and many generalizations; for example on Macdonald polynomials [HHL05], posets

[ER15], quasisymmetric functions [SW10], cyclic sieving [RSW04, AS17], and bijective combinatorics [Foa68, Car75].

Definition 2.2. Given a finite set W and a function $\mathrm{stat}\colon W \to \mathbb{Z}_{\geq 0}$, write the corresponding ordinary generating function as

$$W^{\mathrm{stat}}(q) \coloneqq \sum_{w \in W} q^{\mathrm{stat}(w)}.$$

Definition 2.3. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \vDash n$. We use the following standard q-analogues:

$$\begin{aligned}
[n]_q &\coloneqq 1 + q + \cdots + q^{n-1} = \tfrac{q^n - 1}{q-1}, && (q\text{-integer})\\
[n]_q! &\coloneqq [n]_q [n-1]_q \cdots [1]_q, && (q\text{-factorial})\\
\binom{n}{k}_q &\coloneqq \tfrac{[n]_q!}{[k]_q! [n-k]_q!} \in \mathbb{Z}_{\geq 0}[q], && (q\text{-binomial})\\
\binom{n}{\alpha}_q &\coloneqq \tfrac{[n]_q!}{[\alpha_1]_q! \cdots [\alpha_m]_q!} \in \mathbb{Z}_{\geq 0}[q] && (q\text{-multinomial}).
\end{aligned}$$

Example 2.4. The identity statistic on the set $W = \{0, \ldots, n-1\}$ has generating function $[n]_q$. The "sum" statistic on $W = \prod_{j=1}^{n} \{0, \ldots, j-1\}$ has generating function $[n]_q!$. It is straightforward to show that also $S_n^{\mathrm{inv}} \coloneqq \sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!$.

For $\alpha \vDash n$, let W_α denote the set of all words of content α. A classic result of MacMahon is that maj and inv have the same distribution on W_α which is determined by the corresponding q-multinomial.

Theorem 2.5. [Mac17, §1] *For each* $\alpha \vDash n$*,*

$$\mathrm{W}_\alpha^{\mathrm{maj}}(q) = \binom{n}{\alpha}_q = \mathrm{W}_\alpha^{\mathrm{inv}}(q). \tag{4}$$

Definition 2.6. A polynomial $P(q) = \sum_{i=0}^{n} c_i q^i$ of degree n is *symmetric* if $c_i = c_{n-i}$ for $0 \leq i \leq n$. We generally say $P(q)$ is *symmetric* also if there exists an integer k such that $q^k P(q)$ is symmetric. We say $P(q)$ is *unimodal* if

$$c_0 \leq c_1 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n$$

for some $0 \leq j \leq n$. Furthermore, $P(q)$ has *no internal zeros* provided that $c_j \neq 0$ whenever $c_i, c_k \neq 0$ and $i < j < k$.

From Theorem 2.5 and the definition of the q-multinomials, we see that each $\mathrm{W}_\alpha^{\mathrm{maj}}(q)$ is a symmetric polynomial with constant and leading coefficient 1. Indeed, these polynomials are *unimodal* generalizing the well-known case for Gaussian coefficients [Sta80, Thm 3.1] and [Zei89]. It also follows easily from MacMahon's theorem that $\mathrm{W}_\alpha^{\mathrm{maj}}(q)$ has no internal zeros.

2.2. Partitions and Standard Young Tableaux.

Definition 2.7. A composition $\lambda \vDash n$ such that $\lambda_1 \geq \lambda_2 \geq \ldots$ is called a *partition* of n, written as $\lambda \vdash n$. The *size* of λ is $|\lambda| := n$ and the *length* $\ell(\lambda)$ of λ is the number of non-zero entries. The *Young diagram* of λ is the upper-left justified arrangement of unit squares called *cells* where the ith row from the top has λ_i cells following the English notation; see Figure 1a. The cells of a tableau are indexed by matrix notation when we refer to their row and column. The *hook length* of a cell $c \in \lambda$ is the number h_c of cells in λ in the same row as c to the right of c and in the same column as c and below c, including c itself; see Figure 1b. A *corner* of λ is any cell with hook length 1. A *notch* of λ is any (i, j) not in λ such that both $(i-1, j)$ and $(i, j-1)$ are in λ. Note that notches cannot be in the first row or column of λ. A *bijective filling* of λ is any labeling of the cells of λ by the numbers $[n] = \{1, 2, \ldots, n\}$. The symmetric group S_n acts on bijective fillings of λ by acting on the labels.

(A) Young diagram of λ. (B) Hook lengths of λ.

FIGURE 1. Constructions related to the partition $\lambda = (6, 3, 3) \vdash 12$. The partition has corners at positions $(3, 3)$ and $(1, 6)$ and one notch at position $(2, 4)$.

Definition 2.8. A *skew partition* λ/ν is a pair of partitions (ν, λ) such that the Young diagram of ν is contained in the Young diagram of λ. The cells of λ/ν are the cells in the diagram of λ which are not in the diagram of ν, written $c \in \lambda/\nu$. We identify straight partitions λ with skew partitions $\lambda/\varnothing$ where $\varnothing = (0, 0, \ldots)$ is the empty partition. The *size* of λ/ν is $|\lambda/\nu| := |\lambda| - |\nu|$. The notions of bijective filling, hook lengths, corners, and notches naturally extend to skew partitions as well.

Definition 2.9. Given a sequence of partitions $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$, we identify the sequence with the *block diagonal skew partition* obtained by translating the Young diagrams of the $\lambda^{(i)}$ so that the rows and columns occupied by these components are disjoint, form a valid skew shape, and they appear in order from top to bottom as depicted in Figure 2.

Definition 2.10. A *standard Young tableau* of shape λ/ν is a bijective filling of the cells of λ/ν such that labels increase to the right in rows and down

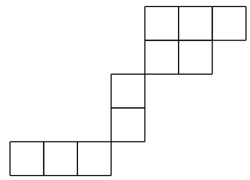


Figure 2. Diagram for the skew partition $\lambda/\nu = 76443/4433$, which is also the block diagonal skew shape $\underline{\lambda} = ((3,2),(1,1),(3))$.

columns; see Figure 3. The set of standard Young tableaux of shape λ/ν is denoted $\mathrm{SYT}(\lambda/\nu)$. The *descent set* of $T \in \mathrm{SYT}(\lambda/\nu)$ is the set $\mathrm{Des}(T)$ of all labels i in T such that $i+1$ is in a strictly lower row than i. The *major index* of T is

$$\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i.$$

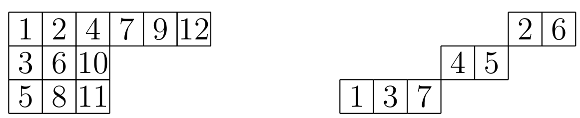


Figure 3. On the left is a standard Young tableau of straight shape $\lambda = (6,3,3)$ with descent set $\{2,4,7,9,10\}$ and major index 32. On the right is a standard Young tableau of block diagonal skew shape $(7,5,3)/(5,3)$ corresponding to the sequence of partitions $((2),(2),(3))$ with descent set $\{2,6\}$ and major index 8.

The block diagonal skew partitions $\underline{\lambda}$ allow us to simultaneously consider words and tableaux as follows. Let W_α be the set of all words with content $\alpha = (\alpha_1, \ldots, \alpha_k)$. Letting $\underline{\lambda} = ((\alpha_k), \ldots, (\alpha_1))$, we have a bijection

$$\phi\colon \mathrm{SYT}(\underline{\lambda}) \xrightarrow{\sim} \mathrm{W}_\alpha \tag{5}$$

which sends a tableau T to the word whose ith letter is the row number in which i appears in T, counting from the *bottom up* rather than top down. For example, using the skew tableau T on the right of Figure 3, we have $\phi(T) = 1312231 \in \mathrm{W}_{(3,2,2)}$. It is easy to see that $\mathrm{Des}(\phi(T)) = \mathrm{Des}(T)$, so that $\mathrm{maj}(\phi(T)) = \mathrm{maj}(T)$.

2.3. **Major Index Generating Functions.** Stanley gave the following analogue of Theorem 2.5 for standard Young tableaux of a given shape. It generalizes the famous Frame–Robinson–Thrall Hook-Length Formula [FRT54, Thm. 1] or [Sta99, Cor. 7.21.6] obtained by setting $q = 1$.

Theorem 2.11. [Sta99, 7.21.5] *Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then*

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q} \tag{6}$$

where $b(\lambda) := \sum (i-1)\lambda_i$ and h_c is the hook length of the cell c.

Remark 2.12. Since $\#\mathrm{SYT}(\lambda)$ typically grows extremely quickly, Stanley's formula offers a practical way to compute $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ even when $n \approx 100$ by expressing both the numerator and denominator, up to a q-shift, as a product of cyclotomic polynomials and canceling all factors from the denominator. We prefer to use cyclotomic factors over linear factors in order to avoid arithmetic in cyclotomic fields.

Example 2.13. For $\lambda = (4,2)$, $b(\lambda) = 2$ and the multiset of hook lengths is $\{1^2, 2^2, 4, 5\}$ so $|\mathrm{SYT}(\lambda)| = 9$ by the Hook-Length Formula. The major index generating function is given by

$$\begin{aligned}\mathrm{SYT}(4,2)^{\mathrm{maj}}(q) &= q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2\\ &= q^2\frac{[6]_q!}{[5]_q[4]_q[2]_q[2]_q} = q^2\frac{[6]_q[3]_q}{[2]_q}.\end{aligned}$$

Note, $\mathrm{SYT}(4,2)^{\mathrm{maj}}(q)$ is symmetric but not unimodal.

For $\lambda = (4,2,1)$, $b(\lambda) = 4$ and the multiset of hook lengths is $\{1^3, 2, 3, 4, 6\}$ so $|\mathrm{SYT}(\lambda)| = 35$ by the Hook-Length Formula. The major index generating function is given by

$$\begin{aligned}\mathrm{SYT}(4,2,1)^{\mathrm{maj}}(q) &= q^{14} + 2q^{13} + 3q^{12} + 4q^{11} + 5q^{10} + 5q^9 + 5q^8 + 4q^7 + 3q^6\\ &\quad + 2q^5 + q^4 = q^4\frac{[7]_q!}{[6]_q[4]_q[3]_q[2]_q} = q^4[7]_q[5]_q.\end{aligned}$$

Note, $\mathrm{SYT}(4,2,1)^{\mathrm{maj}}(q)$ is symmetric and unimodal.

Example 2.14. We recover q-integers, q-binomials, and q-Catalan numbers, up to q-shifts as special cases of the major index generating function for tableaux as follows:

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \begin{cases} q[n]_q & \text{if } \lambda = (n,1),\\ q^{\binom{k+1}{2}}\binom{n}{k}_q & \text{if } \lambda = (n-k+1, 1^k),\\ q^n\frac{1}{[n+1]_q}\binom{2n}{n}_q & \text{if } \lambda = (n,n).\end{cases}$$

The following strengthening of Stanley's formula to $\underline{\lambda}$ is well known (e.g. see [Ste89, (5.6)]), though since it is somewhat difficult to find explicitly in the literature, we include a short proof.

Theorem 2.15. *Let* $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ *where* $\lambda^{(i)} \vdash \alpha_i$ *and* $n = \alpha_1 + \cdots + \alpha_m$*. Then*

$$\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q) = \binom{n}{\alpha_1, \ldots, \alpha_m}_q \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q). \tag{7}$$

Proof. The stable principal specialization of skew Schur functions is given by

$$s_{\lambda/\nu}(1, q, q^2, \ldots) = \frac{\mathrm{SYT}(\lambda/\nu)^{\mathrm{maj}}(q)}{\prod_{j=1}^{|\lambda/\nu|}(1 - q^j)};$$

see [Ste89, Lemma 3.1] or [Sta99, Prop.7.19.11]. On the other hand, it is easy to see from the definition of a skew Schur function as the content generating function for semistandard tableaux of the given shape that

$$s_{\underline{\lambda}}(x_1, x_2, \ldots) = \prod_{i=1}^{m} s_{\lambda^{(i)}}(x_1, x_2, \ldots).$$

The result quickly follows. □

Remark 2.16. Theorem 2.11 and Theorem 2.15 have several immediate corollaries. First, we recover MacMahon's result, Theorem 2.5, from Theorem 2.15 when $\underline{\lambda} = ((\alpha_m), (\alpha_{m-1}), \ldots)$ by using the maj-preserving bijection ϕ in (5). Second, each $\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q)$ is symmetric (up to a q-shift) with leading coefficient 1. In particular, there is a unique "maj-minimizer" and "maj-maximizer" tableau in each $\mathrm{SYT}(\underline{\lambda})$. Moreover,

$$\min \mathrm{maj}(\mathrm{SYT}(\underline{\lambda})) = b(\underline{\lambda}) \tag{8}$$

and

$$\max \mathrm{maj}(\mathrm{SYT}(\underline{\lambda})) = \binom{n}{2} - b(\underline{\lambda}') = b(\underline{\lambda}) + \binom{|\underline{\lambda}| + 1}{2} - \sum_{c \in \underline{\lambda}} h_c \tag{9}$$

where $b(\underline{\lambda}) \coloneqq \sum_i b(\lambda^{(i)})$ and $b(\underline{\lambda}') \coloneqq \sum_i b(\lambda^{(i)\prime})$.

For general skew shapes, $\mathrm{SYT}(\lambda/\nu)^{\mathrm{maj}}(q)$ does not factor as a product of cyclotomic polynomials times q to a power. A "q-Naruse" formula due to Morales–Pak–Panova, [MPP15, (3.4)], gives an analogue of Theorem 2.11 involving a sum over "excited diagrams," though the resulting sum has a single term precisely for the block diagonal skew partitions $\underline{\lambda}$.

2.4. **Complex Reflection Groups.** A *complex reflection group* is a finite subgroup of $\mathrm{GL}(\mathbb{C}^n)$ generated by *pseudo-reflections*, which are elements which pointwise fix a codimension-1 hyperplane. Shephard–Todd, building on work of Coxeter and others, famously classified the complex reflection groups [ST54]. The irreducible representations were constructed by Young, Specht, Lusztig, and others. We now summarize these results and fix some notation.

Definition 2.17. A *pseudo-permutation matrix* is a matrix where each row and column has a single non-zero entry. For positive integers m, n, the *wreath product* $C_m \wr S_n \subset \mathrm{GL}(\mathbb{C}^n)$ is the group of $n \times n$ pseudo-permutation matrices whose non-zero entries are complex mth roots of unity. For $d \mid m$, let $G(m, d, n)$ be the *Shephard–Todd group* consisting of matrices $x \in C_m \wr S_n$ where the product of the non-zero entries in x is an (m/d)th root of unity. In fact, $G(m, d, n)$ is a normal subgroup of $C_m \wr S_n$ of index d with cyclic quotient $(C_m \wr S_n)/G(m, d, n) \cong C_d$ of order d.

Theorem 2.18. [ST54] *Up to isomorphism, the complex reflection groups are precisely the direct products of the groups $G(m, d, n)$, along with 34 exceptional groups.*

Remark 2.19. Special cases of the Shephard–Todd groups include the following. The Weyl group of type A_{n-1}, or equivalently the symmetric group S_n, is isomorphic to $G(1, 1, n)$. The Weyl groups of both types B_n and C_n are $G(2, 1, n)$, the group of $n \times n$ signed permutation matrices. The subgroup of the group of signed permutations whose elements have evenly many negative signs is the Weyl group of type D_n, or $G(2, 2, n)$ as a Shephard–Todd group. We also have that $G(m, m, 2)$ is the dihedral group of order $2m$, and $G(m, 1, 1)$ is the cyclic group C_m of order m.

The complex irreducible representations of S_n were constructed by Young [You77] and are well known to be certain modules S^λ *canonically* indexed by partitions $\lambda \vdash n$. These representations are beautifully described in [Sag91]. Specht extended the construction to irreducibles for $G \wr S_n$ where G is a finite group.

Theorem 2.20. [Spe35] *The complex inequivalent irreducible representations of $C_m \wr S_n$ are certain modules $S^{\underline{\lambda}}$ indexed by the sequences of partitions $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ for which $|\underline{\lambda}| \coloneqq |\lambda^{(1)}| + \cdots + |\lambda^{(m)}| = n$.*

Remark 2.21. The version we give of Theorem 2.20 was stated by Stembridge [Ste89, Thm. 4.1]. The C_m-irreducibles are naturally though non-canonically indexed by $\mathbb{Z}/m$ up to one of $\phi(m)$ additive automorphisms, where $\phi(m)$ is Euler's totient function. Correspondingly, one may identify $\mathbb{Z}/m$ with $\{1, \ldots, m\}$ and obtain $\phi(m)$ different indexing schemes for the $C_m \wr S_n$-irreducibles. The resulting indexing schemes are rearrangements of one another, and our results will be independent of these choices.

Clifford described a method for determining the branching rules of irreducible representations for a normal subgroup of a given finite group [Cli37]. Stembridge combined this method with Specht's theorem to describe the irreducible representations for all Shephard–Todd groups from the $C_m \wr S_n$-irreducible representations.

We use Stembridge's terminology where possible. In particular, for $d \mid m$, the (m/d)-*fold cyclic rotations* are the elements in the subgroup isomorphic to C_d of S_m generated by $\sigma_m^{m/d}$, where $\sigma_m = (1, 2, \ldots, m)$ is the long cycle. Let S_m act on block diagonal partitions of the form $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ by permuting the blocks. This action restricts to $C_d = \langle \sigma_m^{m/d} \rangle$ as well. Let $\{\underline{\lambda}\}^d$ denote the orbit of $\underline{\lambda}$ under the (m/d)-fold cyclic rotations in C_d. Note, the number of block diagonal partitions in such a C_d-orbit, denoted $\#\{\underline{\lambda}\}^d$, always divides d, but could be less than d if $\underline{\lambda}$ contains repeated partitions.

For example, take $d = 2$ and $m = 6$. If $\underline{\lambda} = ((1), (2), (3, 2), (4), (5), (6, 1))$, then $\{\underline{\lambda}\}^2$ has two elements, $\underline{\lambda}$ and $((4), (5), (6, 1), (1), (2), (3, 2))$. If $\underline{\mu} = ((1), (2), (3, 2), (1), (2), (3, 2))$, then $\{\underline{\mu}\}^2$ only contains the element $\underline{\mu}$.

Theorem 2.22. [Ste89, Remark after Prop. 6.1] *The complex inequivalent irreducible representations of $G(m, d, n)$ are certain modules $S^{\{\underline{\lambda}\}^d, c}$ indexed by the pairs $(\{\underline{\lambda}\}^d, c)$ where $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is a sequence of partitions with $|\underline{\lambda}| = n$, $\{\underline{\lambda}\}^d$ is the orbit of $\underline{\lambda}$ under (m/d)-fold cyclic rotations, and c is any positive integer $1 \leq c \leq \frac{d}{\#\{\underline{\lambda}\}^d}$.*

Remark 2.23. As with $C_m \wr S_n$, the indexing scheme is again non-canonical in general up to a choice of orbit representative, though our results relying on this work are independent of these choices. In fact, Stembridge uses $\underline{\lambda} = (\lambda^{(m-1)}, \ldots, \lambda^{(0)})$, which is the most natural setting for Theorem 2.22 and Theorem 2.39 below. The fake degrees for irreducibles $S^{\underline{\lambda}}$ of $C_m \wr S_n$ are invariant up to a q-shift under all permutations of $\underline{\lambda}$ in S_m, so for our purposes the indexing scheme is largely irrelevant. The fake degrees for irreducibles $S^{\{\underline{\lambda}\}^d, c}$ of $G(m, d, n)$, however, are only invariant under the (m/d)-fold cyclic rotations of $\underline{\lambda}$ in general. In this case, strictly speaking our $\lambda^{(i)}$ corresponds to the irreducible cyclic group representation χ^{i-1} defined by $\chi^{i-1}(\sigma_m) = \omega_m^{i-1}$ where ω_m is a fixed primitive mth root of unity in the sense that

$$S^{\underline{\lambda}} \coloneqq \left(\chi^0 \wr S^{\lambda^{(1)}} \otimes \cdots \otimes \chi^{m-1} \wr S^{\lambda^{(m)}}\right)\uparrow_{C_m \wr S_{\alpha(\underline{\lambda})}}^{C_m \wr S_n};$$

see [Ste89, (4.1)]. Since we have no need of these explicit representations, we have used the naive indexing scheme throughout.

Example 2.24. For the type B_n group $G(2, 1, n)$, the irreducible representations are indexed by pairs (λ, μ) since C_1 is the trivial group and so in each case $c = 1$.

Example 2.25. For the type D_n group $G(2, 2, n)$, the irreducible representations can be thought of as being indexed by the sets $\{\lambda, \mu\}$ with $\lambda \neq \mu$ and $|\lambda| + |\mu| = n$ together with the pairs $(\nu, 1)$ and $(\nu, 2)$ where $\nu \vdash n/2$. The orbits alone can be thought of as the 2 element multisets $\{\lambda, \mu\}$ with $|\lambda| + |\mu| = n$.

2.5. **Coinvariant Algebras.** As mentioned in the introduction, Stanley (see [Sta79]) and Lusztig (unpublished) determined the graded irreducible decomposition of the type A coinvariant algebra via the major index generating function on standard Young tableaux. Stembridge was the first to publish a complete proof of this result and extended it to the complex reflection groups $G(m,d,n)$ [Ste89]. We now summarize these results.

Definition 2.26. Any group $G \subset \mathrm{GL}(\mathbb{C}^n)$ acts on the polynomial ring with n variables $\mathbb{C}[x_1,\dots,x_n]$ by identifying $\mathbb{C}^n$ with $\mathrm{Span}_{\mathbb{C}}\{x_1,\dots,x_n\}$ and extending the G-action multiplicatively. The *coinvariant algebra* of G is the quotient of $\mathbb{C}[x_1,\dots,x_n]$ by the ideal generated by homogeneous G-invariant polynomials of positive degree, which is thus a graded G-module.

Definition 2.27. Let R_n denote the coinvariant algebra of S_n. For $\lambda \vdash n$, let $g^\lambda(q)$ be the *fake degree polynomial* whose kth coefficient is the multiplicity of S^λ in the kth degree piece of R_n.

Theorem 2.28 (Lusztig–Stanley, [Sta79, Prop. 4.11])**.** *For a partition λ,*

$$g^\lambda(q) = \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q).$$

Equivalently, the multiplicity of S^λ in the kth degree piece of the type A coinvariant algebra R_n is $b_{\lambda,k}$, the number of standard tableaux of shape $\lambda \vdash n$ with major index k.

Definition 2.29. Let $R_{m,n}$ denote the coinvariant algebra of $C_m \wr S_n$. Set

$$b_{\underline{\lambda},k} \coloneqq \text{the multiplicity of } S^{\underline{\lambda}} \text{ in the } k\text{th degree piece of } R_{m,n}.$$

Write the corresponding *fake degree polynomial* as

$$g^{\underline{\lambda}}(q) \coloneqq \sum_k b_{\underline{\lambda},k} q^k.$$

Definition 2.30. Given a sequence of partitions $\underline{\lambda} = (\lambda^{(1)},\dots,\lambda^{(m)})$, recall

$$b(\alpha(\underline{\lambda})) = \sum_{i=1}^m (i-1)|\lambda^{(i)}|.$$

We continue to identify $\underline{\lambda}$ with a block diagonal skew partition when convenient, as in Definition 2.9. Thus, $\mathrm{SYT}(\underline{\lambda})$ is the set of standard Young tableaux on the block diagonal skew partition $\underline{\lambda}$. We will abuse notation and define $b(\alpha(T)) \coloneqq b(\alpha(\underline{\lambda}))$ for any $T \in \mathrm{SYT}(\underline{\lambda})$, which is not necessary in the next theorem but will be essential for the general Shephard–Todd groups $G(m,d,n)$.

Theorem 2.31. [Ste89, Thm. 5.3] *For $\underline{\lambda} = (\lambda^{(1)},\dots,\lambda^{(m)})$ with $|\underline{\lambda}| = n$,*

$$g^{\underline{\lambda}}(q) = q^{b(\alpha(\underline{\lambda}))}\,\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q^m).$$

Equivalently, the multiplicity of $S^{\underline{\lambda}}$ in the kth degree piece of the $C_m \wr S_n$ coinvariant algebra $R_{m,n}$ is the number of standard tableaux T of block diagonal shape $\underline{\lambda}$ with $k = b(\alpha(T)) + m \cdot \mathrm{maj}(T)$.

Remark 2.32. By (7), we have an explicit product formula for $g^{\underline{\lambda}}(q)$ also. Furthermore, in [BKS20a], we characterize the possible limiting distributions for the coefficients of the polynomials $\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q)$. We show that in most cases, the limiting distribution is the normal distribution. Consequently, that characterization can be interpreted as a statement about the asymptotic distribution of irreducible components in different degrees of the $C_m \wr S_n$ coinvariant algebras.

Corollary 2.33. *In type B_n, the irreducible indexed by (λ, μ) with $|\lambda| = k$ and $|\mu| = n - k$ has fake degree polynomial*

$$g^{(\lambda,\mu)}(q) = q^{|\mu|+2b(\lambda)+2b(\mu)} \binom{n}{k}_{q^2} \frac{[k]_{q^2}!}{\prod_{c\in\lambda}[h_c]_{q^2}} \frac{[n-k]_{q^2}!}{\prod_{c'\in\mu}[h_{c'}]_{q^2}}.$$

Definition 2.34. Let $R_{m,d,n}$ denote the coinvariant algebra of $G(m,d,n)$ assuming $d \mid m$. For an orbit $\{\underline{\lambda}\}^d$ of a sequence of m partitions with n total cells under (m/d)-fold cyclic rotations, set

$$b_{\{\underline{\lambda}\}^d,k} \coloneqq \text{the multiplicity of } S^{\{\underline{\lambda}\}^d,c} \text{ in the } k\text{th degree piece of } R_{m,d,n},$$

which in fact depends only on the orbit $\{\underline{\lambda}\}^d$ and not the number c by [Ste89, Prop. 6.3]. Write the corresponding *fake degree polynomial* as

$$g^{\{\underline{\lambda}\}^d}(q) \coloneqq \sum_k b_{\{\underline{\lambda}\}^d,k} q^k.$$

Theorem 2.35. [Ste89, Cor. 6.4] *Let $\{\underline{\lambda}\}^d$ be the orbit of a sequence of m partitions $\underline{\lambda}$ with $|\underline{\lambda}| = n$ under (m/d)-fold cyclic rotations. Then*

$$g^{\{\underline{\lambda}\}^d}(q) = \frac{\left(\{\underline{\lambda}\}^d\right)^{b\circ\alpha}(q)}{[d]_{q^{nm/d}}} \mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q^m)$$

where

$$\left(\{\underline{\lambda}\}^d\right)^{b\circ\alpha}(q) \coloneqq \sum_{\underline{\mu}\in\{\underline{\lambda}\}^d} q^{b(\alpha(\underline{\mu}))}.$$

Corollary 2.36 ([Lus77, Sect. 2.5], [Ste89, Cor. 6.5])**.** *In type D_n, an irreducible indexed by $\{\underline{\lambda}\}^2$ with $\underline{\lambda} = (\lambda, \mu)$ and $|\lambda| = k$, $|\mu| = n - k$ has fake degree polynomial*

$$g^{\{\underline{\lambda}\}^2}(q) = \kappa_{\lambda\mu} q^{2b(\lambda)+2b(\mu)} \frac{q^k + q^{n-k}}{1+q^n} \binom{n}{k}_{q^2} \frac{[k]_{q^2}!}{\prod_{c\in\lambda}[h_c]_{q^2}} \frac{[n-k]_{q^2}!}{\prod_{c'\in\mu}[h_{c'}]_{q^2}},$$

where $\kappa_{\lambda\mu} = 1$ if $\lambda \neq \mu$ and $\kappa_{\lambda\lambda} = 1/2$.

Observe that Theorem 2.31 gives a direct tableau interpretation of the coefficients of $g^{\underline{\lambda}}(q)$. More generally, Stembridge gave a tableau interpretation of the coefficients of $g^{\{\underline{\lambda}\}^d}(q)$ which we next describe.

Definition 2.37. For a given m, d, n, let $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ be a sequence of m partitions with $|\underline{\lambda}| = n$. Let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under (m/d)-fold rotations. The cyclic group $C_d = \langle \sigma_m^{m/d} \rangle$ acts on the disjoint union $\bigsqcup_{\underline{\mu} \in \{\underline{\lambda}\}^d} \mathrm{SYT}(\underline{\mu})$ as follows. Given $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(m)}) \in \{\underline{\lambda}\}^d$, each $T \in \mathrm{SYT}(\underline{\mu})$ may be considered as a sequence $\underline{T} = (T^{(1)}, \ldots, T^{(m)})$ of fillings of the shapes $\mu^{(i)}$. The group C_d acts by (m/d)-fold rotations of this sequence of fillings. Write the resulting orbit as $\{\underline{T}\}^d$, which necessarily has size d. For such a $\underline{T}$, the largest entry of $\underline{T}$, namely n, appears in some $T^{(k)}$. If among the elements of the orbit $\{\underline{T}\}^d$ of $\underline{T}$ this value k is minimal for $\underline{T}$ itself, then we call $\underline{T}$ the *canonical standard tableau representative* for $\{\underline{T}\}^d$. Let

$$\mathrm{SYT}(\{\underline{\lambda}\}^d) \subset \bigsqcup_{\underline{\mu} \in \{\underline{\lambda}\}^d} \mathrm{SYT}(\underline{\mu})$$

be the *set of canonical standard tableau representatives of orbits* $\{\underline{T}\}^d$ for $T \in \mathrm{SYT}(\underline{\lambda})$. Recall, $b(\alpha(\underline{T})) \coloneqq b(\alpha(\underline{\mu})) = \sum (i-1)|\mu^{(i)}|$ if $\underline{T} \in \mathrm{SYT}(\underline{\mu})$, so $b \circ \alpha$ is not generally constant on $\mathrm{SYT}(\{\underline{\lambda}\}^d)$.

Remark 2.38. When the parts $\lambda^{(i)}$ are all non-empty, the set $\mathrm{SYT}(\{\underline{\lambda}\}^d)$ is the set of standard block diagonal skew tableaux of some shape $\underline{\mu} \in \{\underline{\lambda}\}^d$ where $n = |\underline{\lambda}|$ is in the upper-right-most partition possible among the (m/d)-fold cyclic rotations of its blocks. Since every orbit $\{\underline{T}\}^d$ has size d, we have

$$\# \mathrm{SYT}(\{\underline{\lambda}\}^d) = \frac{\# \{\underline{\lambda}\}^d}{d} \# \mathrm{SYT}(\underline{\lambda}).$$

Theorem 2.39. [Ste89, Thm. 6.6] *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n$. Let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under (m/d)-fold cyclic rotations. Then*

$$g^{\{\underline{\lambda}\}^d}(q) = \mathrm{SYT}(\{\underline{\lambda}\}^d)^{b \circ \alpha + m \cdot \mathrm{maj}}(q).$$

Equivalently, the multiplicity of $S^{\{\underline{\lambda}\}^d, c}$ in the kth degree piece of the $G(m,d,n)$ coinvariant algebra $R_{m,d,n}$ is the number of canonical standard tableaux $\underline{T} \in \mathrm{SYT}(\{\underline{\lambda}\}^d)$ with $k = b(\alpha(\underline{T})) + m \cdot \mathrm{maj}(\underline{T})$.

3. Polynomial Formulas For Fake Degrees

In this section, we briefly show how to construct polynomial formulas for the fake degrees $b_{\lambda,k}$ directly from Stanley's q-hook length formula. We will use these polynomials in the next section for small changes from the minimal major index. Our results extend to a formula for counting permutations of a given inversion number.

Given λ, let

$$H_i(\lambda) = \#\{c \in \lambda : h_c = i\}, \tag{10}$$

$$m_i(\lambda) = \#\{k : \lambda_k = i\}. \tag{11}$$

If λ is understood, we abbreviate $H_i = H_i(\lambda)$. For any nonnegative integer k and polynomial $f(q)$, let $[q^k]f(q)$ be the coefficient of q^k in $f(q)$.

Lemma 3.1. *For every $\lambda \vdash n$ and $k = b(\lambda) + d$, we have*

$$b_{\lambda,k} = [q^{b(\lambda)+d}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{\substack{\mu \vdash d \\ \mu_1 \le n}} \prod_{i=1}^{|\lambda|} \binom{H_i + m_i(\mu) - 2}{m_i(\mu)} \tag{12}$$

which is a polynomial in the H_i's for every positive integer n.

Proof. By Theorem 2.11, we have

$$q^{-b(\lambda)}\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{[n]_q!}{\prod_{c \in \lambda}[h_c]_q} = \prod_{i=1}^{n}(1-q^i)^{-(H_i-1)}. \tag{13}$$

The result follows using the expansion $(1-q^i)^{-j} = \sum_{n=0}^{\infty}\binom{j+n-1}{n}q^{in}$ and multiplication of ordinary generating functions. □

Note that if $H_i(\lambda) = 0$ and $m_i(\mu) = 1$, then the corresponding binomial coefficient in (12) is -1, so it is not obvious from this formula that the coefficients $b_{\lambda,k}$ are all nonnegative, which is clearly true by definition.

Remark 3.2. The first few polynomials are given by

$$\begin{aligned}
[q^{b(\lambda)+1}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) &= H_1 - 1 \\
&= \#\{\text{notches of } \lambda\}, \\
[q^{b(\lambda)+2}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) &= \binom{H_1}{2} + H_2 - 1, \\
[q^{b(\lambda)+3}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) &= \binom{H_1+1}{3} + (H_1-1)(H_2-1) + (H_3-1) \\
[q^{b(\lambda)+4}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) &= \binom{H_1+2}{4} + \binom{H_2}{2} + \binom{H_1}{2}(H_2-1) \\
&\quad + (H_1-1)(H_3-1) + (H_4-1).
\end{aligned}$$

These exact formulas hold for all $|\lambda| \ge 4$. For smaller size partitions some terms will not appear.

It is interesting to compare these polynomials to the ones described by Knuth for the number of permutations with $k \le n$ inversions in S_n in [Knu73, p.16]. See also [Sta12, Ex. 1.124] and [OEI18, A008302]. We can extend Knuth's formulas to all $0 \le k \le \binom{n}{2}$ using the same idea.

Corollary 3.3. *For fixed positive integers k and n, we have*

$$\#\{w \in S_n : \operatorname{inv}(w) = d\} = \sum (-1)^{\#\{\mu_i > 1\}} \binom{n + m_1(\mu) - 2}{m_1(\mu)} \tag{14}$$

where the sum is over all partitions $\mu \vdash d$ such that $\mu_1 \leq n$ and all of the parts of μ larger than 1 are distinct.

The proof follows in exactly the same way from the formula

$$\sum_{w \in S_n} q^{\operatorname{inv}(w)} = \prod_{i=1}^{n} [i]_q = \prod_{i=1}^{n} [i]_q / [1]_q = (1-q)^{-n} \prod_{i=1}^{n} (1 - q^i).$$

In essence, this is the case of the q-hook length formula when all of the hooks are of length 1.

Remark 3.4. Let $T(d,n)$ be the number of partitions $\mu \vdash d$ such that $\mu_1 \leq n$ and all of the parts of μ larger than 1 are distinct. The triangle of numbers $T(d,n)$ for $1 \leq n \leq d$ is [OEI18, A318806].

4. Type A Internal Zeros Classification

As a corollary of Stanley's formula, we know that for every partition $\lambda \vdash n \geq 1$ there is a unique tableau with minimal major index $b(\lambda)$ and a unique tableau with maximal major index $\binom{n}{2} - b(\lambda')$. These two agree for shapes consisting of one row or one column, and otherwise they are distinct. It is easy to identify these two tableaux in $\mathrm{SYT}(\lambda)$; see Definition 4.1 below. Then, we classify all of the values k such that $b(\lambda) < k < \binom{n}{2} - b(\lambda')$ and the fake degree $b_{\lambda,k} = 0$. We refer to such k as *internal zeros*, meaning the location of zeros in the fake degree sequence for λ between the known minimal and maximal nonzero locations.

Definition 4.1.

(1) The *max-maj tableau* for λ is obtained by filling the outermost, maximum length, vertical strip in λ with the largest possible numbers $|\lambda|, |\lambda| - 1, \ldots, |\lambda| - \ell(\lambda) + 1$ starting from the bottom row and going up, then filling the rightmost maximum length vertical strip containing cells not previously used with the largest remaining numbers, etc.

(2) The *min-maj tableau* of λ is obtained similarly by filling the outermost, maximum length, horizontal strip in λ with the largest possible numbers $|\lambda|, |\lambda| - 1, \ldots, |\lambda| - \lambda_1 + 1$ going right to left, then filling the lowest maximum length horizontal strip containing cells not previously used with the largest remaining numbers, etc.

See Figure 4 for an example. Note that the max-maj tableau of λ is the transpose of the min-maj tableau of λ'.

The $q^{b(\lambda)+1}$ coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ can be computed as in Lemma 3.1 or Remark 3.2, resulting in the following.

$$\begin{array}{cccccc} 1 & 2 & 3 & 5 & 9 & \boxed{13} \\ 4 & 6 & 10 & \boxed{14} \\ 7 & 11 & \boxed{15} \\ 8 & 12 & \boxed{16} \\ \boxed{17} \end{array}$$

(A) A max-maj tableau and its outermost vertical strip.

$$\begin{array}{cccccc} 1 & 3 & 4 & 11 & \boxed{16} & \boxed{17} \\ 2 & 6 & 7 & \boxed{15} \\ 5 & 9 & 10 \\ 8 & \boxed{13} & \boxed{14} \\ \boxed{12} \end{array}$$

(B) A min-maj tableau and its outermost horizontal strip.

FIGURE 4. Max-maj tableau and min-maj tableau for $\lambda = (6,4,3,3,1)$.

Corollary 4.2. *We have* $[q^{b(\lambda)+1}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = 0$ *if and only if* λ *is a rectangle. If* λ *is a rectangle with more than one row and column, then* $[q^{b(\lambda)+2}]\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = 1$.

A similar statement holds for $\mathrm{maj}(T) = \binom{n}{2} - b(\lambda') - 1$ by symmetry. Thus, $\mathrm{SYT}^{\mathrm{maj}}(q)$ has internal zeros when λ is a rectangle with at least two rows and columns. We will show these are the only internal zeros of type A fake degrees, proving Theorem 1.1.

Definition 4.3. Let $\mathcal{E}(\lambda)$ denote the set of *exceptional* tableaux of shape λ consisting of the following elements.

(i) For all λ, the max-maj tableau for λ.
(ii) If λ is a rectangle, the min-maj tableau for λ.
(iii) If λ is a rectangle with at least two rows and columns, the unique tableau in $\mathrm{SYT}(\lambda)$ with major index equal to $\binom{n}{2} - b(\lambda') - 2$. It is obtained from the max-maj tableau of λ by applying the cycle $(2, 3, \ldots, \ell(\lambda) + 1)$, which reduces the major index by 2.

For example, $\mathcal{E}(64331)$ consists of just the max-maj tableau for 64331 in Figure 4a, while $\mathcal{E}(555)$ has the following three elements:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 7 & 10 & 13 \\ 3 & 5 & 8 & 11 & 14 \\ 4 & 6 & 9 & 12 & 15 \end{array} \qquad \begin{array}{ccccc} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14. \\ 3 & 6 & 9 & 12 & 15 \end{array}$$

We prove Theorem 1.1 by constructing a map

$$\varphi\colon \mathrm{SYT}(\lambda) \smallsetminus \mathcal{E}(\lambda) \longrightarrow \mathrm{SYT}(\lambda) \tag{15}$$

with the property

$$\mathrm{maj}(\varphi(T)) = \mathrm{maj}(T) + 1. \tag{16}$$

For *most* tableaux T, we can find another tableau T' of the same shape such that $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$ by applying some *simple cycle* to the values in T, meaning a permutation whose cycle notation is either $(i, i+1, \ldots, k-1, k)$ or $(k, k-1, \ldots, i+1, i)$ for some $i < k$. We will show there are 5 additional rules that must be added to complete the definition.

We note that technically the symmetric group S_n does not act on $\mathrm{SYT}(\lambda)$ for $\lambda \vdash n$ since this action will not generally preserve the row and column strict requirements for standard tableaux. However, S_n acts on the set of all bijective fillings of λ using the alphabet $\{1, 2, \ldots, n\}$ by acting on the values. We will only apply permutations to tableaux after locating all values in some interval $[i, j] = \{i, i+1, \ldots, j\}$ in T. The reader is encouraged to verify that the specified permutations always maintain the row and column strict properties.

4.1. Rotation Rules.

We next describe certain configurations in a tableau which imply that a simple cycle will increase maj by 1. Recall, the cells of a tableau are indexed by matrix notation.

Definition 4.4. Given $\lambda \vdash n$ and $T \in \mathrm{SYT}(\lambda)$, a *positive rotation* for T is an interval $[i, k] \subset [n]$ such that if $T' := (i, i+1, \ldots, k-1, k) \cdot T$, then $T' \in \mathrm{SYT}(\lambda)$ and there is some j for which

$$\{j\} = \mathrm{Des}(T') - \mathrm{Des}(T) \qquad \text{and} \qquad \{j-1\} = \mathrm{Des}(T) - \mathrm{Des}(T').$$

Intuitively, a positive rotation is one for which $j - 1 \in \mathrm{Des}(T)$ becomes $j \in \mathrm{Des}(T')$ and all other entries remain the same. Consequently, $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$. We call j the *moving descent* for the positive rotation.

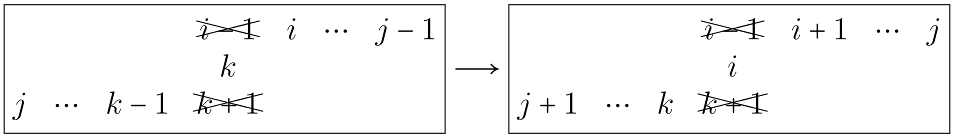
The positive rotations can be characterized explicitly as follows. The proof is omitted since it follows directly from the pictures in Figure 5.

Lemma 4.5. *An interval $[i, k]$ is a positive rotation for $T \in \mathrm{SYT}(\lambda)$ if and only if $i < k$ and there is some necessarily unique moving descent j with $1 \le i \le j \le k \le n$ such that*

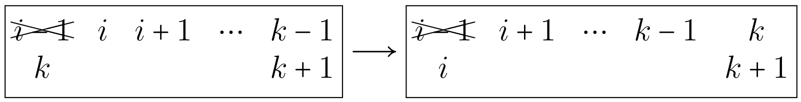
(a) $i, \ldots, j-1$ form a horizontal strip, $j-1, j$ form a vertical strip, and $j, j+1, \ldots, k$ form a horizontal strip;
(b) if $i < j$, then i appears strictly northeast of k and $i-1$ is not in the rectangle bounding i and k;
(c) if $i = j$, then $i-1$ appears in the rectangle bounding i and k;
(d) if $j < k$, then k appears strictly northeast of $k-1$ and $k+1$ is not in the rectangle bounding k and $k-1$; and
(e) if $j = k$, then $k+1$ appears in the rectangle bounding k and $k-1$.

See Figure 5 for diagrams summarizing these conditions.

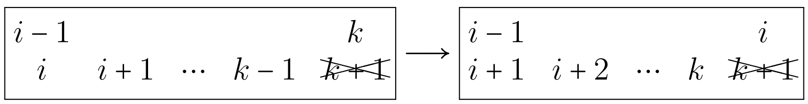
In addition to the *positive rotations* above, we can also apply *negative rotations*, which are defined exactly as in Definition 4.4 with $(i, i+1, \ldots, k-1, k)$ replaced by $(k, k-1, \ldots, i+1, i)$ and the rest unchanged. Combinatorially,



(A) Schematic of a positive rotation with $i < j < k$.



(B) Schematic of a positive rotation with $i < j = k$.



(C) Schematic of a positive rotation with $i = j < k$.

FIGURE 5. Summary diagrams for positive rotations.

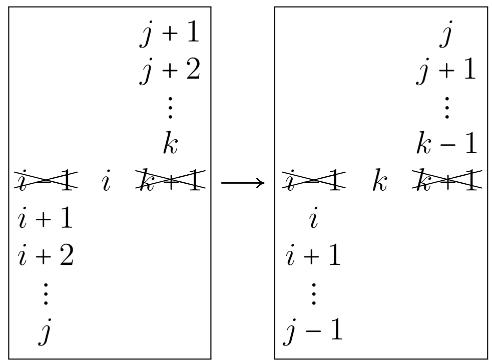
negative rotations can be obtained from positive rotations by applying *inverse-transpose* moves, that is, by applying negative cycles $(k, k-1, \ldots, i)$ to the transpose of the configurations in Figure 5 and reversing the arrows. Explicitly, we have the following analogue of Lemma 4.5. See Figure 6 for the corresponding diagrams.

Lemma 4.6. *An interval* $[i, k]$ *is a negative rotation for* $T \in \mathrm{SYT}(\lambda)$ *if and only if* $i < k$ *and there is some necessarily unique moving descent* j *with* $1 \le i \le j \le k \le n$ *such that*

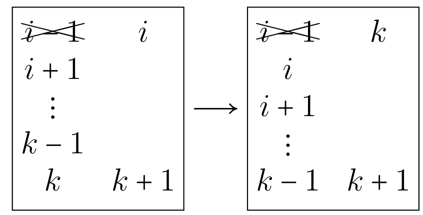
(a) $i, \ldots, j$ *form a vertical strip,* $j, j+1$ *form a horizontal strip, and* $j+1, \ldots, k$ *form a vertical strip;*
(b) if $i < j$*, then* $i+1$ *appears strictly southwest of* i *and* $i-1$ *is not in the rectangle bounding* i *and* $i+1$*;*
(c) if $i = j$*, then* $i-1$ *appears in the rectangle bounding* i *and* $i+1$*;*
(d) if $j < k$*, then* i *appears strictly southwest of* k *and* $k+1$ *is not in the rectangle bounding* i *and* k*; and*
(e) if $j = k$*, then* $k+1$ *appears in the rectangle bounding* i *and* k*.*

Example 4.7. The tableau

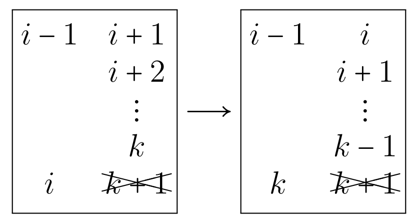
1	2	6	7	9
3	4	8	13	
5	11	12	15	
10	14			



(A) $i < j < k$.



(B) $i < j = k$.



(C) $i = j < k$.

Figure 6. Summary diagrams for negative rotations.

allows positive rotation rules with $[i,k] \in \{[5,6],[8,9],[8,10],[8,11],[9,13]\}$, and the tableau

$$\begin{array}{ccccc} 1 & 3 & 8 & 10 & 15 \\ 2 & 4 & 9 & 11 & \\ 5 & 7 & 13 & 14 & \\ 6 & 12 & & & \end{array}$$

allows negative rotation rules with $[i,k] \in \{[4,6],[6,7],[11,12]\}$.

It turns out that for the vast majority of tableaux, some negative rotation rule applies. The positive rotations can be applied in many of the remaining cases. For example, among the 81,081 tableaux in SYT(5442), there are only 24 (i.e., 0.03%) on which we cannot apply any positive or negative rotation rule. For example, no rotation rules can be applied to the following two tableaux:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & \\ 14 & 15 & & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & 2 & 3 & 8 & 12 \\ 4 & 6 & 9 & 13 & \\ 5 & 7 & 10 & 14 & \\ 11 & 15 & & & \end{array}.$$

The following lemma and its corollary give a partial explanation for why negative rotation rules are so common. Given a tableaux T, let $T|_{[z]}$ denote the restriction of T to those values in $[z]$.

Lemma 4.8. *Let $T \in \mathrm{SYT}(\lambda) \setminus \mathcal{E}(\lambda)$. Suppose z is the largest value such that $T|_{[z]}$ is contained in* $\mathrm{maxmaj}(\mu)$ *for some μ. If $T|_{[z+1]}$ is not of the form*

$$\begin{matrix} 1 & 2 & \cdots & i \\ i+1 & z+1 & & \\ i+2 & & & \\ \vdots & & & \\ z & & & \end{matrix}$$

then some negative rotation rule applies to T.

Proof. Since $T \notin \mathcal{E}(\lambda)$, T is not $\mathrm{maxmaj}(\lambda)$, so λ is not a one row or column shape. We have $z \geq 2$ since both two-cell tableaux are the max-maj tableau of their shape. Since $\mathrm{maxmaj}(\mu)$ is built from successive, outermost, maximal length, vertical strips as in Figure 4a, the same is true of $T|_{[z]}$.

First, suppose z is not in the lowest row of $T|_{[z]}$. Let i be the value in the topmost corner cell in $T|_{[z]}$ which is strictly below z. Let $j \geq i$ be the bottommost cell in the vertical strip of $T|_{[z]}$ which contains i. See Figure 7a. We verify the conditions of Lemma 4.6, so the negative $[i, z]$-rotation rule applies with moving descent j. By construction, $i, \ldots, j$ form a vertical strip, $j, j+1$ form a horizontal strip, and $j+1, \ldots, z$ form a vertical strip. If $i < j$, then since i is a corner cell, $i+1$ appears strictly southwest of i, and $i-1$ is above both i and $i+1$ so $i-1$ is not in the rectangle bounding i and $i+1$. If $i = j$, we see that $i-1$ appears in the rectangle bounded by i and $i+1$. We also see that i appears strictly southwest of z, and $z+1$ is not in the rectangle bounding i and z since i is a topmost corner and z is maximal.

Now suppose z is in the lowest row of $T|_{[z]}$. In this case, $T|_{[z]}$ is the max-maj tableau of its shape, so that $z < |\lambda|$ and $z+1$ exists in T since $T \notin \mathcal{E}(\lambda)$. By maximality of z, $z+1$ cannot be in row 1 or below z. Let $i < z$ be the value in the rightmost cell of $T|_{[z]}$ in the row immediately above $z+1$. See Figure 7b. We check that the negative $[i, z]$-rotation rule applies with moving descent $j = z$ using the conditions in Lemma 4.6. By construction, $i, \ldots, z$ form a vertical strip. Since $z+1$ is not below z, we see that $z, z+1$ form a horizontal strip. Since $z+1$ is in the row below i, $i+1$ appears strictly southwest of i. We also see that $z+1$ appears in the rectangle bounded by i and z by choice of i. It remains to show that $i-1$ is not in the rectangle bounding i and $i+1$. Suppose to the contrary that $i-1$ is in the rectangle bounding i and $i+1$. Then i would have to be in row 1 by choice of $i < z$. Consequently $i+1$ is in row 2 and strictly west of i, forcing $i-1$ to be in row 1 also. It follows from the choice of z that $T|_{[i]}$ is a single row, the values $i, i+1, \ldots z$ form a vertical strip, and $T|_{[z+1]}$ is of the above forbidden form, giving a contradiction. ☐

Corollary 4.9. *If $T \in \mathrm{SYT}(\lambda) \setminus \mathcal{E}(\lambda)$ and $1 \in \mathrm{Des}(T)$, then some negative rotation rule applies to T.*

$$\begin{array}{llll} 1 & 3 & 6 & 11 \\ 2 & 4 & 7 & 12 \\ 5 & 8 & & \\ 9 & 13 & & \\ 10 & & & \end{array} \quad \longrightarrow \quad \begin{array}{llll} 1 & 3 & 6 & 10 \\ 2 & 4 & 7 & 11 \\ 5 & 12 & & \\ 8 & 13 & & \\ 9 & & & \end{array}$$

(A) For the tableau on the left above, $i = 8$ and $z = 12$ since $T|_{[12]}$ is contained the max-maj tableau of shape 44322, 12 is not in the lowest row, 8 is in the closet corner to 12 in $T|_{[12]}$ and below 12. Apply the negative rotation $(12, 11, 10, 9, 8)$ to get the tableau on the right, and observe maj has increased by 1. The moving descent is $j = 10$.

$$\begin{array}{lll} 1 & 3 & 6 \\ 2 & 4 & 7 \\ 5 & 8 & 11 \\ 9 & & \\ 10 & & \end{array} \quad \longrightarrow \quad \begin{array}{lll} 1 & 3 & 6 \\ 2 & 4 & 10 \\ 5 & 7 & 11 \\ 8 & & \\ 9 & & \end{array}$$

(B) For the tableau on the left above, $i = 7$ and $z = 10$ since $T|_{[10]}$ is the max-maj tableau of shape 33211, 10 is in the lowest row, 11 is in row 3, and 7 is the largest value in $T|_{[10]}$ in row 2. Apply the negative rotation $(10, 9, 8, 7)$ to get the tableau on the right, and observe maj has increased by 1. The moving descent is $j = z = 10$.

FIGURE 7. Examples of the negative rotations obtained from Lemma 4.8.

Proof. Let z be as in Lemma 4.8. Clearly $z \geq 2$ and $T|_{[2]}$ is a single column, so $T|_{[z+1]}$ cannot possibly be of the forbidden form. □

We also have the following variation on Lemma 4.8. It is based on finding the largest value q such that $T|_{[q]}$ is contained in an exceptional tableau of type (iii). The proof is again a straightforward verification of the conditions in Lemma 4.6, and is omitted.

Lemma 4.10. *Let $T \in \mathrm{SYT}(\lambda) \smallsetminus \mathcal{E}(\lambda)$. Suppose the initial values of T are of the form*

$$\begin{array}{cc} 1 & 2 \\ 3 & p+1 \\ 4 & \vdots \\ \vdots & q \\ \vdots & \xcancel{q+1} \\ p & \end{array} \qquad \textit{or} \qquad \begin{array}{cccccc} 1 & 2 & \ell+1 & \cdots & \vdots & p+1 \\ 3 & z+1 & \vdots & \vdots & \vdots & \vdots \\ 4 & z+2 & \vdots & \vdots & \vdots & q \\ \vdots & \vdots & \vdots & \vdots & \vdots & \xcancel{q+1} \\ z & \ell & m & \cdots & p & \end{array}.$$

In either case, the $[p, q]$-negative rotation rule applies to T.

4.2. **Initial Block Rules.** Here we describe a collection of five additional *block rules* which may apply to a tableau that is not in the exceptional set. In each case, if the rule applies, then we specify a permutation of the entries so that we either add 1 into the descent set and leave the other descents unchanged, or we add 1 into the descent set, increase one existing descent by 1, and decrease one existing descent by 1. Thus, maj will increase by 1 in all cases. While these additional rules are certainly not uniquely determined by these criteria, they are also not arbitrary.

Example 4.11. For a given $T \in \mathrm{SYT}(\lambda)$, one may consider all $T' \in \mathrm{SYT}(\lambda)$ where $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$. If $T' = \sigma \cdot T$ where σ is a simple cycle, then one of the rotation rules may apply to T. Table 1 summarizes five particular T for which *no* rotation rules apply. These examples have guided our choices in defining the block rules. In all but one of these examples, there is a unique T' with $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$, though in the third case there are two such T', one of which ends up being easier to generalize.

Tableau T	Tableaux T'	σ	Block rule
$\begin{array}{llll} 1&2&3&7 \\ 4&5&6&8 \end{array}$	$\begin{array}{llll} 1&3&4&6 \\ 2&5&7&8 \end{array}$	$(2,3,4)(6,7)$	B1
$\begin{array}{llll} 1&2&3&4 \\ 5&6&7 \end{array}$	$\begin{array}{llll} 1&3&4&7 \\ 2&5&6 \end{array}$	$(2,3,4,7,6,5)$	B2
$\begin{array}{lll} 1&2&3 \\ 4&6 \\ 5&7 \end{array}$	$\begin{array}{lll} 1&3&6 \\ 2&4 \\ 5&7 \end{array}, \begin{array}{lll} 1&4&5 \\ 2&6 \\ 3&7 \end{array}$	$(2,3,6,4), (2,4)(3,5)$	B3, —
$\begin{array}{lll} 1&2&7 \\ 3&5&8 \\ 4&6&9 \\ 10 \end{array}$	$\begin{array}{lll} 1&4&8 \\ 2&5&9 \\ 3&6&10 \\ 7 \end{array}$	$(2,4,3)(7,8,9,10)$	B4
$\begin{array}{ll} 1&2 \\ 3&5 \\ 4&6 \\ 7 \end{array}$	$\begin{array}{ll} 1&5 \\ 2&6 \\ 3&7 \\ 4 \end{array}$	$(2,5,6,7,4,3)$	B5

TABLE 1. Some tableaux $T \in \mathrm{SYT}(\lambda)$ together with all $T' = \sigma \cdot T \in \mathrm{SYT}(\lambda)$ where $\mathrm{maj}(T') = \mathrm{maj}(T) + 1$. See Definition 4.13 for an explanation of the final column.

In the remainder of this subsection, we describe the block rules, abbreviated B-rules. Then, we prove that if no rotation rules are possible for a tableau then either it is in the exceptional set or we can apply one of the B-rules. The B-rules cover disjoint cases so no tableau admits more than one block rule. To state the B-rules precisely, assume $T \in \mathrm{SYT}(\lambda) \smallsetminus \mathcal{E}(\lambda)$ and no rotation rule applies.

Notation 4.12. Let c be the largest possible value such that $T|_{[c]}$ is contained in the min-maj tableau of a rectangle shape with a columns and b rows.

Consequently, the first a numbers in row i, $1 \le i \le b-1$, of T are $(i-1)a+1, \ldots, ia$, and row b begins with $(b-1)a+1, (b-1)a+2, \ldots, c$.

Assuming $1 \notin \mathrm{Des}(T)$ and $T \notin \mathcal{E}(\lambda)$, we know $a, b \ge 2$ and $c \ge 3$. If $c+1$ is in T, then it must be either in position $(1, a+1)$ or $(b+1, 1)$. If $c = ab$, then $c < |\lambda|$ since $T \notin \mathcal{E}(\lambda)$, otherwise $c = |\lambda|$ is possible. For example, the tableaux

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 16 \\ 6 & 7 & 8 & 9 & 10 & 17 \\ 11 & 12 & 13 & 14 & 15 & \end{array}, \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 9 & & \\ 6 & 10 & & \\ 7 & & & \\ 8 & & & \end{array}, \quad \begin{array}{cccc} 1 & 2 & 7 & 10 \\ 3 & 5 & 8 & 11 \\ 4 & 6 & 9 & 12 \\ 13 & & & \end{array}$$

have (a, b, c) equal to $(5, 3, 15)$, $(5, 3, 13)$, $(4, 2, 5)$, and $(2, 2, 3)$, respectively.

Definition 4.13. Using Notation 4.12, we identify the *block rules* with further required assumptions as follows. See Figure 8 for summary diagrams.

- **Rule B1**: Assume $c = ab$, $T_{(1,a+1)} = c+1$, $T_{(2,a+1)} = c+2$, and $a < c-2$. In this case, we perform the rotations $(2, \ldots, a+1)$ and $(c, c+1)$ which are sufficiently separated by hypothesis. Then, $1, a+1$ and c become descents, and a and $c+1$ are no longer descents, so the major index is increased by 1. The B1 rule is illustrated here with $a = 5$, $b = 3$:

$$\text{B1:} \quad \begin{array}{cccccc} 1 & {\color{red}2} & {\color{red}3} & {\color{red}4} & \boxed{\color{red}5} & \boxed{\color{blue}16} \\ {\color{red}6} & 7 & 8 & 9 & 10 & 17 \\ 11 & 12 & 13 & 14 & {\color{blue}15} & \end{array} \quad \begin{array}{c} {\color{red}\circlearrowright} \\ {\color{blue}\circlearrowright} \end{array} \quad \begin{array}{cccccc} \boxed{1} & {\color{red}3} & {\color{red}4} & {\color{red}5} & \boxed{\color{red}6} & \boxed{\color{blue}15} \\ {\color{red}2} & 7 & 8 & 9 & 10 & 17 \\ 11 & 12 & 13 & 14 & {\color{blue}16} & \end{array}$$

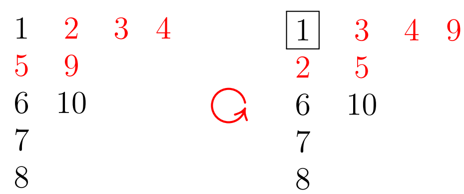
  The boxed numbers represent descents of the tableau on the left/right that are not descents of the tableau on the right/left. The elements not shown (i.e. $18, 19, \ldots, |\lambda|$) can be in any position.

- **Rule B2**: Assume $c < ab$ and there exists a $1 \le k < a$ such that $T_{(b,k)} = c$ and $T_{(b,k+1)} \ne c+1$. In this case, we perform the rotation $(2, 3, \ldots, a, 2a, 3a, \ldots, a(b-1), c, c-1, \ldots, c-k+1 = a(b-1)+1, a(b-2)+1, \ldots, 2a+1, a+1)$ around the perimeter of $T|_{[c]}$. Now 1 becomes a descent, and the other descents stay the same so the major index again increases by 1. The B2 rule is illustrated by the following (here $a = 5$, $b = 2$ and $k = 3$):

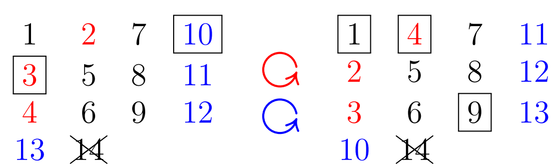
$$\text{B2:} \quad \begin{array}{ccccc} 1 & {\color{red}2} & {\color{red}3} & {\color{red}4} & {\color{red}5} \\ {\color{red}6} & 7 & 8 & 9 & {\color{red}10} \\ {\color{red}11} & {\color{red}12} & {\color{red}13} & \xcancel{14} & \end{array} \quad {\color{red}\circlearrowright} \quad \begin{array}{ccccc} \boxed{1} & {\color{red}3} & {\color{red}4} & {\color{red}5} & {\color{red}10} \\ {\color{red}2} & 7 & 8 & 9 & {\color{red}13} \\ {\color{red}6} & {\color{red}11} & {\color{red}12} & \xcancel{14} & \end{array}$$

  The crossed out number 14 means that 14 is not in position $(3, 4)$: it can either be in positions $(1, 6)$ or $(4, 1)$, or it can be that $\lambda = 553$. Again, the numbers $15, \ldots, |\lambda|$ can be anywhere in T.

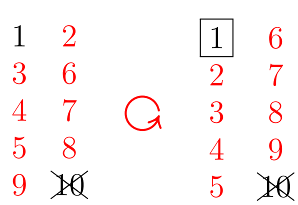
- **Rule B3**: Assume $a \geq 3$, $c = a + 1$, and there exists $k \geq 2$ such that $T_{(2,2)} = a + k + 1$, $T_{(3,2)} = a + k + 2$, and for all $i \in \{1, 2, \ldots, k\}$ we have $T_{(i+1,1)} = a + i$. Thus $b = 2$. Then we apply the rotation $(2, 3, \ldots, a, a + k + 1, a + 1)$. Now 1 becomes a descent, and the rest of the descent set is unchanged so the major index again increases by 1. The B3 rule is illustrated by the following (here $a = 4$, $k = 4$):

B3: 

- **Rule B4**: Assume that $a = 2$, $c = 3$, and there exists $k \geq 2$ such that $\{3, 4, \ldots, k + 1\}$ appear in column 1 of T, $\{k + 2, k + 3, \ldots, 2k\}$ appear in column 2 in T. Further assume that the set $\{2k + 1, 2k + 2, \ldots, 3k\}$ appears in column 3, $\{3k + 1, 3k + 2, \ldots, 4k\}$ appears in column 4, etc., until column l for some $l > 2$ and $T_{(k+1,1)} = kl + 1$ and $T_{(k+1,2)} \neq kl + 2$. Thus, $b = 2$. In this case, we can perform the two rotations $(k + 1, k, \ldots, 3, 2)$ and $(k(l-1) + 1, k(l-1) + 2, \ldots, kl, kl + 1)$. Now 1, $k + 1$ and $k(l-1)$ enter the descent set, and k and $k(l-1) + 1$ leave it, so the major index increases by 1. The B4 rule is illustrated by the following (here $k = 3$ and $l = 4$):

B4: 

- **Rule B5**: Assume that $a = 2$, $c = 3$, and there exists $k > 3$ such that $\{3, 4, \ldots, k\}$ appear in column 1 of T, $\{k + 1, k + 2, \ldots, 2k - 2\}$ appear in column 2 in T. Furthermore, assume $T_{(k,1)} = 2k - 1$ and $T_{(k,2)} \neq 2k$. Thus, $b = 2$. Then apply the cycle $(k, k-1, \ldots, 3, 2, k+1, k+2, \ldots, 2k-1)$ to T. Now 1 becomes a descent, and the rest of the descent set remains unchanged, so the major index increases by 1. The B5 rule is illustrated by the following (here $k = 5$):

B5: 

Lemma 4.14. *If $T \in \mathrm{SYT}(\lambda)$, $T \notin \mathcal{E}(\lambda)$, and $1, 2 \notin \mathrm{Des}(T)$, then either some rotation rule applies to T or a B1, B2 or B3 rule applies.*

Proof. Let c be the largest possible value such that $T|_{[c]}$ is contained in the min-maj tableau of a rectangle shape with a columns and b rows, see Notation 4.12.

$$\begin{array}{|ccccc|}\hline 1 & 2 & \cdots & a & ab+1 \\ a+1 & a+2 & \cdots & 2a & ab+2 \\ \vdots & \vdots & \ddots & \vdots & \\ a(b-1)+1 & a(b-1)+2 & \cdots & ab & \\ \hline \end{array}$$
$$\downarrow$$
$$\begin{array}{|ccccc|}\hline 1 & 3 & \cdots & a+1 & ab \\ 2 & a+2 & \cdots & 2a & ab+2 \\ \vdots & \vdots & \ddots & \vdots & \\ a(b-1)+1 & a(b-1)+2 & \cdots & ab+1 & \\ \hline \end{array}$$

(A) B1.

$$\begin{array}{|ccccccc|}\hline 1 & 2 & \cdots & \cdots & \cdots & \cdots & a \\ a+1 & a+2 & \cdots & \cdots & \cdots & \cdots & 2a \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a(b-2)+1 & a(b-2)+2 & \cdots & \cdots & \cdots & \cdots & a(b-1) \\ a(b-1)+a & a(b-1)+2 & \cdots & c-1 & c & \cdots & \cancel{ab} \\ \hline \end{array}$$
$$\downarrow$$
$$\begin{array}{|ccccccc|}\hline 1 & 3 & \cdots & \cdots & \cdots & \cdots & 2a \\ 2 & a+2 & \cdots & \cdots & \cdots & \cdots & 3a \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a(b-3)+1 & a(b-2)+2 & \cdots & \cdots & \cdots & \cdots & c \\ a(b-2)+1 & a(b-1)+1 & \cdots & c-2 & c-1 & \cdots & \cancel{ab} \\ \hline \end{array}$$

(B) B2.

$$\begin{array}{|ccccc|}\hline 1 & 2 & \cdots & a-1 & a \\ a+1 & a+k+1 & & & \\ a+2 & a+k+2 & & & \\ \vdots & & & & \\ a+k & & & & \\ \hline \end{array} \longrightarrow \begin{array}{|ccccc|}\hline 1 & 3 & \cdots & a & a+k+1 \\ 2 & a+1 & & & \\ a+2 & & & & \\ \vdots & & & & \\ a+k & & & & \\ \hline \end{array}$$

(C) B3.

$$\begin{array}{|ccccc|} 1 & 2 & 2k+1 & \cdots & k(\ell-1)+1 \\ 3 & k+2 & 2k+2 & \cdots & k(\ell-1)+2 \\ 4 & k+3 & 2k+3 & \cdots & k(\ell-1)+3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k+1 & 2k & 3k & \cdots & k\ell \\ k\ell+1 & \not{k\ell+2} & & & \end{array}$$

$$\downarrow$$

$$\begin{array}{|ccccc|} 1 & k+1 & 2k+1 & \cdots & k(\ell-1)+2 \\ 2 & k+2 & 2k+2 & \cdots & k(\ell-1)+3 \\ 3 & k+3 & 2k+3 & \cdots & k(\ell-1)+4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & 2k & 3k & \cdots & k\ell+1 \\ k(\ell-1)+1 & \not{k\ell+2} & & & \end{array}$$

(D) B4.

$$\begin{array}{|cc|} 1 & 2 \\ 3 & k+1 \\ 4 & k+2 \\ \vdots & \vdots \\ k-1 & 2k-3 \\ k & 2k-2 \\ 2k-1 & \not{2k} \end{array} \longrightarrow \begin{array}{|cc|} 1 & k+1 \\ 2 & k+2 \\ 3 & k+3 \\ \vdots & \vdots \\ k-2 & 2k-2 \\ k-1 & 2k-1 \\ k & \not{2k} \end{array}$$

(E) B5.

FIGURE 8. Summary diagrams for block rules.

Since $1, 2 \notin \mathrm{Des}(T)$ and $T \notin \mathcal{E}(\lambda)$, we know $1, 2, 3$ are in the first row of T so $a \geq 3$, $b \geq 2$, and $a + 2 \leq |\lambda|$. By construction, we have $T_{(2,1)} = a + 1$ and $a + 2$ must appear in position $(1, a + 1)$, $(2, 2)$, or $(3, 1)$ in T.

Case 1: $T_{(1,a+1)} = a + 2$**.** Observe that

$$T|_{[a+2]} = \begin{array}{cccccc} 1 & 2 & 3 & \cdots & a & a+2 \\ a+1 & & & & & \end{array}$$

and $z \geq a + 2$. Consequently, $T|_{[z+1]}$ cannot be of the form forbidden by Lemma 4.8, so a negative rotation rule applies.

Case 2: $T_{(2,2)} = a + 2$**.** First suppose $c = ab$, then $T_{(1,a+1)} = c + 1$ by choice of c. Now consider the two subcases, $T_{(2,a+1)} = c + 2$ and $T_{(2,a+1)} \neq c + 2$. In the former case, as in Figure 8a, the B1 rule applies to T. In the latter case, one may check that an $[i, c+1]$-positive rotation rule applies to T where $i = T_{(b,1)}$. On the other hand, if $c < ab$, then a B2 rule applies to T as in Figure 8b.

Case 3: $T_{(3,1)} = a + 2$**.** Let $k = \min\{j \geq 2 \mid a + j \notin \mathrm{Des}(T)\}$ so $T_{(k+1,1)} = a + k$ and $T_{(k+2,1)} \neq a + k + 1$. Since $T \notin \mathcal{E}(\lambda)$, we know $a + k + 1$ exists in T either in position $(1, a + 1)$ or $(2, 2)$, so $T|_{[a+k+1]}$ looks like

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & a & a+k+1 \\ a+1 \\ a+2 \\ \vdots \\ a+k \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} 1 & 2 & 3 & \cdots & a \\ a+1 & a+k+1 \\ a+2 \\ \vdots \\ a+k \end{array} .$$

If $T_{(1,a+1)} = a + k + 1$, then Lemma 4.8 shows that a negative rotation rule applies to T. On the other hand, if $T_{(2,2)} = a + k + 1$, then observe that either a B3 move applies or the rotation $(a + k, a + k + 1)$ applies to T, depending on whether $T_{(3,2)} = a + k + 2$ or not. ☐

Lemma 4.15. *If* $T \in \mathrm{SYT}(\lambda)$*,* $T \notin \mathcal{E}(\lambda)$*,* $1 \notin \mathrm{Des}(T)$*, and* $2 \in \mathrm{Des}(T)$*, then either some rotation rule applies to* T *or a B1, B2, B4 or B5 rule applies.*

Proof. Let $k = \min\{j \geq 3 \mid j \notin \mathrm{Des}(T)\}$ so the consecutive sequence $[3, k]$ appears in the first column of T and $k + 1$ does not. By definition of k and the fact that $T \notin \mathcal{E}(\lambda)$, T must have $k+1$ in position $(1, 3)$ or $(2, 2)$. If $T_{(1,3)} = k+1$, then a negative rotation rule holds by Lemma 4.8.

Assume $T_{(2,2)} = k+1$. Let ℓ be the maximum value such that $[k+1, \ell]$ appears as a consecutive sequence in column 2 of T. If $\ell < 2(k - 1)$, then the negative rotation rule for $(\ell, \ell - 1, \dots, k)$ applies to T by the first case of Lemma 4.10.

If $\ell = 2(k-1)$ and $T_{(1,3)} = \ell+1$, let m be the maximum value such that $[\ell+1, m]$ appears as a consecutive sequence in column 3 of T. We subdivide on cases for m again. If $m < 3(k - 1)$, then the negative rotation rule $(m, m - 1, \dots, \ell)$ applies to T by the second case of Lemma 4.10. If $m = 3(k - 1)$, we consider the maximal sequence of columns containing a consecutive sequence in rows $[1, k - 1]$ to the right of column 2 until one of two conditions hold

$$\boxed{\begin{array}{ccccc} 1 & 2 & \ell+1 & \cdots & \vdots \\ 3 & k+1 & \vdots & \vdots & p \\ \vdots & \vdots & \vdots & \ddots & \xcancel{p+1} \\ k & \ell & m & \cdots \end{array}} \qquad \boxed{\begin{array}{ccccc} 1 & 2 & \ell+1 & \cdots \\ 3 & k+1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & \ell & m & \cdots & p \\ p+1 \end{array}}$$

In the first picture, $T|_{[p]}$ is not a rectangle, so we may apply a negative rotation by the second case of Lemma 4.10, so consider the second picture. In the second picture, $T|_{[p]}$ is a rectangle and we know $p + 1$ exists in T since $T|_{[p]}$

is an exceptional tableau for a rectangle shape. If $p+2$ is in row k, column 2, a $(p, p+1)$ rotation rule applies. If $p+2$ is not in row k, column 2, then a B4-move applies.

Finally, consider the case $\ell = 2(k-1)$ and $T_{(k,1)} = \ell + 1$. If $T_{(k,2)} \neq \ell + 2$ and $k > 3$, then a B5-move applies. If $T_{(k,2)} = \ell + 2$ and $k > 3$, then the rotation $(\ell, \ell+1)$ applies to T since $\ell - 1$ is above ℓ. If $T_{(k,2)} = \ell + 2$ and $k = 3$, then $\ell = 4 = T_{(2,2)}$ and $T_{(3,1)} = 5$ so T contains

$$\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{matrix} .$$

In this case, consider the subcases $c = ab$ or $c < ab$ with $a = 2$. If $c = ab$, then $T_{(1,3)} = c+1$ since $T \notin \mathcal{E}(\lambda)$. Either a B1-move applies if $T_{(2,3)} = c+2$ and a $(c, c+1)$ rotation applies otherwise. On the other hand, if $c < ab$ then a B2-rule applies. □

We may finally define the map φ from (15). The proof of Theorem 1.1 from the introduction follows immediately from this definition and the last few lemmas.

Definition 4.16. Given $T \in \mathrm{SYT}(\lambda) - \mathcal{E}(\lambda)$, we define $\varphi(T)$ as follows. If $1 \in \mathrm{Des}(T)$, define $\varphi(T) = (z, z-1, \dots, i)T$ as in Corollary 4.9. If $1, 2 \notin \mathrm{Des}(T)$, then Lemma 4.14 applies, so define $\varphi(T)$ using the specific B1, B2, B3 or rotation rule identified in the proof of that lemma. If $1 \notin \mathrm{Des}(T)$ and $2 \in \mathrm{Des}(T)$, then Lemma 4.15 applies, so define $\varphi(T)$ using the specific B1, B2, B4, B5, or negative rotation rule identified in the proof of that lemma. These rules cover all possible cases. By contruction, $\mathrm{maj}(\varphi(T)) = \mathrm{maj}(T) + 1$.

We may define two poset structures on standard tableaux of a given shape using the preceding combinatorial operations. We call them "strong" and "weak" in analogy with the strong and weak Bruhat orders on permutations. Recall an *inverse-transpose* block rule is a block rule obtained from transposing the diagrams in Figure 8 and reversing the arrows.

Definition 4.17. As sets, let $P(\lambda)$ and $Q(\lambda)$ be either

$$\mathrm{SYT}(\lambda) \smallsetminus \{\mathrm{minmaj}(\lambda), \mathrm{maxmaj}(\lambda)\}$$

if λ is a rectangle with at least two rows and columns, or $\mathrm{SYT}(\lambda)$ otherwise.

- (Strong SYT Poset) Let $P(\lambda)$ be the partial order with covering relations given by rotations, block rules, and inverse-transpose block rules increasing maj by 1.
- (Weak SYT Poset) Let $Q(\lambda)$ be the partial order with covering relations given by $S \lessdot T$ if $\varphi(S) = T$ or $\varphi(T') = S'$ where S', T' are the transpose of S, T, respectively.

Corollary 4.18. *As posets, $P(\lambda)$ and $Q(\lambda)$ are ranked with a unique minimal and maximal element. If λ is not a rectangle, the rank function is given by* $\mathrm{rk}(T) = \mathrm{maj}(T) - b(\lambda)$. *If λ is a rectangle with at least two rows and columns, then the rank function is given by* $\mathrm{rk}(T) = \mathrm{maj}(T) - b(\lambda) - 2$.

Proof. By Corollary 4.2, $P(\lambda)$ and $Q(\lambda)$ have a single element of minimal maj and of maximal maj. Any element T besides these is covered by $\varphi(T)$ and covers $\varphi(T')'$, so is not maximal or minimal. By construction maj increases by 1 under covering relations. The result follows. □

In Figure 9, we show an example of both the Weak SYT Poset and the Strong SYT poset for $\lambda = (3, 2, 1)$. More examples of these partial orders are given at `https://sites.math.washington.edu/~billey/papers/syt.posets`.

Remark 4.19. Observe that both the positive and negative rotation rules apply equally well to any skew shape tableaux in $\mathrm{SYT}(\lambda/\nu)$. The block rules apply to skew shape tableaux as well when T_z is a straight shape tableau. However, in order to define the analogous posets on $\mathrm{SYT}(\lambda/\mu)$, one must include additional block moves. This is part of an ongoing project.

Remark 4.20. Lascoux–Schützenberger [LS81] defined an operation called *cyclage* on semistandard tableaux, which decreases *cocharge* by 1. The *cyclage poset* on the set of semistandard tableaux arises from applying cyclage in all possible ways. Cyclage preserves the *content*, i.e. the number of 1's, 2's, etc. See also [SW00, Sect. 4.2]. Restricting to standard tableaux, cocharge coincides with maj, so the cyclage poset on $\mathrm{SYT}(n)$ is ranked by maj. However, cyclage does not necessarily preserve the shape, so it does not suffice to prove Theorem 1.1. For example, restricting the cyclage poset to $\mathrm{SYT}(32)$ gives a poset which has two connected components and is not ranked by maj, while both of our poset structures on $\mathrm{SYT}(32)$ are chains. A reviewer of [BKS20b] posed an interesting question: is there any relation between the cyclage poset covering relations restricted to $\mathrm{SYT}(\lambda)$ and the two ranked poset structures used to prove Theorem 1.1? We have not found one, but such a connection would be interesting if found.

5. Internal zeros for des on $\mathrm{SYT}(\lambda)$

The results of Section 4 show that $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ almost never has internal zeros. Adin–Elizalde–Roichman analogously considered the internal zeros of the descent number generating functions $\mathrm{SYT}(\lambda/\nu)^{\mathrm{des}}(q)$ where $\mathrm{des}(T)$ is the number of descents in a tableau T.

Question 5.1. [AER18, Problem 7.5] *Is* $\{\mathrm{des}(T) : T \in \mathrm{SYT}(\lambda/\nu)\}$ *an interval consisting of consecutive integers, for any skew shape λ/ν? That is, does* $\mathrm{SYT}(\lambda/\nu)^{\mathrm{des}}(q)$ *ever have internal zeros?*

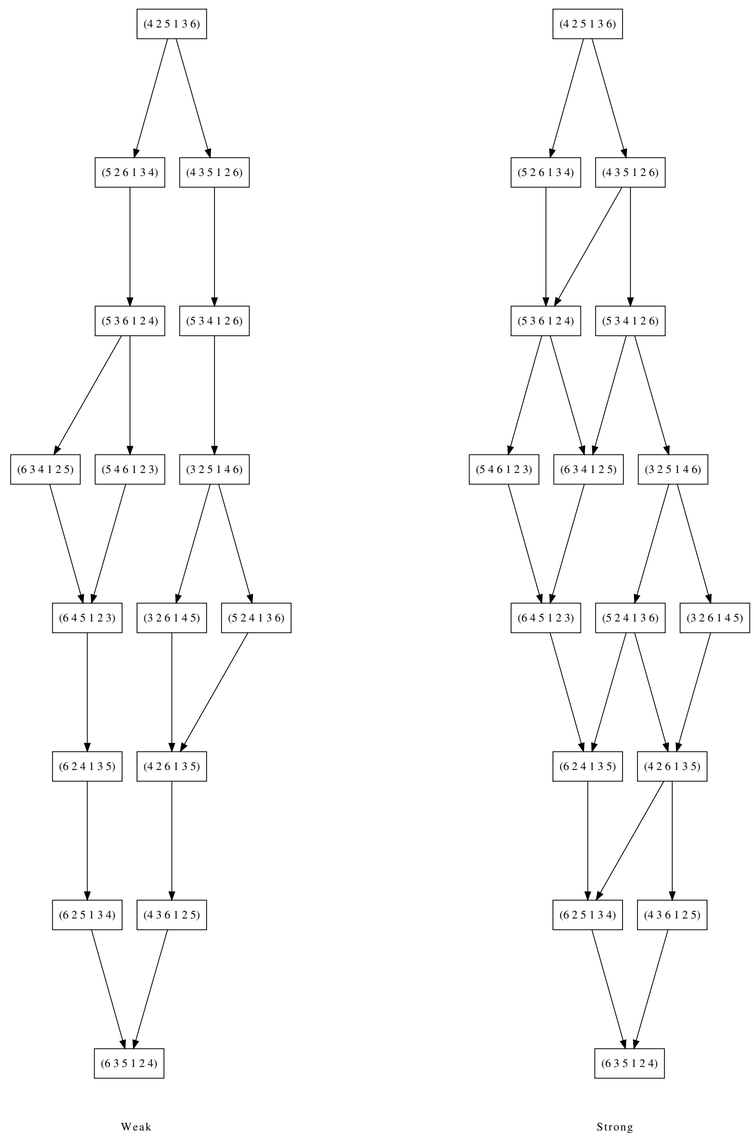


FIGURE 9. Hasse diagram of the Weak SYT Poset and the Strong SYT Poset of $\lambda = (3,2,1)$. Each tableau is represented by its row reading word in these pictures.

The minimum and maximum descent numbers are easily described as follows. The argument involves constructions similar to the sequences of vertical and horizontal strips used in Definition 4.1.

Lemma 5.2. [AER18, Lemma 3.7(1)] *Let λ/ν be a skew shape with n cells. Let c be the maximum length of a column and r be the maximum length of a* *row. Then*

$$\min \operatorname{des}(\mathrm{SYT}(\lambda/\nu)) = c - 1$$
$$\max \operatorname{des}(\mathrm{SYT}(\lambda/\nu)) = (n-1) - (r-1) = n - r.$$

Indeed, it is easy to see that $\operatorname{minmaj}(\lambda)$ constructed as in Definition 4.1 has $\lambda_1' - 1 = c - 1$ descents, and symmetrically that $\operatorname{maxmaj}(\lambda)$ has $n - r$ descents. The arguments in Section 4 consequently resolve Question 5.1 affirmatively in the straight-shape case.

Corollary 5.3. *For $\lambda \vdash n$, we have*

$$\{\operatorname{des}(T) : T \in \mathrm{SYT}(\lambda)\} = \{\lambda_1' - 1, \lambda_1', \ldots, n - \lambda_1 - 1, n - \lambda_1\}.$$

In particular, $\mathrm{SYT}(\lambda)^{\operatorname{des}}(q)$ *has no internal zeros.*

Proof. First suppose λ is not a rectangle with at least two rows and columns. Iterating the φ map creates a chain from $\operatorname{minmaj}(\lambda)$ to $\operatorname{maxmaj}(\lambda)$. At each step, φ either applies a rotation rule or a block rule. Rotation rules preserve descent number. Block rules always increase the descent number by exactly 1. Since $\operatorname{minmaj}(\lambda)$ and $\operatorname{maxmaj}(\lambda)$ have the minimum and maximum number of descents possible, the result follows.

If λ is a rectangle with at least two rows and columns, it is easy to see that the unique tableau of major index $b(\lambda) + 2$ has exactly one more descent than $\operatorname{minmaj}(\lambda)$. The result follows as before by iterating the φ map. □

Remark 5.4. The same argument shows that $\mathrm{SYT}(\lambda)^{\operatorname{maj}-\operatorname{des}}(q)$ also has no internal zeros. Indeed, applying a rotation rule increases $\operatorname{maj} - \operatorname{des}$ by 1 while fixing des, and applying a B-rule fixes $\operatorname{maj} - \operatorname{des}$ and increases des by 1. In this sense, the strong or weak posets $P(\lambda)$ and $Q(\lambda)$ have a $\mathbb{Z} \times \mathbb{Z}$ ranking given by $(\operatorname{maj} - \operatorname{des}, \operatorname{des})$.

6. Internal zeros for fake degrees of $C_m \wr S_n$

In this section, we classify which irreducible representations appear in which degrees of the corresponding coinvariant algebras for all finite groups of the form $C_m \wr S_n$. The goal is to classify when the fake degrees $b_{\underline{\lambda},k} \neq 0$. We will use the following helpful lemma which is straightforward to prove.

Lemma 6.1. *Suppose that f and g are polynomials in $\mathbb{Z}[q]$ with non-negative coefficients, that f has no internal zeros and has at least two non-zero coefficients, and that g has no adjacent internal zeros. Then, fg has no internal zeros.*

Lemma 6.2. *Let $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ be a sequence of partitions. The polynomial* $\mathrm{SYT}(\underline{\lambda})^{\operatorname{maj}}(q)$ *has no internal zeros except when $\underline{\lambda}$ has a single non-empty* *block $\lambda^{(i)}$ which is a rectangle with at least two rows and columns. In this latter case, the only internal zero up to symmetry occurs at $k = b(\lambda^{(i)}) + 1$.*

Proof. If $\underline{\lambda}$ has only one nonempty partition, then the characterization of internal zeros follows from Theorem 1.1, so assume $\underline{\lambda}$ has two or more nonempty partitions. From Theorem 2.15, we have

$$\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q) = \binom{n}{|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|}_q \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q). \tag{17}$$

By MacMahon's Theorem 2.5, we observe that the q-multinomial coefficients have no internal zeros. Furthermore, the q-multinomial in (17) is not constant whenever $\underline{\lambda}$ has two or more non-empty partitions. From Theorem 1.1, we know $\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q)$ has no *adjacent* internal zeros for any $1 \le i \le m$. Consequently, by Lemma 6.1, the overall product in (17) has no internal zeros. □

Theorem 6.3. *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n$, and assume $g^{\underline{\lambda}}(q) = \sum_k b_{\underline{\lambda},k} q^k$. Then for $k \in \mathbb{Z}$, $b_{\underline{\lambda},k} \neq 0$ if and only if*

$$\frac{k - b(\alpha(\underline{\lambda}))}{m} - b(\underline{\lambda}) \in \left\{0, 1, \ldots, \binom{n+1}{2} - \sum_{c \in \underline{\lambda}} h_c \right\} \smallsetminus \mathcal{D}_{\underline{\lambda}},$$

where $\mathcal{D}_{\underline{\lambda}}$ is empty unless $\underline{\lambda}$ has a single non-empty partition $\lambda^{(i)}$ which is a rectangle with at least two rows and columns, in which case

$$\mathcal{D}_{\underline{\lambda}} = \left\{1, \binom{n+1}{2} - \sum_{c \in \lambda^{(i)}} h_c - 1 \right\}.$$

Proof. By Theorem 2.31,

$$g^{\underline{\lambda}}(q) = q^{b(\alpha(\underline{\lambda}))}\, \mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q^m)$$

which implies $b_{\underline{\lambda},k} \neq 0$ only if $k - b(\alpha(\underline{\lambda}))$ is a multiple of m. By Lemma 6.2, we know $\mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q)$ has either no internal zeros or internal zeros at degree $1 + b(\underline{\lambda})$ and degree one less than the maximal major index for $\underline{\lambda}$ in the case of a rectangle with at least 2 rows and columns. By (8) and (9), the minimal major index for $\underline{\lambda}$ is $b(\underline{\lambda}) := \sum_i b(\lambda^{(i)})$, and the maximal major index is $b(\underline{\lambda}) + \binom{|\underline{\lambda}|+1}{2} - \sum_{c \in \underline{\lambda}} h_c$. Hence, the result follows. □

Corollary 6.4. *In type B_n, the irreducible representation indexed by (λ, μ) with $|\lambda| + |\mu| = n$ appears in degree k of the coinvariant algebra of $G(2,1,n)$ if and only if*

$$\frac{k - |\mu|}{2} - b(\lambda) - b(\mu) \in \left\{0, 1, \ldots, \binom{n+1}{2} - \sum_{c \in \lambda} h_c - \sum_{c' \in \mu} h_{c'} \right\} \smallsetminus \mathcal{D}_{(\lambda,\mu)}.$$

Example 6.5. Consider the type B_6 case, where $m = 2, d = 1, n = 6$. For $\underline{\lambda} = ((2),(31))$, we have $b(\underline{\lambda}) = 1$ and

$$g^{\underline{\lambda}}(q) = q^{26} + 2q^{24} + 4q^{22} + 5q^{20} + 7q^{18} + 7q^{16} + 7q^{14} + 5q^{12} + 4q^{10} + 2q^8 + q^6.$$

For $\underline{\mu} = (\varnothing,(33))$, we have $b(\underline{\mu}) = 3$ and

$$g^{\underline{\mu}}(q) = q^{24} + q^{20} + q^{18} + q^{16} + q^{12}.$$

In both cases, the nonzero coefficients are determined by Corollary 6.4.

7. Deformed Gaussian Multinomial Coefficients

We now turn our attention to extending Theorem 6.3 to general Shephard–Todd groups $G(m,d,n)$. We begin by introducing a deformation of the q-multinomial coefficients arising from Theorem 2.35 in the special case when $\underline{\lambda} = ((\alpha_1),(\alpha_2),\ldots,(\alpha_m))$ is a sequence of one row partitions. After several lemmas, we give an alternative formulation for these deformed q-multinomials in terms of inversion generating functions on words with a bounded first letter.

Definition 7.1. Let $\alpha = (\alpha_1,\ldots,\alpha_m) \vDash n$ be a weak composition of n with m parts. Recall the long cycle $\sigma_m = (1,2,\ldots,m) \in S_m$, so

$$\sigma_m \cdot \alpha = (\alpha_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}).$$

Let $d \mid m$, $\tau = \sigma_m^{m/d}$, and $C_d = \langle \tau \rangle = \langle \sigma_m^{m/d} \rangle$ so C_d acts on length m compositions by (m/d)-fold cyclic rotations as in Definition 2.37. Set

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d} \coloneqq \frac{\sum_{\sigma \in C_d} q^{b(\sigma\cdot\alpha)}}{[d]_{q^{nm/d}}} \binom{n}{\alpha}_{q^m} \tag{18}$$

where

$$b(\alpha) \coloneqq \sum_{i=1}^{m} (i-1)\alpha_i.$$

Note that when $q = 1$, we have $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{1;d} = \binom{n}{\alpha}$, and when $d = 1$, we have $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;1} = q^{b(\alpha)}\binom{n}{\alpha}_{q^m}$, where m is the number of parts of α. As usual, we also write $\begin{bmatrix} n \\ k \end{bmatrix}_{q;d} \coloneqq \begin{bmatrix} n \\ k,n-k \end{bmatrix}_{q;d} = \begin{bmatrix} n \\ n-k,k \end{bmatrix}_{q;d}$, where $m = 2$ in this case. Note that $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d}$ is invariant under the C_d-action on α, though this is not typically true of general permutations of α.

Example 7.2. Observe that $\binom{n}{\alpha}_{q^m}$ alone is generally not divisible by $[d]_{q^{nm/d}}$. For example, if $n = 5$, $\alpha = (2,1,1,1)$, and $d = 2$, we have

$$\binom{5}{2,1,1,1}_{q^4} = q^{36} + 3q^{32} + 6q^{28} + 9q^{24} + 11q^{20} + 11q^{16} + 9q^{12} + 6q^8 + 3q^4 + 1$$

which is not divisible by $[2]_{q^{5\cdot4/2}} = q^{10}+1$. However, $\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)} = q^8+q^6$ and $(q^8+q^6)\binom{5}{2,1,1,1}_{q^4}$ is divisible by $q^{10}+1$ giving

$$\begin{aligned}\begin{bmatrix}5\\2,1,1,1\end{bmatrix}_{q;2} &= q^{34}+q^{32}+3q^{30}+3q^{28}+6q^{26}+5q^{24}+8q^{22}\\ &+6q^{20}+8q^{18}+5q^{16}+6q^{14}+3q^{12}+3q^{10}+q^8+q^6.\end{aligned}$$

See Figure 10 for a larger example.

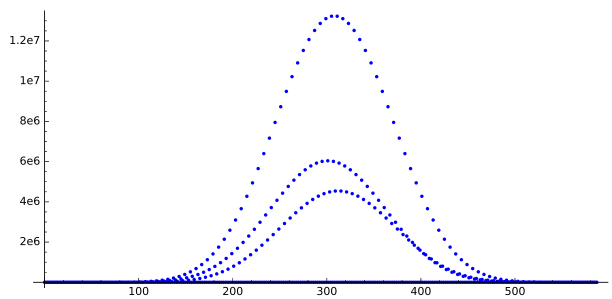


FIGURE 10. A plot of the coefficients for the deformed q-multinomial $\begin{bmatrix}n\\\alpha\end{bmatrix}_{q;d}$ with $\alpha = (2,1,3,1,4,5)$ and $d = 3$.

Lemma 7.3. *Given* $\alpha = (\alpha_1,\dots,\alpha_m) \vDash n$, *we have*

$$b(\sigma_m\cdot\alpha) - b(\alpha) = n - m\alpha_m,$$

and

$$b(\tau\cdot\alpha) - b(\alpha) = nm/d - m(\alpha_m+\alpha_{m-1}+\cdots+\alpha_{m-m/d+1}).$$

Proof. The second claim follows by iterating the first for $\tau = \sigma_m^{m/d}$. For the first, we have

$$\begin{aligned}b(\sigma_m\cdot\alpha)-b(\alpha) &= (\alpha_1+2\alpha_2+\cdots+(m-1)\alpha_{m-1})\\ &\quad-(\alpha_2+2\alpha_3+\cdots+(m-1)\alpha_m)\\ &= \alpha_1+\alpha_2+\cdots+\alpha_{m-1}-(m-1)\alpha_m,\end{aligned}$$

which simplifies to $n - m\alpha_m$. □

If $\alpha \vDash n$, let $\downarrow_i\alpha$ be the vector obtained from α by decreasing α_i by 1. Extend the definition of $\binom{n}{\alpha}_q$ to m-tuples of integers by declaring $\binom{n}{\alpha}_q = 0$ if any α_i is negative. The following lemma is well known but we include a proof for completeness.

Lemma 7.4. *We have the following recurrence for q-multinomial coefficients,*

$$\binom{n}{\alpha_1, \dots, \alpha_m}_q = \sum_{i=1}^{m} q^{\alpha_1+\cdots+\alpha_{i-1}} \binom{n-1}{\downarrow_i \alpha}_q .$$

Proof. By MacMahon's Theorem, the left-hand side is the inversion number generating function on length n words with α_i copies of the letter i for each i. If the first letter in such a word is i, the number of inversions involving the first letter is $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}$, from which the result quickly follows. □

The non-trivial deformation of the q-binomial coefficients in Definition 7.1 has the following more explicit form. In particular, these rational functions are polynomials with non-negative integer coefficients that satisfy a Pascal-type formula.

Lemma 7.5. *In the case $d = m = 2$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q;2} = \frac{q^k + q^{n-k}}{1+q^n} \binom{n}{k}_{q^2} = q^{n-k} \binom{n-1}{k-1}_{q^2} + q^k \binom{n-1}{k}_{q^2} \in \mathbb{Z}_{\geq 0}[q]. \tag{19}$$

Proof. The first equality is immediate from Definition 7.1. For the second, we use the well-known "q-Pascal" identities

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q ,$$

which arise from Lemma 7.4. Thus,

$$q^k \binom{n}{k}_{q^2} = q^k \binom{n-1}{k}_{q^2} + q^{n+n-k} \binom{n-1}{k-1}_{q^2}$$

and

$$q^{n-k} \binom{n}{k}_{q^2} = q^{n+k} \binom{n-1}{k}_{q^2} + q^{n-k} \binom{n-1}{k-1}_{q^2} .$$

Hence,

$$(q^k + q^{n-k}) \binom{n}{k}_{q^2} = (1+q^n) \left(q^k \binom{n-1}{k}_{q^2} + q^{n-k} \binom{n-1}{k-1}_{q^2} \right)$$

so the second equality in (19) holds. □

We next generalize Lemma 7.5 to all $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d}$ for any $\alpha \vDash n$. The proof that follows is independent of Theorem 2.35, which can also be used to prove they are polynomials with non-negative integer coefficients.

Theorem 7.6. *Let α be a weak composition of n with m parts, and let $d \mid m$. Then*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d} = \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} \sum_{v=1}^{m/d} q^{m \cdot ((\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{v-1})} \binom{n-1}{\downarrow_v (\sigma \cdot \alpha)}_{q^m} .$$

In particular, $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d}$ *is a polynomial with non-negative coefficients.*

Proof. Observe from the definition that $\binom{n}{\alpha}_q = \binom{n}{\sigma\cdot\alpha}_q$ for any $\sigma \in C_d$. Thus, by Lemma 7.4, we can rewrite the numerator of $\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d}$ as

$$\begin{aligned}\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\binom{n}{\alpha}_{q^m} &= \sum_{j=1}^{d} q^{b(\tau^j\cdot\alpha)}\binom{n}{\tau^j\cdot\alpha}_{q^m} \\ &= \sum_{j=1}^{d}\sum_{i=1}^{m} q^{\epsilon(i,j,\alpha)}\binom{n-1}{\downarrow_i(\tau^j\cdot\alpha)}_{q^m}\end{aligned}$$

where

$$\epsilon(i,j,\alpha) \coloneqq b(\tau^j\cdot\alpha) + m\cdot((\tau^j\cdot\alpha)_1 + \cdots + (\tau^j\cdot\alpha)_{i-1}). \tag{20}$$

It is straightforward to check that $\downarrow_i(\sigma_m\cdot\alpha) = \sigma_m\cdot\downarrow_{i-1}\alpha$, so that $\downarrow_i(\tau^j\cdot\alpha) = \tau^j\cdot\downarrow_{i-jm/d}\alpha$, where indices are taken modulo m. Thus,

$$\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\binom{n}{\alpha}_{q^m} = \sum_{i=1}^{m}\sum_{j=1}^{d} q^{\epsilon(i,j,\alpha)}\binom{n-1}{\downarrow_{i-jm/d}\alpha}_{q^m}.$$

Group the terms on the right according to the value $i - jm/d \equiv_m t \in [m]$. Note that $j \in [d]$ could be equivalently represented as $j \in \mathbb{Z}/d$, though $i \in [m]$ cannot be treated similarly here. One may check that the set of $(i,j) \in [m]\times\mathbb{Z}/d$ such that $i - jm/d \equiv_m t$ can be described as

$$\{(t + sm/d, s) : s \in [-p_t, d-1-p_t]\}$$

where $t = p_t m/d + v_t$ for some unique $p_t \in [0, d-1]$ and $v_t \in [m/d]$. Consequently,

$$\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\binom{n}{\alpha}_{q^m} = \sum_{t=1}^{m}\left(\sum_{s=-p_t}^{d-1-p_t} q^{\epsilon(t+sm/d,s,\alpha)}\right)\binom{n-1}{\downarrow_t\alpha}_{q^m}.$$

Next, we evaluate the incremental change

$$\epsilon(t + (s+1)m/d, s+1, \alpha) - \epsilon(t + sm/d, s, \alpha)$$

for a given s. Let $\beta = \tau^s\cdot\alpha$. By Lemma 7.3,

$$\begin{aligned}b(\tau^{s+1}\cdot\alpha) - b(\tau^s\cdot\alpha) &= b(\tau\cdot\beta) - b(\beta) \\ &= nm/d - m\cdot(\beta_m + \cdots + \beta_{m-m/d+1}).\end{aligned}$$

We also find

$$(\tau\cdot\beta)_1 + \cdots + (\tau\cdot\beta)_{t+(s+1)m/d-1} = \beta_{m-m/d+1} + \cdots + \beta_m + \beta_1 + \cdots + \beta_{t+sm/d-1}$$

so

$$\begin{aligned}&(\tau\cdot\beta)_1 + \cdots + (\tau\cdot\beta)_{t+(s+1)m/d-1} - \beta_1 - \cdots - \beta_{t+sm/d-1} \\ &\quad= \beta_{m-m/d+1} + \cdots + \beta_m.\end{aligned}$$

Combining these observations,

$$\epsilon(t+(s+1)m/d, s+1, \alpha) - \epsilon(t+sm/d, s, \alpha) = nm/d.$$

It follows that

$$\begin{aligned}\sum_{s=-p_t}^{d-1-p_t} q^{\epsilon(t+sm/d,s,\alpha)} &= q^{\epsilon(t-p_t m/d,-p_t,\alpha)}[d]_{q^{nm/d}} \\ &= q^{\epsilon(v_t,-p_t,\alpha)}[d]_{q^{nm/d}}.\end{aligned}$$

Since we have a bijection $[0,d-1]\times[m/d]\to[m]$ given by $(p,v)\mapsto pm/d+v$, we have

$$\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\binom{n}{\alpha}_{q^m} = [d]_{q^{nm/d}}\sum_{p=0}^{d-1}\sum_{v=1}^{m/d} q^{\epsilon(v,-p,\alpha)}\binom{n-1}{\downarrow_{v+pm/d}\alpha}_{q^m}, \tag{21}$$

proving the polynomiality of $\begin{bmatrix} n \\ \alpha\end{bmatrix}_{q;d}$.

We can further refine (21). From (20), we observe that

$$\epsilon(v,-p,\alpha) = \epsilon(v,0,\tau^{-p}\cdot\alpha),$$

and since $\tau = \sigma_m^{m/d}$, we have

$$\downarrow_{v+pm/d}\alpha = \tau^p\cdot\downarrow_v(\tau^{-p}\cdot\alpha).$$

So,

$$q^{\epsilon(v,-p,\alpha)}\binom{n-1}{\downarrow_{v+pm/d}\alpha}_{q^m} = q^{\epsilon(v,0,\tau^{-p}\cdot\alpha)}\binom{n-1}{\downarrow_v(\tau^{-p}\cdot\alpha)}_{q^m},$$

which implies

$$\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\binom{n}{\alpha}_{q^m} = [d]_{q^{nm/d}}\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)}\sum_{v=1}^{m/d} q^{m\cdot((\sigma\cdot\alpha)_1+\cdots+(\sigma\cdot\alpha)_{v-1})}\binom{n-1}{\downarrow_v(\sigma\cdot\alpha)}_{q^m}.$$

The result follows by dividing by $[d]_{q^{nm/d}}$. □

In light of Theorem 7.6, we define the following polynomials.

Definition 7.7. Let $\alpha = (\alpha_1,\ldots,\alpha_m)\vDash n$, and say $1\le k\le m$. Define the *α, k-partial sum multinomial* by

$$p_\alpha^{(k)}(q) = \sum_{i=1}^{k} q^{\alpha_1+\cdots+\alpha_{i-1}}\binom{n-1}{\downarrow_i\alpha}_q.$$

Remark 7.8. By Lemma 7.4, $p_\alpha^{(m)} = \binom{n}{\alpha}_q$, and more generally the same argument shows that

$$p_\alpha^{(k)}(q) = \{w\in \mathrm{W}_\alpha : w_1\le k\}^{\mathrm{inv}}(q) \tag{22}$$

is an inversion number generating function.

It is very well-known that the multinomial coefficients can be written as a product of binomial coefficients. More generally, q-multinomial coefficients can be written as a product of q-binomial coefficients. This holds true even for the α, k-partial sum multinomials as follows.

Lemma 7.9. *Let $\alpha = (\alpha_1, \dots, \alpha_m) \vDash n$ and $1 \le k \le m$. We have*

$$p_\alpha^{(k)}(q) = \prod_{i=1}^{k} \binom{\alpha_1 + \cdots + \alpha_i}{\alpha_i}_q \cdot \prod_{i=k+1}^{m} \binom{\alpha_1 + \cdots + \alpha_i - 1}{\alpha_i}_q. \tag{23}$$

Proof. Recall that $p_\alpha^{(k)}(q) = \{w \in \mathrm{W}_\alpha : w_1 \le k\}^{\mathrm{inv}}(q)$. Partition the set $\{w \in \mathrm{W}_\alpha : w_1 \le k\}$ into $\binom{n-1}{\alpha_1+\cdots+\alpha_k-1, \alpha_{k+1}, \dots, \alpha_m}$ subsets according to the placement of all $k+1, k+2, \dots, m$'s in positions $2, 3, \dots, n$. For each such placement, there are $\binom{\alpha_1+\cdots+\alpha_k}{\alpha_1, \dots, \alpha_k}$ ways to place numbers $1, 2, \dots, k$ in the remaining positions. Since each inversion in a word $w \in \mathrm{W}_\alpha$ is between two letters $\le k$, between two letters $\ge k+1$, or between a letter $\le k$ and a letter $\ge k+1$, it follows that

$$\{w \in \mathrm{W}_\alpha : w_1 \le k\}^{\mathrm{inv}}(q) = \binom{\alpha_1 + \cdots + \alpha_k}{\alpha_1, \dots, \alpha_k}_q \binom{n-1}{\alpha_1 + \cdots + \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_m}_q \tag{24}$$

by MacMahon's Theorem 2.5. Factoring each q-multinomial in (24) into q-binomials gives (23). □

Corollary 7.10. *Let $\alpha = (\alpha_1, \dots, \alpha_m) \vDash n$ and $1 \le k \le m$. The α, k-partial sum multinomial $p_\alpha^{(k)}(q)$ is symmetric and unimodal.*

Proof. A result of Andrews [And76, Thm. 3.9] states that the product of symmetric, unimodal polynomials with non-negative coefficients is symmetric and unimodal with non-negative coefficients. The q-binomials are symmetric with non-negative coefficients, and it is a well-known, non-trivial fact that they are also unimodal. See [Zei89] for a combinatorial proof of this fact and further historical references. The result now follows from Lemma 7.9. □

Lemma 7.11. *Let $\alpha = (\alpha_1, \dots, \alpha_m) \vDash n$ and $1 \le k \le m$. Then $p_\alpha^{(k)}(q) \ne 0$ if and only if $\alpha_1 + \cdots + \alpha_k > 0$. In this case, $p_\alpha^{(k)}(q)$ has constant coefficient 1, degree $D_\alpha - \alpha_{k+1} - \cdots - \alpha_m$ where $D_\alpha = \binom{n}{2} - \sum \binom{\alpha_i}{2}$ is the degree of $\binom{n}{\alpha}_q$, and has no internal zeros. Furthermore, $p_\alpha^{(k)}(q)$ is non-constant except when*

- *$\alpha_1 + \cdots + \alpha_k = 0$, in which case $p_\alpha^{(k)}(q) = 0$;*
- *$\alpha_1 + \cdots + \alpha_k = 1$ and $\alpha_i = n-1$ for some $i > k$, in which case $p_\alpha^{(k)}(q) = 1$; or*
- *$\alpha_i = n$ for some $i \le k$, in which case $p_\alpha^{(k)}(q) = 1$.*

Proof. Each claim follows easily from the fact that $p_\alpha^{(k)}(q)$ is the inversion generating function for $\{w \in \mathrm{W}_\alpha : w_1 \le k\}$. Alternatively, one may use Lemma 7.9. □

We have the following summary statement.

Corollary 7.12. *Let α be weak composition of n with m parts, and let $d \mid m$. Let $\underline{\alpha} = ((\alpha_1), (\alpha_2), \ldots, (\alpha_m))$ be the corresponding sequence of one row partitions. Then,*

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q;d} = \frac{\sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)}}{[d]_{q^{nm/d}}} \binom{n}{\alpha}_{q^m} = \sum_{\sigma \in C_d} q^{b(\sigma\cdot\alpha)} p_{\sigma\cdot\alpha}^{(m/d)}(q^m) = \frac{d}{\#\{\underline{\alpha}\}^d} \, g^{\{\underline{\alpha}\}^d}(q).$$

Proof. The first equality is just the definition. The second equality follows from Theorem 7.6. The third equality follows from Theorem 2.35 and the fact that

$$\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha)} = \frac{d}{\#\{\underline{\alpha}\}^d} \left(\{\underline{\alpha}\}^d\right)^{b\circ\alpha}(q).$$

□

Remark 7.13. We note that since $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]_{q;d} = \frac{d}{\#\{\underline{\alpha}\}^d} \, g^{\{\underline{\alpha}\}^d}(q)$, we knew from Stembridge's work that the deformed multinomial coefficients are polynomials in q even though they are defined as rational functions. Our proof in Theorem 7.6 gives an alternate, direct proof of this fact without going through representation theory. Furthermore, we use the summation formula in Corollary 7.12 to characterize the internal zeros of $g^{\{\underline{\lambda}\}^d}(q)$ in the next section.

8. Internal zeros for $G(m, d, n)$

We can now extend the results of Section 6 to all Shephard–Todd groups $G(m, d, n)$. We thus give a remarkably simple and completely general description for which irreducible representations appear in which degrees of the coinvariant algebras of essentially arbitrary complex reflection groups. Recall the notation established in Section 2.4. Let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under (m/d)-fold rotations in $C_d = \langle \sigma_m^{m/d} \rangle$.

Definition 8.1. Given a sequence $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ with $|\underline{\lambda}| = n$, let

$$\alpha(\underline{\lambda}) \coloneqq (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|) \vDash n.$$

Similarly, let $\underline{\alpha}(\underline{\lambda})$ be the length m sequence of partitions whose ith partition is the single row partition of size $|\lambda^{(i)}|$.

The map $\underline{\alpha}$ may not be injective on $\{\underline{\lambda}\}^d$, though it has constant fiber sizes since $\underline{\alpha}$ is C_d-equivariant. For example, when $m = 4, d = 4$, we have

$$\begin{aligned} \underline{\alpha}\colon ((2), \varnothing, (1^2), \varnothing) &\mapsto ((2), \varnothing, (2), \varnothing) \\ (\varnothing, (2), \varnothing, (1^2)) &\mapsto (\varnothing, (2), \varnothing, (2)) \\ ((1^2), \varnothing, (2), \varnothing) &\mapsto ((2), \varnothing, (2), \varnothing) \\ (\varnothing, (1^2), \varnothing, (2)) &\mapsto (\varnothing, (2), \varnothing, (2)). \end{aligned}$$

Generalizing Theorem 2.15, we have the following corollary of Stembridge's Theorem 2.35 and Definition 7.1.

Corollary 8.2. *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n$. Let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under (m/d)-fold cyclic rotations. Then*

$$g^{\{\underline{\lambda}\}^d}(q) = \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d} \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m).$$

Proof. By Theorem 2.35,

$$g^{\{\underline{\lambda}\}^d}(q) = \frac{\left(\{\underline{\lambda}\}^d\right)^{b\circ\alpha}(q)}{[d]_{q^{nm/d}}} \mathrm{SYT}(\underline{\lambda})^{\mathrm{maj}}(q^m).$$

We have

$$\left(\{\underline{\lambda}\}^d\right)^{b\circ\alpha}(q) = \frac{\#\{\underline{\lambda}\}^d}{\#\{\underline{\alpha}(\underline{\lambda})\}^d} \cdot \left(\{\underline{\alpha}(\underline{\lambda})\}^d\right)^{b\circ\alpha}(q)$$

and

$$\left(\{\underline{\alpha}(\underline{\lambda})\}^d\right)^{b\circ\alpha}(q) = \frac{\#\{\underline{\alpha}(\underline{\lambda})\}^d}{d} \sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha(\underline{\lambda}))}.$$

Consequently, using Theorem 2.15 and Definition 7.1, we have

$$\begin{aligned} g^{\{\underline{\lambda}\}^d}(q) &= \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \frac{\sum_{\sigma\in C_d} q^{b(\sigma\cdot\alpha(\underline{\lambda}))}}{[d]_{q^{nm/d}}} \cdot \binom{n}{\alpha(\underline{\lambda})}_{q^m} \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m) \\ &= \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \begin{bmatrix} n \\ \alpha(\underline{\lambda}) \end{bmatrix}_{q;d} \cdot \prod_{i=1}^{m} \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m). \end{aligned}$$

□

We will now prove the general classification theorem for nonzero fake degrees as mentioned in the introduction. The reader may find it useful to compare the statement to the type A case in Theorem 1.1 and the $C_m \wr S_n$ case in Theorem 6.3.

Theorem 8.3. *Let $\underline{\lambda}$ be a sequence of m partitions with $|\underline{\lambda}| = n \geq 1$, let $d \mid m$, and let $\{\underline{\lambda}\}^d$ be the orbit of $\underline{\lambda}$ under the group C_d of (m/d)-fold cyclic rotations. Then $b_{\{\underline{\lambda}\}^d,k} \neq 0$ if and only if for some $\underline{\mu} \in \{\underline{\lambda}\}$ we have $|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| > 0$ and*

$$\frac{k - b(\alpha(\underline{\mu}))}{m} - b(\underline{\mu}) \in \left\{0, 1, \ldots, |\mu^{(1)}| + \cdots + |\mu^{(m/d)}| + \binom{n}{2} - \sum_{c\in\underline{\mu}} h_c\right\} \smallsetminus \mathcal{D}_{\underline{\mu};d}.$$

Here $\mathcal{D}_{\underline{\mu};d}$ is empty unless either

(1) $\underline{\mu}$ has a partition μ of size n; or

(2) $\underline{\mu}$ has a partition μ of size $n-1$ and $|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| = 1$,

where in both cases μ must be a rectangle with at least two rows and columns. In case (1), we have

$$\mathcal{D}_{\underline{\mu};d} \coloneqq \left\{1, \binom{n+1}{2} - \sum_{c \in \mu} h_c - 1\right\},$$

and in case (2) we have

$$\mathcal{D}_{\underline{\mu};d} \coloneqq \left\{1, \binom{n}{2} - \sum_{c \in \mu} h_c\right\}.$$

Proof. Let $\alpha = \alpha(\underline{\lambda})$. Using Corollary 8.2 and Corollary 7.12, we have

$$g^{\{\underline{\lambda}\}^d}(q) = \frac{\#\{\underline{\lambda}\}^d}{d} \cdot \sum_{\sigma \in C_d} q^{b(\sigma\cdot\alpha)} p_{\sigma\cdot\alpha}^{(m/d)}(q^m) \cdot \prod_{i=1}^m \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m). \tag{25}$$

Thus, we consider the locations of the nonzero terms in

$$p_{\sigma\cdot\alpha}^{(m/d)}(q) \prod_{i=1}^m \mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q). \tag{26}$$

Recall that $p_{\sigma\cdot\alpha}^{(m/d)}(q) = 0$ whenever $(\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} = 0$, so assume $(\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} > 0$. Since $\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q) \neq 0$ for all partitions $\lambda^{(i)}$, we can also assume (26) is not zero.

By Lemma 7.11, $p_{\sigma\cdot\alpha}^{(m/d)}(q)$ has no internal zeros, degree $\binom{n}{2} - \sum\binom{\alpha_i}{2} - (\sigma\cdot\alpha)_{m/d+1} - \cdots - (\sigma\cdot\alpha)_m$, and constant term $p_{\sigma\cdot\alpha}^{(m/d)}(0) = 1$. Thus the minimal degree term of (26) with nonzero coefficient is $q^{b(\underline{\lambda})}$ by (8), and the maximal degree term is q to the power

$$\binom{n}{2} - \sum\binom{\alpha_i}{2} - (\sigma\cdot\alpha)_{m/d+1} - \cdots - (\sigma\cdot\alpha)_m + \sum \deg(\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q)). \tag{27}$$

Since $\alpha_i = |\lambda^{(i)}|$, we know by (9) that $\deg(\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q)) = \binom{\alpha_i}{2} - b(\lambda^{(i)\prime})$, so (27) simplifies to

$$\binom{n}{2} - (\sigma\cdot\alpha)_{m/d+1} - \cdots - (\sigma\cdot\alpha)_m - b(\underline{\lambda}') \tag{28}$$

where $b(\underline{\lambda}') \coloneqq \sum_i b(\lambda^{(i)\prime})$. From (9), we also know $\binom{n}{2} - b(\underline{\lambda}') = b(\underline{\lambda}) + \binom{n+1}{2} - \sum_{c\in\underline{\lambda}} h_c$ and $\sigma\cdot\alpha \vDash n$, so we conclude that the maximal degree of (26) is

$$b(\underline{\lambda}) + (\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} + \binom{n}{2} - \sum_{c\in\underline{\lambda}} h_c. \tag{29}$$

If $p_{\sigma\cdot\alpha}^{(m/d)}(q) \neq 1$, then the product in (26) also has no internal zeros by Theorem 1.1 and Lemma 6.1. The cases where $p_{\sigma\cdot\alpha}^{(m/d)}(q) = 1$ are listed in Lemma 7.11. By assumption, $(\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} > 0$ so the remaining cases are when $(\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} = 1$ and some $\alpha_i = n-1$ for $i > m/d$, or $\alpha_i = n$ for some $i \le m/d$. In either of the remaining cases, determining the nonzero coefficients of (26) reduces to the case of a single partition described in Theorem 1.1.

To finish the proof, we observe from (25) that $b_{\{\underline{\lambda}\}^d,k} \neq 0$ if and only if there exists some $\sigma \in C_d$ such that $(\sigma\cdot\alpha)_1 + \cdots + (\sigma\cdot\alpha)_{m/d} > 0$ and the corresponding product

$$q^{b(\sigma\cdot\alpha)} p^{(m/d)}_{\sigma\cdot\alpha}(q^m)\cdot\prod_{i=1}^{m}\mathrm{SYT}(\lambda^{(i)})^{\mathrm{maj}}(q^m)$$

has nonzero coefficient of q^k. Thus, by our analysis of the location of nonzero coefficients in (26), we observe that $b_{\{\underline{\lambda}\}^d,k} \neq 0$ if and only if

$$\frac{k-b(\sigma\cdot\alpha)}{m} - b(\underline{\lambda}) \in \left\{0,1,\ldots,(\sigma\cdot\alpha)_1+\cdots+(\sigma\cdot\alpha)_{m/d}+\binom{n}{2}-\sum_{c\in\underline{\lambda}}h_c\right\}\setminus\mathcal{D}_{\underline{\mu};d}$$

where $\underline{\mu} = \sigma\cdot\underline{\lambda}$. Observing that $b(\underline{\lambda}) = b(\underline{\mu})$, $\sum_{c\in\underline{\lambda}} h_c = \sum_{c\in\underline{\mu}} h_c$, and $b(\sigma\cdot\alpha) = b(\alpha(\underline{\mu}))$ completes the proof of the theorem. □

Corollary 8.4. *In type D_n, an irreducible with orbit $\{\lambda,\mu\}$ where $|\lambda|+|\mu| = n$ appears in degree k of the coinvariant algebra of $G(2,2,n)$ if and only if either $\lambda \neq \varnothing$ and*

$$\frac{k-|\mu|}{2} - b(\lambda) - b(\mu) \in \left\{0,1,\ldots,|\lambda|+\binom{n}{2}-\sum_{c\in\lambda}h_c-\sum_{c'\in\mu}h_{c'}\right\}\setminus\mathcal{D}_{(\lambda,\mu);2}$$

or $\mu \neq \varnothing$ and

$$\frac{k-|\lambda|}{2} - b(\lambda) - b(\mu) \in \left\{0,1,\ldots,|\mu|+\binom{n}{2}-\sum_{c\in\lambda}h_c-\sum_{c'\in\mu}h_{c'}\right\}\setminus\mathcal{D}_{(\mu,\lambda);2}.$$

Example 8.5. Consider the type D_6 case, where $m = 2, d = 2, n = 6$. For $\underline{\lambda} = ((2),(31))$, we have $b(\underline{\lambda}) = 1$ and

$$g^{\{\underline{\lambda}\}^2}(q) = q^{20}+3q^{18}+6q^{16}+8q^{14}+9q^{12}+8q^{10}+6q^{8}+3q^{6}+q^{4}.$$

On the other hand, if $\underline{\mu} = (\varnothing,(33))$ then $b(\underline{\mu}) = 3$ and

$$g^{\{\underline{\mu}\}^2}(q) = q^{18}+q^{14}+q^{12}+q^{10}+q^{6}.$$

In both cases, the nonzero coefficients are determined by Corollary 8.4. One may notice that $g^{\{\underline{\mu}\}^2}(q) \neq g^{\{\underline{\mu}\}^1}(q)$, which appeared in Example 6.5. However, for $\underline{\nu} = ((33),\varnothing)$, one can check $\{\underline{\mu}\}^2 = \{\underline{\nu}\}^2$ and $g^{\{\underline{\mu}\}^2}(q) = g^{\{\underline{\nu}\}^2}(q) = g^{\{\underline{\nu}\}^1}(q)$.

9. Future Work

A sequence $a_0, a_1, a_2, \ldots$ is *parity-unimodal* if $a_0, a_2, a_4, \ldots$ and $a_1, a_3, a_5, \ldots$ are each unimodal. Stucky [Stu18, Thm. 1.3] recently showed that the q-Catalan polynomials, namely $\mathrm{SYT}((n,n))^{maj}(q)$ up to a q-shift, are parity-unimodal. The argument involves constructing an $\mathfrak{sl}_2$-action on rational Cherednik algebras. See [§3.1, Haiman94] for a prototype of the argument in a highly related context. Recent work of Gaetz–Gao [GG20] constructed an $\mathfrak{sl}_2$-module on $\mathbb{C}S_n$ and strongly related work of Hamaker–Pechenik–Speyer–Weigandt [HPSW20] constructed an $\mathfrak{sl}_2$-module on R_n, though neither of these structures are capable of producing internal zeros and they do not respect the isotypic decomposition. Nonetheless, based on Stucky's result, our internal zeros classification, and a brute-force check for $n \leq 50$, we conjecture the following.

Conjecture 9.1. *The fake-degree polynomials $f^\lambda(q)$ are parity-unimodal for all λ.*

When W is a Weyl group, the Hilbert series of the coinvariant algebra R_W is symmetric and unimodal by the Hard Lefschetz Theorem since R_W presents the cohomology of the associated flag variety G/B. One referee asked the following interesting question.

> Is there an algebraic or geometric witness to the fact that the $f^\lambda(q)$ so rarely have internal zeros in their coefficient sequences? More precisely, let $\ell = c_1x_1 + \cdots + c_nx_n$ be a linear form in $\mathbb{C}[x_1, \ldots, x_n]$ with $c_i \neq c_j$ whenever $i \neq j$. These are precisely the Lefschetz elements when $W = S_n$. If $V_\lambda = \oplus_{d=0}^{n(n-1)/2}(V_\lambda)_d$ is the λ-isotypical component of R_n, decomposed by polynomial degree, for each d we have a linear map
>
> $$(V_\lambda)_d \overset{\times \ell}{\to} (R_n)_{d+1} \overset{\epsilon_\lambda}{\to} (V_\lambda)_{d+1},$$
>
> where we first multiply by ℓ and then we act by the Young symmetrizer $\epsilon_\lambda \in \mathbb{C}[S_n]$. Initial computer verifications suggest this composite linear map is nonzero whenever $f^\lambda(q)$ does not have an internal zero, at least for some special choices of the coefficients c_i. Could this correspond to some property in geometry?

Remark 9.2. We note that Stanley used the Hard Lefschetz Theorem to prove that Bruhat orders are rank symmetric, rank unimodal, and have a symmetric chain decomposition, so they are Sperner [Sta84]. This theorem was part of our motivation for defining the Weak SYT Poset. We were looking for a subposet of the Strong SYT Poset which is a symmetric chain decomposition, though we did not find one. The Weak SYT Poset is not a disjoint union of chains in general, though it is the most natural subposet we could find. The names "Weak" SYT Poset and "Strong" SYT Poset simply imply that one is a subposet of the other in the same way that the weak order is a subposet of "strong" Bruhat order.

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