On a Representation of Commuting Maps by Tensor Products

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Abstract

It is shown that a representation by tensor products of degree \( n^2 \) exists for every pair of commuting linear maps on an \( n \)-dimensional vector space, but in general, not a representation of degree less than \( n^2 \).

1 Introduction

In his paper [4] Chandler Davis introduced a representation of a pair of commuting linear operators by tensor products. When the underlying Hilbert space has finite dimension, say \( n \), then the tensor product operators constructed in [4] act on a vector space of dimension \( n^3 \). Davis asked whether a representation by tensor products in dimension less than \( n^3 \) is possible. In general we cannot expect that this dimension, called the degree of representation, is less than \( n^2 \) (see [7, Example 1.28] and our Example below). Here we show that a pair of commuting linear maps on an \( n \)-dimensional vector space can be represented by tensor products on a vector space of dimension \( n^2 \). Thus it follows that, in general, \( n^2 \) is the minimal possible degree for such a representation which answers Davis’ question. Our construction is similar to the one given originally in [4]. However coalgebraic techniques enable us to reduce the degree of the representation. We remark that this construction has a direct generalizations for a \( k \)-tuple of commuting linear maps on \( V \). The representation we obtain then has degree less than or equal to \( n^k \), also best possible in general.

We introduce the notion of a representation by tensor products and discuss the above mentioned Example in §2. In §3 we introduce coalgebras and comodules and in §4 we present some of the properties of the coalgebra of representative functionals on polynomial algebras. Comodules associated to a pair of commuting linear maps are studied in §5. The main result is proved in §6. We only outline the properties of coalgebras and comodules needed. For details we refer to [1, 6, 8, 9].
Representations by tensor products for commuting operators on Hilbert spaces were also studied by Fong and Sourour [5, Theorem 3.2], while De Boor and Rice [2] considered special pairs of commuting matrices that have a representation by tensor products on the original vector space.

2 Preliminaries

We assume throughout that $V$ is a vector space of dimension $n$ over a field $F$. We consider a pair $A_1, A_2$ of commuting linear maps on $V$. Suppose that there exist vector spaces $W_1$ and $W_2$, an injective linear map $T : V \rightarrow W$, where $W = W_1 \otimes W_2$, and linear maps $B_i : W_i \rightarrow W_i$ ($i = 1, 2$) such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{A_1} & V \\
\downarrow T & & \downarrow T \\
W_1 \otimes W_2 & \xrightarrow{B_i^1} & W_1 \otimes W_2 \\
\end{array}
$$

(1)

commutes. Here

$$B_1^1 = B_1 \otimes I \text{ and } B_2^1 = I \otimes B_2.$$ (2)

Such a construction is called a representation of the pair $A_1, A_2$ by tensor products and the dimension of the vector space $W$ is the degree of the representation.

If we already have $A_i = B_i^1$ ($i = 1, 2$) for some $B_i^1$ as in (2) then the pair $A_1, A_2$ has a representation by tensor products already on the original space and hence the smallest degree of a representation by tensor products for $A_1, A_2$ is $n$. In general this is not the case. We consider next an example of a pair of commuting matrices for which the smallest degree of a representation by tensor products is $n^2$. It is a special case of [7, Example 1.28].

Example Suppose that $V = F^n$ and that

$$A_1 = A_2 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.$$

Now choose the vector space $W = F^n \otimes F^n$ and the subspace $\mathcal{M} \subset W$ spanned by the set $\mathcal{B} = \{x_j = \sum_{i=0}^{j-1} v_{i+1} \otimes v_{j-i}\}_{j=1}^n$ where $v_i$ ($i = 1, 2, \ldots, n$), denote the standard basis vectors in $F^n$, i.e., $v_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ where 1 is in the $i$-th position. For $j = 1, 2, \ldots, n$ we have $B_1^1 x_j = x_{j-1}$, where $x_0 = 0$, $B_1^1 = A_1 \otimes I$ and $B_2^1 = I \otimes A_2$. Next define a linear map $T : F^n \rightarrow W$ by $T(v_j) = x_j$. Then $T A_i = B_i^1 T$ is a representation by tensor products for $A_i$. It is of the smallest degree possible because for the tensor product space $F^p \otimes F^q$ where $p < n$ or $q < n$ there do not exist two $n \times n$ matrices $C_1$ and $C_2$ such that both $C_1 \otimes I$ and $I \otimes C_2$ have a Jordan chain of length $n$. (This is a consequence of the Aitken-Roth theorem, see e.g. [3, Theorem 4.6].) □
3 Coalgebras and Comodules

A coalgebra $C$ is a vector space with a structure dual to that of an algebra, i.e., with a counit $\varepsilon : C \to F$ and a comultiplication $\delta : C \to C \otimes C$, which are linear maps such that the diagrams

\[
\begin{align*}
F \otimes C & \cong \varepsilon I_C & C & \cong C \otimes F \\
\delta & \downarrow \alpha & \alpha \downarrow \\
C \otimes C & \to C \otimes C \\
\delta & \downarrow \delta_1 \otimes \delta_2 & \alpha \downarrow \\
C \otimes C & \to C \otimes C \otimes C
\end{align*}
\]

commute. We call the pair of maps $\varepsilon$ and $\delta$ also the structure maps of $C$. The first of the above diagrams is the counit law and the second is coassociativity. Here we use the symbol $I_C$ to denote the identity map of $C$. If $C_1$ and $C_2$ are two coalgebras with structure maps $\varepsilon_1, \delta_1$ and $\varepsilon_2, \delta_2$, respectively, then $C_1 \otimes C_2$ is a coalgebra with structure maps $\varepsilon_1 \otimes \varepsilon_2$ and $\delta_1 \otimes \delta_2 + \sigma_{12} (\delta_1 \otimes \delta_2)$, where $\sigma_{ij}$ switches the $i$-th and the $j$-th tensor factor. All the coalgebras considered in this paper are cocommutative, i.e., $\delta = \sigma_{12} \delta$.

A notion dual to the notion of a module over an algebra is the notion of a comodule over a coalgebra. Suppose that $C$ is a coalgebra. Then a vector space $N$ is a $C$-comodule if there is a linear map $\alpha : N \to N \otimes C$, called a coaction of $C$ on $N$, such that

\[
\begin{align*}
N & \cong N \otimes F \\
\alpha & \downarrow I_N \otimes \varepsilon \\
N \otimes C & \to N \otimes C
\end{align*}
\]

commute. If $V$ is a vector space and $C$ a coalgebra then the comultiplication $\delta$ of $C$ induces a coaction $\alpha = I_V \otimes \delta$ on $V \otimes C$. Such a comodule $V \otimes C$ is called cofree. If $M$ and $N$ are $C$-comodules then a linear map $\varphi : M \to N$ is a comodule homomorphism if the diagram

\[
\begin{align*}
M & \alpha_M \downarrow \varphi \\
M \otimes C & \varphi I_C \downarrow \\
N \otimes C & \alpha_N \downarrow
\end{align*}
\]

commutes. For further details on the theory of coalgebras, comodules and Hopf algebras we refer to the books of Abe [1] and Sweedler [9].

4 The Coalgebra of Functionals on Polynomials

A linear functional $f : F[x] \to F$ is called a representative functional if its kernel contains an ideal of finite codimension. The vector space $F[x]^0$ of all representative functionals on the polynomial ring $F[x]$ has a (topological) basis $\{e_m\}_{m=0}^\infty$, where $e_m (x^n) = \delta_{mn}$ and $\delta_{mn}$ is the Kronecker symbol. An element $f \in F[x]^0$ has an infinite series representation $f = \sum_{m=0}^\infty a_m e_m$, where $\{a_m\}_{m=0}^\infty$ forms a linearly recursive sequence. The canonical coalgebra structure on $F[x]^0$ is defined by

\[
\begin{align*}
\varepsilon (e_m) & = \delta_{0m} \\
\delta (e_m) & = \sum_{r+s=m} e_r \otimes e_s.
\end{align*}
\]
The structure maps (4) are extended on the whole of \( F[x]^0 \) by (infinite) linearity, for example \( \delta \left( \sum_{m=0}^{\infty} a_m e_m \right) = \sum_{m=0}^{\infty} a_m \delta(e_m) \).

The linear map \( D : F[x]^0 \to F[x]^0 \), defined by

\[
D f (p) = f (xp)
\]

for \( p \in F[x] \) is dual to the map \( M_x : F[x] \to F[x] \) given by \( M_x p = xp \).

Next suppose next that \( p(x) = x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \ldots - a_0 \) is an arbitrary monic polynomial and that \((p)\) is the ideal generated by \( p \) in \( F[x] \). The dual space

\[
C_p = \left( F[x] / (p) \right)^* \tag{6}
\]

is a subcoalgebra of \( F[x]^0 \), called the subcoalgebra associated with \( p \). If \( f_r \in C_p \) is defined by \( f_r (x^i) = \delta_{rs} (r, s = 0, 1, \ldots, d - 1) \), where \( d = \deg p \), then

\[
\{f_r\}_{r=0}^{d-1}
\]

is a basis for \( C_p \). Furthermore we have

\[
D f_0 = a_0 f_{d-1} \quad \text{and} \quad D f_r = f_{r-1} + a_r f_{d-1} \quad (r = 1, 2, \ldots, d - 1) . \tag{8}
\]

The interested reader will find more information about the structure of \( F[x]^0 \) in [6, 8].

## 5 Comodules Associated with Linear Maps

Now we turn our attention to a pair of commuting linear maps \( A_1, A_2 \). Suppose that \( p_i \) is the minimal polynomial of \( A_i \) \((i = 1, 2)\). Let \( C_i \) be the coalgebra associated with the polynomial \( p_i \), of degree \( d_i \), as in (6) and \( C = C_1 \otimes C_2 \). Then we have

\[
\dim C_i = d_i \leq n . \tag{9}
\]

The restriction of the map (5) to the coalgebra \( C_i \) is denoted by \( D_i \). We write \( D_1^i = D_1 \otimes I_{C_2} \) and \( D_2^i = I_{C_1} \otimes D_2 \) for the induced maps on \( C \). But for the map induced by \( D_i \) (resp. \( D_i^i \)) on the cofree comodules \( V \otimes C_i \) and \( V \otimes C \) we use the same symbol \( D_i \) (resp. \( D_i^i \)). Similarly the map induced by \( A_i \) on \( V \otimes C_i \) and \( V \otimes C \) is still denoted by \( A_i \).

Because \((A_i - D_i) : V \otimes C_i \to V \otimes C_i \) and \((A_i - D_i^i) : V \otimes C \to V \otimes C \) are comodule homomorphisms it follows that their kernels \( r_i \) and \( R_i \) are subcomodules of \( V \otimes C_i \) and \( V \otimes C \), respectively. Moreover,

\[
R_1 = r_1 \otimes C_2 \quad \text{and} \quad R_2 = C_1 \otimes r_2 . \tag{10}
\]

If \( v \) is in \( R_{12} = R_1 \cap R_2 \) then \( A_i v = D_i^i v \quad (i = 1, 2) \), and so the diagram

\[
\begin{array}{ccc}
R_{12} & \xrightarrow{A_1} & R_1 \\
\downarrow I_{V \otimes C} & & \downarrow I_{V \otimes C} \\
V & \xrightarrow{A_1} & V
\end{array}
\]

\[
\tag{11}
\]
commutes.

**Lemma** Let \( \mathcal{R} \) be either \( r_1, r_2 \) or \( \mathcal{R}_{12} \), and let \( \varepsilon : \mathcal{R} \to V \) be the restriction of \( I_V \otimes \varepsilon \) to \( \mathcal{R} \) in each case. Then \( \varepsilon \) is invertible. Moreover, \( \dim \mathcal{R} = n \) and \( \dim \mathcal{R}_i = nd_i \) for \( i = 1, 2 \).

**Proof.** The second part of the lemma follows from the first part and (*10*). We prove the first part of the lemma for \( \mathcal{R} = \mathcal{R}_{12} \). The proof for \( r_1 \) and \( r_2 \) is similar.

Suppose that \( \varepsilon v = 0 \). Because (*11*) commutes it follows that \( 0 = A^r_1A^s_2\varepsilon v = \varepsilon D^r_1D^s_2v \) (\( r = 0, 1, \ldots, d_1 - 1 \) and \( s = 0, 1, \ldots, d_2 - 1 \)). By the induction on \( r \) and \( s \) one shows that \( v = 0 \), and so \( \varepsilon \) is one-to-one.

Now choose \( v \in V \) and consider \( v = \sum_{r=0}^{d_1-1} \sum_{s=0}^{d_2-1} A^r_1A^s_2v \otimes f^1_r \otimes f^2_s \in V \otimes C \), where \( \{ f^1_r \}_{r=0}^{d_1-1} \) is the basis for \( C_i \) given by (*7*). We write \( p_i(x) = x^{d_i} - a_{i,d_i-1}x^{d_i-1} - \ldots - a_{i,0} \), and then using (*8*), we see that

\[
D^i_1v = \sum_{r=1}^{d_1-1} \sum_{s=0}^{d_2-1} A^r_1A^s_2v \otimes f^1_r \otimes f^2_s + \sum_{r=0}^{d_1-1} \sum_{s=1}^{d_2-1} A^r_1A^s_2v \otimes a_{ir}f^1_{d_1-1} \otimes f^2_s. \tag{12}
\]

Since \( p_i(A_i) = 0 \), the right-hand side of (*12*) is equal to

\[
\sum_{r=0}^{d_1-2} \sum_{s=0}^{d_2-1} A^{r+1}_1A^s_2v \otimes f^1_r \otimes f^2_s + \sum_{s=0}^{d_2-1} A^d_1A^s_2v \otimes f^1_{d_1-1} \otimes f^2_s = A_1v.
\]

Hence \( v \in \mathcal{R}_1 \). Similarly we show that \( v \in \mathcal{R}_2 \). Because \( \varepsilon v = v \) it follows that \( \varepsilon \) is also surjective.

\[\square\]

## 6 Representation by Tensor Products

The following is our main theorem.

**Theorem** Every pair of commuting linear maps \( A_1 \) and \( A_2 \) on \( V \) has a representation by tensor products of degree less than or equal to \( n^2 \). In particular, there exists such a representation of degree \( n \cdot \min \{d_1, d_2\} \).

**Proof.** By the lemma \( \varepsilon : \mathcal{R}_{12} \to V \) is invertible. Define \( T : V \to r_1 \otimes C_2 \) by \( T = \iota \circ \varepsilon^{-1} \), where \( \iota : \mathcal{R}_{12} \hookrightarrow r_1 \otimes C_2 \) is the inclusion. The diagram

\[
\begin{array}{ccc}
V & \xrightarrow{A_1} & V \\
\downarrow T & & \downarrow T \\
r_1 \otimes C_2 & \xrightarrow{D^i_1} & r_1 \otimes C_2
\end{array}
\tag{13}
\]

commutes for \( i = 1, 2 \) because diagram (*11*) commutes. Then (*13*) is a representation by tensor products for \( A_1, A_2 \) as defined in (*1*). The analog of diagram (*13*), where \( r_1 \otimes C_2 \) is replaced by \( C_1 \otimes r_2 \) also commutes. By our Lemma it follows then that there exists a representation of the required degree.

\[\square\]

We remark that \( T \) is as in [4]. However our approach reveals further structure of Davis' construction and this enables us to find a representation of best possible degree.
References


