$G^1$ interpolation by rational cubic PH curves in $\mathbb{R}^3$

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Abstract

In this paper the $G^1$ interpolation of two data points and two tangent directions with spatial cubic rational PH curves is considered. It is shown that interpolants exist for any true spatial data configuration. The equations that determine the interpolants are derived by combining a closed form representation of a ten parameter family of rational PH cubics given in [1], and the Gram matrix approach. The existence of a solution is proven by using a homotopy analysis, and numerical method to compute solutions is proposed. In contrast to polynomial PH cubics for which the range of $G^1$ data admitting the existence of interpolants is limited, a switch to rationals provides a practically useful interpolation scheme with no restrictions.

Keywords: Pythagorean-hodograph, cubic rational curves, $G^1$ interpolation, homotopy analysis

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1. Introduction

Pythagorean-hodograph (or shortly PH) curves form a special subclass of parametric curves. They are characterized by the property that their unit vector field of tangents is rational. Polynomial PH curves thus have a (piecewise) polynomial arc length, planar PH curves possess rational offset curves and spatial PH curves are equipped with rational orthonormal frames. This makes PH curves very useful for many practical applications in CAGD, CAD/CAM systems, CNC

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machining, robotics, animations, etc. Polynomial PH curves were first introduced in [2] and since then many approximation and interpolation schemes that involve these curves have been developed (see [3] and the references therein). On the other hand, not much work has been done on rational PH curves, since the extension from the polynomial ones is not straightforward. Planar rational PH curves were derived in [4], [5], and independently in [6]. Some interpolation schemes involving these curves can be found in [7].

The first step to spatial rational PH curves is carried out as a short note in [5] based on the dual representation of rational curves. More extensive study can be found in [8], where the construction of spatial rational PH curves is presented and illustrated from a geometric point of view. The spatial rational PH curve is determined by prescribing a rational unit vector field of binormals obtained from a rational unit field of tangents, and a rational function that prescribes the signed distance of the osculating plane from the origin. Rational unit tangents can be constructed using stereographic projection or Euler-Rodrigues frame (ERF) associated to a quaternion polynomial. In [1] it is shown, based on the dual approach and ERF, that such a construction leads in general to curves of relatively high degrees. More precisely, the connection between degrees of quaternion polynomials, dual representation and rational PH curves is revealed, and the question how to obtain low degree curves is considered in detail. Based on quadratic quaternion polynomials rational PH curves with the dual representation of degree $m = 3, 4, 5, 6$, having $2m + 4$ degrees of freedom are constructed. In particular, cubic rational PH curves with nonconstant denominator depending on ten free parameters are presented in a closed form. Since a family of cubic polynomial PH curves is also ten parametric, rationals do not provide additional degrees of freedom, but, as shown in this paper, cubic rational PH curves are superior to the polynomial ones when used for $G^1$ interpolation. However, a switch from polynomial to rational curves has also a drawback. Rational PH curves do not have rational arc length in general.

The $G^1$ interpolation of two data points and two tangent directions at these two points by spatial cubic PH curves is important for practical applications and has therefore been considered for polynomial curves in quite a few papers (e.g. [9], [10], [11], [12], [13]). Since the problem is nonlinear, the analysis of the existence and the number of interpolants according to different $G^1$ data is quite a difficult task. The conditions that provide the existence of cubic polynomial PH interpolants for many but not all possible data configurations can be found in [9], [10], [11]. A complete characterization of the existence conditions for all possible $G^1$ data has later been established in [12], and further in [13], where nice
geometrically intuitive necessary and sufficient conditions on the data have been derived.

Unfortunately, the range of data configurations for which \( G^1 \) cubic polynomial PH interpolants do not exist is quite large (see Fig. 1), which is a disadvantage if we want to make a scheme practically useful. In this paper we show that \( G^1 \) cubic rational PH interpolants exist for all possible true spatial \( G^1 \) data. They are determined by nonlinear equations derived by combining a closed form curve representation given in [1], and the Gram matrix approach (used also in [13]). The existence of the solution is proven by using a homotopy analysis. It is shown that the number of rational interpolants with nonconstant denominator is always odd, and that rational interpolants are separated from the polynomial ones for all possible spatial data. But, as discussed further, if the data are planar, cubic rational PH curves provide the interpolant only in very particular setup. Numerical procedures to compute the solution are proposed too, together with some examples that confirm the theoretical results.

The paper is organized as follows. In the next section the interpolation problem and the main result are presented. Section 3 recalls the quaternion approach from [1] and derives some basic properties of interpolants. In Section 4 the Gram matrix approach is presented and in Section 5 the basic PH equations are derived. Homotopy analysis is used in the next section to confirm the existence of a solution for true spatial data. Technically demanding, but tedious proofs needed in the section 6 are left to [14]. In Section 7 the algebraic approach and the planar case are briefly considered. Section 8 discusses the numerical solution of the PH equations. An algorithm is proposed in which one has to find real roots of a couple of single variable degree 6 polynomials only. Numerical examples of the last section conclude the main part of the paper.

2. Interpolation problem

The \( G^1 \) interpolation problem is as follows. Suppose that the directions

\[
d_0, d_i, \quad \|d_i\| = 1, \quad d_i \in \mathbb{R}^3,
\]

and the points \( P_0, P_1 \in \mathbb{R}^3 \) are prescribed. The data define a true space problem if if the vectors \( d_0, \Delta P_0 := P_1 - P_0, d_1 \) are linearly independent. A parametric curve \( r : [0, 1] \rightarrow \mathbb{R}^3 \) interpolates the data in the \( G^1 \) way if

\[
r'(0) = \lambda_0 d_0, \quad r(0) = P_0, \quad r(1) = P_1, \quad r'(1) = \lambda_1 d_1,
\] (1)
where the tangent lengths \( \lambda_i = \| r'(i) \| \), \( i = 0, 1 \), should be positive. The main result of the paper is the following assertion.

**Theorem 1.** Suppose that the prescribed data \( d_0, \Delta P_0, \) and \( d_1 \) are not coplanar. Then there exists a cubic rational PH curve that solves the \( G^1 \) interpolation problem (1).

**Remark 1.** Here and throughout the paper the term true space problem denotes the case when problem can't be reduced to a planar one. Similarly, true rational curve refers a rational curve that is not polynomial too.

3. Relations inherited from the quaternion approach

In [1] the general spatial cubic rational PH curve closed form was derived. The approach that led to it is based upon Euler-Rodrigues frame obtained from a quadratic quaternion polynomial. The closed form depends on ten free parameters: four of them come from the quadratic quaternion polynomial coefficients, three are hidden in the rotation matrix \( R \), and the last three determine the point on the curve at a particular parameter value. Let us recall the main theorem ([1, Theorem 7]) in short. Suppose that the parameters

\[
\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \quad \nu_i \in \mathbb{R},
\]

satisfy

\[
\nu_2^4 \geq 16 \left( \nu_3^2 + \nu_4^2 \right) > 0
\]

(2)

and

\[
4\nu_4^4 - (\nu_3^2 + 8\nu_0\nu_2^2)\nu_1^2 + \nu_2^2 \left( \nu_3^2 + \nu_4^2 + \nu_0 \left( \nu_2^2 + 4\nu_0 \right) \right) = 0.
\]

(3)

This equation can be solved on \( \nu_0 \), and we obtain

\[
\nu_0 = \frac{1}{8} \left( \frac{8\nu_2^4}{\nu_2^2} - \nu_2^2 \pm \sqrt{\nu_2^4 - 16 \left( \nu_3^2 + \nu_4^2 \right)} \right).
\]

(4)

Then, with

\[
q(t) := 16\nu_2^3 \left( \nu_3^2 + \nu_4^2 \right) \left( \nu_2 t^2 + 2\nu_2\nu_1 t + 2\nu_1^2 - \nu_0\nu_2^2 \right)
\]

(5)

and

\[
p(t) := (p_1(t), p_2(t), p_3(t))^T, \quad \text{where}
\]

\[
p_1(t) := \nu_2 \left( 4\nu_4^4 - (\nu_3^2 + 8\nu_0\nu_2^2) \nu_1^2 + \nu_2^2 \left( \nu_0\nu_2^2 + 5 \left( \nu_3^2 + \nu_4^2 \right) \right) \right)
\]

\[-4\nu_1\nu_2^4 \left( \nu_2^2 + 2\nu_0 \right) t - \left( 8\nu_1^2\nu_2^2 - 4\nu_0\nu_2^4 \right) t^2 \right) t,
\]

(6)

\[
p_2(t) := -4\nu_2^4 \left( 2\nu_2 \left( \nu_1\nu_3 + \nu_0\nu_4 \right) + \left( \nu_3^2 + 2\nu_1\nu_3 \right) t \right) t,
\]

\[
p_3(t) := -4\nu_2^4 \left( 2\nu_2 \left( \nu_1\nu_4 - \nu_0\nu_3 \right) + \left( \nu_3^2 + 2\nu_1\nu_3 \right) t \right) t,
\]
the cubic rational space PH curve \( r \) is given as
\[
r(t) := \frac{1}{q(t)} R p(t) + r(0). \tag{7}
\]
Here, \( R \) denotes a rotation matrix that depends on three free parameters. The rotation matrix \( R \) admits several standard representations. As an example, it could be expressed as a product of rotations around the Cartesian coordinate axes depending on the Euler angles.

Though the curve given by (7) is in the closed form, we have found it quite unsuitable to cope with the \( G^1 \) interpolation problem (1). Only interpolating one point it is straightforward since it is achieved by the curve translation. All the other equations that follow from (1) and (7) turn out highly nonlinear, and can’t be significantly simplified. Even for the numerical data, finding the solution was quite a task. So we switched to the Gram matrix approach introduced in [13]. Nevertheless, a brief look at the closed form (7) simplifies to amenable extent the approach followed in the paper. First of all, observe that the denominator \( q \) is of degree 2 only. This is due to a particularly chosen parametrization of the curve that maps a singular curve point to infinity. The crucial observations are given as the following lemma.

**Lemma 1.** Let \( r \) be the curve defined in (5), (6), and (7). Let (2) and (3) hold. The curve denominator \( q \) is positive on \( \mathbb{R} \). The hodograph norm is a rational function of degree two only,
\[
\|r'(t)\| = 4|\nu_2| (2\nu_1^2 - \nu_0\nu_2^2) \frac{\nu_2^2 t^2 + 2\nu_2\nu_1 t + \nu_2^2 (\nu_2^2 + 3\nu_0) - 2\nu_1^2}{q(t)}. \tag{8}
\]
The curve \( r \) is regular.

**PROOF.** The discriminant of \( q \) equals
\[
1024\nu_2^{10} (\nu_0\nu_2^2 - \nu_1^2) (\nu_2^2 + \nu_1^2)^2,
\]
and it can’t be nonnegative since (2) and (4) imply
\[
8 (\nu_2^2 - \nu_0\nu_2^2) = \nu_2^2 \left( \nu_2^2 \pm \sqrt{\nu_2^4 - 16 (\nu_0^2 + \nu_1^2)} \right) > 0. \tag{9}
\]
Also \( q(0) = 16\nu_2^4 (\nu_3^2 + \nu_1^2) (2\nu_1^2 - \nu_0\nu_2^2) > 0 \), and the first assertion is proved. From (7) we obtain
\[
r' = \frac{1}{q} R p' - \frac{q'}{q^2} R p = R \left( \frac{1}{q} p' - \frac{q'}{q^2} p \right).
\]
Since $\mathcal{R}^T\mathcal{R} = I$, we observe

$$(r')^T r' = \frac{1}{q^4} \left( q^2 \|p'\|^2 - 2qq' (p')^T p + (q')^2 \|p\|^2 \right).$$

It is straightforward to confirm that the numerator factorizes in

$$q^2 \|p'\|^2 - 2qq' (p')^T p + (q')^2 \|p\|^2 = q^2 \pi_1^2,$$

with $\pi_1(t) := 4|\nu_2| (2\nu_1^2 - \nu_0 \nu_2^2) (\nu_2^2 t^2 + 2\nu_2 \nu_1 t + \nu_2^2 (\nu_0^2 + 3\nu_0) - 2\nu_1^2)$. This confirms (8). The discriminant of $\pi_1$ equals $64\nu_1^2 (2\nu_1^2 - \nu_0 \nu_2^2)^2 (3 (\nu_1^2 - \nu_0 \nu_2^2) - \nu_1^2)$, and note that the equation (3) could be written also in the form

$$\frac{1}{9} (\nu_2^4 + 12 (\nu_1^2 - \nu_0 \nu_2^2)) (3 (\nu_1^2 - \nu_0 \nu_2^2) - \nu_1^2) + \frac{\nu_2^2}{9} + (\nu_2^2 + \nu_2^2) \nu_1^2 = 0.$$

So clearly by (9) the discriminant should be negative and since

$$\frac{1}{|\nu_2|} \pi_1(0) = \nu_2^2 (\nu_2^2 + 3\nu_0) - 2\nu_1^2 = \nu_2^2 + \frac{5\nu_2^4}{8} - \frac{3\nu_2^4}{8} \sqrt{\nu_2^4 - 16(\nu_3^2 + \nu_4^2)} > 0,$$

it follows $\pi_1 > 0$. The proof is completed.

4. Interpolating curve in Lagrange basis

Let us consider now a cubic rational curve $r$ that satisfies the interpolation conditions (1). A convenient way to construct $r$ is to use the Lagrange basis that corresponds to the data $d_0, \Delta P_0, d_1, P_0$. The interpolating curve written in such a basis would be

$$r(t) := \lambda_0 d_0 \ell_0(t) + \Delta P_0 \ell_1(t) + \lambda_1 d_1 \ell_2(t) + P_0,$$

where the basis $(\ell_0(t), \ell_1(t), \ell_2(t), 1)$ should be chosen in such a way that the interpolation conditions (1) are fulfilled. By Lemma 1, the denominator of the functions $\ell_i$ should be quadratic, and we write it in the Bézier form

$$q(t) := w_0 (1-t)^2 + 2w_1 t(1-t) + w_2 t^2. \quad (10)$$

This reveals the basis functions as

$$\ell_0(t) := \frac{w_0}{q(t)} (1-t)^2 t,$$

$$\ell_1(t) := \frac{1}{q(t)} (2(1-t)w_1 + w_2) t^2,$$

$$\ell_2(t) := \frac{w_2}{q(t)} (t-1)t^2.$$
Since the basis is homogeneous in the weights \( w := (w_0, w_1, w_2)^T \) there are only four free parameters that pin down the curve \( r: \lambda_0, \lambda_1 \), and two of the weights \( w_i \).

It is straightforward to compute the basis derivatives:

\[
\ell'_0(t) = \frac{w_0}{q(t)^2} \left( t^2 ((t + 1)w_2 - 2(t - 1)w_1) + (t - 1)^3w_0 \right) (t - 1), \\
\ell'_1(t) = \frac{2}{q(t)^2} \left( w_1(t - 1)(2tw_1 - (t - 2)w_0) - w_2(t(t + 1)w_1 + w_0) \right) (t - 1)t, \\
\ell'_2(t) = \frac{w_2}{q(t)^2} \left( w_2t^2 + (w_0(t - 2) - 2w_1) (1 - t)^2 \right) t.
\]

(11)

Let us incorporate all the unknowns in the modified basis vector

\[
\ell := (\lambda_0\ell_0, \ell_1, \lambda_1\ell_2)^T = \text{diag} \ (\lambda_0, 1, \lambda_1) \ (\ell_0, \ell_1, \ell_2)^T.
\]

The hodograph of the curve can be written as

\[
r' = (d_0, \Delta P_0, d_1) \ell',
\]

and its squared norm as

\[
\|r'\|^2 = (r')^T r' = (\ell')^T G \ell' =: \frac{\pi_2}{q^2},
\]

(12)

where (11) implies that \( \pi_2 \) is a polynomial of degree \( \leq 8 \). Here, \( G \) denotes the data Gram matrix,

\[
G := \begin{pmatrix}
\|d_0\|^2 & d_0^T \Delta P_0 & d_0^T d_1 \\
\Delta P_0^T d_0 & \|\Delta P_0\|^2 & \Delta P_0^T d_1 \\
d_1^T d_0 & d_1^T \Delta P_0 & \|d_1\|^2
\end{pmatrix}.
\]

With the notation

\[
d_0 := \|\Delta P_0\|, \quad c_{01} := \frac{1}{\delta_0} d_0^T \Delta P_0, \quad c_{12} := \frac{1}{\delta_0} \Delta P_0^T d_1, \quad c_{02} := d_0^T d_1,
\]

the Gram matrix obtains the cosine form

\[
G = \begin{pmatrix}
1 & \delta_0 c_{01} & \delta_0 c_{02} \\
\delta_0 c_{01} & \delta_0^2 & \delta_0 \delta_0 c_{12} \\
c_{02} & \delta_0 c_{12} & 1
\end{pmatrix}.
\]

(13)
Let us define

\[ b_c := b_c(c_{01}, c_{12}) := (1 - c_{01}^2) (1 - c_{12}^2) - (c_{02} - c_{01} c_{12})^2 = b_c(c_{12}, c_{01}) . \]  

(14)

The constants \( c_{ij} \) are not independent ([15]), and they should belong to the set \( \overline{\mathcal{P}} \), the closure of the set

\[ \mathcal{P} := \{(c_{01}, c_{02}, c_{12}) \in \mathbb{R}^3 | |c_{ij}| \leq 1, \ b_c > 0 \} . \]  

(15)

The boundary \( \partial \mathcal{P} := \overline{\mathcal{P}} \setminus \mathcal{P} \) is reached if and only if the corresponding vectors \( d_0, \Delta P_0, \) and \( d_1 \) are coplanar, i.e., \( b_c = 0 \). Note that the set \( \mathcal{P} \) in (15) has a clear geometric meaning. Its points satisfy

\[ \frac{(c_{01} + c_{12})^2}{2(1 + c_{02})} + \frac{(c_{01} - c_{12})^2}{2(1 - c_{02})} \leq 1, \quad \text{if } c_{02} \in (-1, 1), \]

or

\[ c_{02} = \pm 1, \quad c_{01} = \pm c_{12} . \]

For every fixed \( |c_{02}| < 1 \), the boundary of \( \mathcal{P} \) is an ellipse that reduces to a line segment at \( |c_{02}| = 1 \). The following figure (Figure 1) demonstrates that for quite a few cosine values only the true rational solution of Theorem 1 is to be expected.

![Figure 1: The red area denotes cosine parameters for which both polynomial and true rational solution exist. The blue part has the true rational solution only.](image-url)
5. Basic PH equations derived

A straightforward approach to equations that represent PH conditions is obvious. The polynomial \( \pi_2 \), introduced in (12), should satisfy \( \pi_2 = \pi^2 \) for some polynomial \( \pi \) of degree four. A simple way to achieve this is to compute five coefficients of \( \pi \), expressed in the Bernstein basis, from the equations

\[
\pi_2^{(i)}(0) = (\pi^2)^{(i)}(0), \quad i = 0, 1, 2, \quad \pi_2^{(i)}(1) = (\pi^2)^{(i)}(1), \quad i = 0, 1.
\]

This determines \( \pi \) in a reasonably simple closed form, and the remaining part of the PH conditions is given by

\[
\pi_2^{(i)}(0) = (\pi^2)^{(i)}(0), \quad i = 3, 4, \quad \pi_2^{(i)}(1) = (\pi^2)^{(i)}(1), \quad i = 2, 3.
\]  

Unfortunately, if expanded these four equations would require several pages of the paper, and we have found them hard to analyse. But Lemma 1 paves the way to simpler and stronger conditions since the polynomial \( \pi_2 \) should be divisible by \( \pi \). First of all, let us introduce new variables by

\[
\begin{align*}
\mu_0 &:= \frac{1}{\delta_0} \sqrt{\frac{w_0}{w_2}} \lambda_0, \\
\mu_1 &:= \frac{1}{\delta_0} \sqrt{\frac{w_2}{w_0}} \lambda_1, \\
\omega_0 &:= \sqrt{\frac{w_2}{w_0}}, \\
\omega_1 &:= \frac{w_1}{\sqrt{w_0 w_2}}.
\end{align*}
\]  

In new variables, the curve is determined by

\[
\tilde{q}(t) := \frac{1}{\mu_0 \mu_1} q(t) = \frac{1}{\omega_0} (1 - t)^2 + 2 \omega_1 t (1 - t) + \omega_0 t^2,
\]

\[
\ell(t) = \frac{1}{\tilde{q}(t)} \left( (\delta_0 \mu_0 (1 - t)^2 t, (\omega_0 - 2(t - 1) \omega_1) t^2, \delta_0 \mu_1 (t - 1) t^2 \right)^T.
\]  

If we rescale \( \pi_2 \) introduced in (12) by

\[
\tilde{\pi}_2 := \frac{1}{(\delta_0 \omega_0 \omega_2)^2} \pi_2 = \frac{1}{\delta_0^2} \tilde{q}^3 (\ell')^T G \ell',
\]

the PH conditions based upon Lemma 1 are given as

\[
\pi_3 := \tilde{\pi}_2 - \tilde{q}^2 \tilde{\pi}_1^2 \equiv 0, \quad \pi_1(t) = s_0 (1 - t)^2 + 2 s_1 t (1 - t) + s_2 t^2,
\]  

where \( s_i \) are the free parameters to be determined. Note that by Lemma 1 polynomials \( \tilde{q} \) and \( \pi_1 \) should both be positive. From (12), (13), and (18) we observe that the changes

\[
t \rightarrow 1 - t, \quad \mu_0 \rightleftharpoons \mu_1, \quad \omega_0 \rightarrow \frac{1}{\omega_0}, \quad c_{01} \rightleftharpoons c_{12},
\]  

9
leave $\pi_2$ and $\tilde{q}$ unchanged. So it is natural to impose the PH conditions (19) by annihilating the Hermite basis coefficients of $\pi_3$ at the boundaries 0 and 1,

$$\pi_3^{(i)}(0) = 0, \quad i = 0, 1, \ldots, 4, \quad \pi_3^{(i)}(1) = 0, \quad i = 0, 1, 2, 3.$$  

From

$$\pi_3(0) = \frac{1}{\omega_0} (\mu_0^2 - s_0^2), \quad \pi_3(1) = \omega_0^2 (\mu_1^2 - s_2^2),$$

and the positivity of $\pi_1$ we determine $s_0 = \mu_0$, $s_2 = \mu_1$, and the derivatives imply

$$\pi_3'(0) = 0 : s_1 = -\omega_0 \mu_0 c_0 + (\omega_0 + 2 \omega_1) c_01, \quad (21)$$

$$\pi_3'(1) = 0 : s_1 = -\omega_0 \mu_1 c_0 + \left(\frac{1}{\omega_0} + 2 \omega_1\right) c_12. \quad (22)$$

Since both values $s_1$ should be equal, we obtain all of the equations as

$$e := (e_1, e_2, \ldots, e_6)^T$$

$$:= \left(\pi_3'(0) - \pi_3'(1), \pi_3^{(2)}(0), \pi_3^{(2)}(1), \pi_3^{(3)}(0), \pi_3^{(3)}(1), \pi_3^{(4)}(0)\right)^T = 0. \quad (23)$$

A straightforward evaluation gives

$$\pi_2^{(2)}(0) = \frac{4}{\omega_0^2} (\mu_0 (\mu_1 (13 - 2 \omega_0 \omega_1) c_{02} + 2(\omega_0 (\omega_1 (\omega_0 + 2 \omega_1) - 5) - 13 \omega_1) c_{01})$$

$$+ 2 (\mu_1^2 - 2 \mu_1 (\omega_0 + 2 \omega_1) c_{12} + (\omega_0 + 2 \omega_1)^2) + \mu_0^2 (14 - 2 \omega_1 \omega_0 - \omega_0^2)),$$

$$\pi_2^{(3)}(0) = \frac{24}{\omega_0^2} (\mu_0 (\mu_1 \omega_0^2 + 8 \omega_0 \omega_1 - 18) c_{02} + (36 \omega_1 - \omega_0 ((\omega_0 + 4 \omega_1)^2 - 10)) c_{01})$$

$$+ \mu_1 (2 \omega_0 (\omega_0 (7 - 4 \omega_1 (\omega_0 + 2 \omega_1))) c_{12} + 2 \mu_0^2 (\omega_0^2 + 3 \omega_1 \omega_0 - 7)$$

$$+ 2 \omega_0 (\omega_0 (\mu_1^2 + (\omega_0 + 2 \omega_1)^2) - 7) - 10 \omega_1) - 5 \mu_1^2 - 2 \omega_0^2),$$

$$\pi_2^{(4)}(0) = \frac{24}{\omega_0^2} (2 \mu_0 (\mu_1 (\omega_0 (\omega_0 (\omega_0 (\omega_0 + 2 \omega_1) - 2) - 24 \omega_1) + 55) c_{02})$$

$$- 2 (55 \omega_1 + \omega_0 (\omega_0 \omega_0^2 + 4 \omega_1^2 - 1) \omega_0^2 + 4 \omega_1 (\omega_0^2 - 4) \omega_0 - 48 \omega_1^2 + 10)) c_{01})$$

$$- 4 \mu_1 (41 \omega_1 + \omega_0 (\omega_0 (2 \omega_1 \omega_0 - 11) - 36 \omega_1) + 9)) c_{12}$$

$$+ \mu_0^2 (\omega_0 (\omega_0 (\omega_0 + 2 \omega_1)^2 - 10) - 60 \omega_1) + 70) + 4 \omega_1 (\omega_0 (\mu_1^2 (\omega_0 \omega_1 - 9)$$

$$+ (\omega_0 + 2 \omega_1) (\omega_0 (\omega_0 (\omega_0 + 2 \omega_1) - 2) - 18 \omega_1) + 18) + 41 \omega_1) + 41 \mu_1^2 + 4 \omega_0^2).$$
Further, the derivatives of the polynomial \( \pi_4 := q^2 \pi_1^2 \) at \( t = 0 \) are obtained from the following equations where the value of \( s_1 \) is replaced by (21),

\[
\begin{align*}
\pi_4^{(2)}(0) &= \frac{4}{\omega_0^2} \left( \mu_0 \left( \omega_0 \left( 2 \omega_0 \omega_1 - 14 \omega_1 + \omega_0 \right) + 14 \right) + \mu_1 \right) \\
&\quad + 2 \mu_0 s_1 \left( 4 \omega_0 \omega_1 - 7 \right) + 2 s_1^2 , \\
\pi_4^{(3)}(0) &= \frac{24}{\omega_0^2} \left( \mu_0 \left( \mu_1 \left( 2 \omega_0 \omega_1 - 3 \right) + \mu_0 \left( 21 \omega_1 + \omega_0 \left( (\omega_0 - 6 \omega_1) \omega_1 - 3 \right) - 14 \right) \right) + s_1 \left( \mu_0 \left( (4 \omega_1^2 + 2) \omega_0^2 - 24 \omega_1 \omega_0 + 21 \right) + \mu_1 \right) + s_1^2 \left( 4 \omega_0 \omega_1 - 6 \right) \right) , \\
\pi_4^{(4)}(0) &= \frac{24}{\omega_0^2} \left( 2 \mu_0 \left( \mu_1 \left( (4 \omega_1^2 + 2) \omega_0^2 - 20 \omega_1 \omega_0 + 15 \right) + \mu_0^2 \left( 60 \omega_0 \omega_1^2 \right) - 20 \omega_1 \omega_0 + 15 \right) \right) + s_1 \left( \mu_0 \left( 2 \omega_0 \left( 30 \omega_1 + \omega_0 \left( 2 \left( \omega_0 - 5 \omega_1 \right) \omega_1 - 5 \right) - 35 \right) \right) + 4 s_1 \left( (4 \omega_1^2 + 2) \omega_0^2 - 20 \omega_1 \omega_0 + 15 \right) \right) .
\end{align*}
\] (25)

From (24) and (25) the derivatives \( \pi_3^{(i)}(0) = \pi_2^{(i)}(0) - \pi_4^{(i)}(0) \) that are needed in (23) are derived. Finally note that the derivatives at \( t = 1 \) follow from

\[
\pi_3^{(j)}(1) = (-1)^j \pi_3^{(j)}(0) \bigg|_{\text{replacements (20)}}, \quad j = 2, 3.
\]

In order to preserve the symmetry, we insert in these two equations \( s_1 \) determined in (22). Thus we have obtained the system of six rational equations (23) for the four unknowns \( \mu_0, \mu_1, \omega_0, \) and \( \omega_1 \).

The polynomial form, usually preferred in numerical computations, is obtained by a diagonal multiplication

\[
\tilde{e} := \text{diag} \left( \omega_0, \omega_0^2, 1, \omega_0, \omega_0 \right) e = 0.
\] (26)

6. Homotopy analysis

This section is devoted to the proof of Theorem 1. The proofs of the lemmas are quite technical and tedious. As strongly suggested, we moved them to [14]. The reader that is interested in the applications part of the paper mainly can skip them entirely. The equations (23) depend on three parameters \( (c_{01}, c_{02}, c_{12}) \in P \) only, and so do the solutions \( (\mu_0, \mu_1, \omega_0, \omega_1) \in \mathbb{R}^4 \). But only some of them are admissible. Since by Lemma 1 the denominator \( q \) is positive for \( t \in \mathbb{R} \), we observe \( \omega_0 > 0 \), and its discriminant \( 4 (\omega_1^2 - \frac{3}{6}) \) should be negative. From (17)
and \(0 < \lambda_i < \infty\) we conclude that \(\mu_i\) has to be positive too, and bounded. This reveals the set of admissible solutions as

\[
\mathcal{D} := \{(\mu_0, \mu_1, \omega_0, \omega_1) \in \mathbb{R}^4 \mid 0 < \mu_0 < \infty, 0 < \mu_1 < \infty, 0 < \omega_0 < \infty, |\omega_1| < 1\}.
\] (27)

The following lemma determines the admissible solution for a particular data set.

**Lemma 2.** Suppose the cosine parameters satisfy \(c_{01} = c_{12} = 0, |c_{02}| < 1\). The system (23) has precisely one admissible solution, given in a closed form. With the help of

\[
u := u(c_{02}) := (c_{02} + 1) \left( 8c_{02}^5 - 14c_{02}^4 - 19c_{02}^3 + 80c_{02}^2 - 95c_{02} + 44 \right.
- \left. 3\sqrt{3} (1 - c_{02})^2 \sqrt{c_{02} (c_{02} (3 - 4 (c_{02} - 5) c_{02}) - 70) + 59} \right)
\]

the solution reads

\[
\begin{align*}
\mu_0 &= \mu_1 = \sqrt{\frac{2(1 - c_{02} + c_{02}^2)u^{1/3} + (1 + c_{02})^2(7 - 10c_{02} + 4c_{02}^2) + u^{2/3}}{3\sqrt{u} (1 - c_{02})}}, \\
\omega_0 &= 1, \quad \omega_1 = -\frac{1 + c_{02}\mu_0^2}{1 + \mu_0^2}.
\end{align*}
\] (28)

Note that the particular data triple of Lemma 2, and any other one can be joined by a line segment

\[
s(\zeta) := (1 - \zeta) (0, c_{02}, 0) + \zeta (c_{01}, c_{02}, c_{12}), \quad 0 \leq \zeta \leq 1, \quad s \in \mathcal{P},
\] (29)

a homotopy path. The homotopy argument that counts the number of solutions is very simple: if no rational solution along \(s\) approaches \(\partial \mathcal{D}\), the total number of rational solutions along the homotopy path \(s\) could change only by an even number. Since the number of solutions is odd at the particular data set of Lemma 2, is must be odd at the general one too. It remains now to prove that a rational solution can’t near \(\partial \mathcal{D}\) unless the data turn almost planar. Lemma 3 proves that a rational solution can’t approach a particular point of the boundary \(\partial \mathcal{D}\), i.e., \(\omega_0 = 1, \omega_1 = 1\), and Lemma 4 covers the rest of the boundary.

**Lemma 3.** Let \((c_{01}, c_{02}, c_{12}) \in \mathcal{P}\) be any true space data triple. The corresponding polynomial and the rational solutions are separated.
Lemma 4. A true rational solution \((\mu_0, \mu_1, \omega_0, \omega_1) \in \mathcal{D}\) of the system (23) may appear arbitrarily close to the boundary \(\partial \mathcal{D}\) only if the corresponding data \((c_{01}, c_{02}, c_{12}) \in \mathcal{P}\) are arbitrarily close to the planar case \(\partial \mathcal{P}\).

So, the number of solutions for all the data in \(\mathcal{P}\) stayes odd. This completes the proof of Theorem 1.

7. Briefly on algebraic approach

One is clearly tempted to study the properties of the system (23) for general cosines \(c_{ij}\) and the data point distance \(\delta_0\) by the entirely algebraic approach directly. But as expected the straightforward Gröbner basis computation fails in this symbolic setup. Nevertheless, it works for the data prescribed, what gives a numerical approach as one of the computational procedures described in the next section.

However, stubborn as we are, we have not given up the symbolic algebraic approach entirely, and the interplay of human and computer elimination steps that took quite a while resulted in the first Gröbner basis polynomial. Suppose that the Gröbner basis of \(e\) in (23) is computed with respect to variables \(\mu_0, \mu_1, \omega_1, \) and \(\omega_0\) in the lexicographic monomial order. Then

\[
(\omega_0^2 - 1) q_6(\omega_0^2) q_8(\omega_0^2)
\]

determines the same variety as the first Gröbner basis polynomial. The polynomials \(q_6\) and \(q_8\) are of degree 6 and 8 in \(\omega_0^2\) respectively. Since they are too large to be included in the paper, one can find them in [14]. Note that the equations (23) determine both polynomial and true rational solutions. Let us separate both varieties. Let

\[
c_\omega := \left(\sqrt{\omega_0} - \frac{1}{\sqrt{\omega_0}}\right)^2 + 2 (1 - \omega_1).
\]

By Lemma 3, the additional equation \(c_\omega = 0\) with the only real solution \(\omega_0 = \omega_1 = 1\) that satisfies \(|\omega_1| \leq 1\) narrows the solution set to polynomial solutions ([13], [12]). The true rational solutions are this way obtained by an extended equation set

\[
e_E := (e_1, e_2, \ldots, e_6, \omega_0 c_\omega u - 1)^T = 0, \quad u \in \mathbb{R},
\]

where \(u\) denotes a new variable to be determined. If we compute Gröbner basis of \(e_E\) with respect to \(u, \mu_0, \mu_1, \omega_1, \) and \(\omega_0\), the first function simplifies from (30)
to the polynomial \( q_6 \) only, up to a constant and the possible multiplicity. There is no need to follow all the elimination steps to verify this. The coefficients of \( q_6 \) are polynomials in \( c_{01}, c_{02}, \) and \( c_{12} \), and the verification is confirmed if it holds for enough independent numerical values of the cosine parameters ([14]). The polynomial \( q_6 \) implies the following observation.

**Lemma 5.** The possible candidates for the \( \omega_0 \)-part of an admissible true rational solution of (23) are determined by the set

\[
\{ x \in \mathbb{R}_+ | q_6 (x^2) = 0 \}. \tag{33}
\]

The polynomial \( q_6 \) has an even number \( \geq 2 \) positive roots.

**Proof.** Recall \( b_c \) introduced in (14) and let

\[
g_1 := -(c_{02} - 2c_{01}c_{12} + 1) = -\frac{b_c + (c_{01} - c_{12})^2}{1 - c_{02}} < 0. \tag{34}
\]

Then

\[
q_6 (x) = (1 - c_{01}^2) b_c^2 g_1^2 + \cdots + (1 - c_{12}^2) b_c^2 g_1^2 x^6, \tag{35}
\]

and

\[
q_6 (1) = -(c_{01} - c_{12})^6 \tilde{g}_1, \tag{36}
\]

where

\[
\tilde{g}_1 := 24b_c + 8 \left( 1 - c_{02}^2 \right) - 3 \left( c_{01}^2 (c_{01} - c_{12}) + (1 - c_{02}^2) (c_{01} + c_{12}) \right). \tag{37}
\]

It is straightforward to verify

\[
\tilde{g}_1 - 24b_c \geq 0, \quad (c_{01}, c_{02}, c_{12}) \in \mathbb{F}. \tag{38}
\]

So if the data are not planar one yields

\[
q_6 (0) > 0, \quad q_6 (1) \leq 0, \quad q_6 (\infty) > 0,
\]

and the set (33) gives an even number of candidates. \( \Box \)

In the section 6 it was shown that the number of admissible true rational solutions is odd. So at least one of the solutions of (33) has to be extraneous.

The following observation covers a particular data case that will be used in the next section.
Lemma 6. For the true space problem, the polynomial $q_6$ vanishes at $\omega_0 = 1$ iff the cosines $c_{12}$ and $c_{01}$ are equal.

Proof. Since $b_c > 0$, (34) and (38) show $g_1 < 0$, and $\bar{g}_1 > 0$ respectively. Thus $q_6(1) = 0$ by (36) implies $c_{12} = c_{01}$. But if $c_{12} = c_{01}$ then $q_6$ simplifies to

$$q_6(\omega_0^2) = (1 - c_{01}^2) (1 - c_{02})^2 (1 + c_{02} - 2c_{01}^2)^4 (\omega_0^2 - 1)^6$$

$$= (1 - c_{01}^2) b^2 c_1 g_1^2 (\omega_0^2 - 1)^6,$$

and the conclusion $\omega_0 = 1$ follows. $\square$

Let us consider now the planar data case $b_c = 0$. This equation solved on $c_{02}$ yields

$$c_{02} = c_{01} c_{12} \mp \sqrt{(1 - c_{01}^2) (1 - c_{12}^2)}.$$

(40)

The planar case simplifies the polynomial $q_6$ to

$$q_6(x) = -\frac{1}{(1 - c_{02})} (c_{01} - c_{12})^6 q_4(x)$$

where (14)

$$q_4(x) = (1 - c_{02}^2)^2 + (1 - c_{01}^2) ((c_{01} + c_{12}) (c_{01} + (2c_{02} - 1) c_{12}) - 4c_{02}) x +$$

$$4 (1 - c_{12}^2) c_{01}^4 + 2c_{12} (4c_{02} (c_{12}^2 - 1) + 1) c_{01}^3 -$$

$$\frac{1}{2} (8c_{12}^4 - 20c_{12}^2 + c_{02} + 15) c_{01}^2 + c_{12} (2 (1 - 4c_{02}) c_{12}^2 + 9c_{02} - 5) c_{01} +$$

$$4c_{12}^4 - \frac{1}{2} (c_{02} + 15) c_{12}^2 + 6) x^2 +$$

$$(1 - c_{12}^2) ((c_{01} + c_{12}) (c_{01} (2c_{02} - 1) + c_{12}) - 4c_{02}) x^3 + (1 - c_{12}^2)^2 x^4.$$

Let us apply (40). The algebraic cylindrical decomposition reveals that $q_4$ has no positive root for the parameters $-1 \leq c_{01}, c_{12} \leq 1$, $c_{01} \neq c_{12}$, regardless of the choice of sign in (40). This proves the following lemma.

Lemma 7. Suppose that the cosines $c_{01}, c_{02},$ and $c_{12}$ determine planar data. If $c_{12} \neq c_{01}$ there is no admissible true rational solution of equations (23).

If the cosines define planar data, i.e., $b_c = 0$, and $c_{12} = c_{01}$, what implies $g_1 = 0$ too, the polynomial $q_6$ in (39) vanishes identically, imposing no conditions on $\omega_0$. So this case has to be studied separately.
Theorem 2. In the planar case $b_c = 0$, the equations (23) have an admissible true rational solution iff $-1 < c_{01} = c_{12} \leq 1$. For a fixed $c_{01}$, the solution is a unique circle segment, up to reparameterization.

Proof. The necessary part is given as Lemma 7. Let now $c_{01} = c_{12}$. Then

$$b_c = (1 + c_{02} - 2c_{01}^2)(1 - c_{02}) = 0.$$  \hspace{1cm} (41)

If either one of the factors in (41) vanishes, the solution of (32) can be derived in a closed form. If $c_{02} = 2c_{01}^2 - 1$, $-1 < c_{01} < 1$, the system (32) has four solutions, but only

$$\mu_0 = \mu_1 = 1, \omega_1 = c_{01}, u = \frac{1}{\omega_0 c_\omega}$$

is admissible. Since the data are planar, and $\delta_0 \neq 0$, we may assume they are of the form

$$P_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d_0 = d_1 = \begin{pmatrix} c_{01} \\ \sqrt{1 - c_{01}^2} \\ 0 \end{pmatrix}$$

without losing generality. The resulting curve reads

$$r(t) = \frac{1}{(1 - t)^2 + 2\omega_0 c_{01}(1 - t)t + \omega_0^2 t^2} \begin{pmatrix} \omega_0 (c_{01}(1 - t) + \omega_0 t)t \\ \omega_0 \sqrt{1 - c_{01}^2}(1 - t)t \\ 0 \end{pmatrix},$$

and with the reparameterization

$$t = t(u) = \frac{u}{u + \omega_0(1 - u)}$$

simplifies to $\omega_0$-independent form

$$r(t(u)) = \frac{1}{(1 - u)^2 + 2c_{01}(1 - u)u + u^2} \begin{pmatrix} (c_{01}(1 - u) + u)u \\ \sqrt{1 - c_{01}^2}(1 - u)u \\ 0 \end{pmatrix},$$

clearly a circle with the center at

$$\left(\frac{1}{2}, -\frac{c_{01}}{2\sqrt{1 - c_{01}^2}}, 0\right)^T,$$
and of the radius \( \frac{1}{2\sqrt{1-c_{01}^2}} \). If \( c_{01} = 1 \), the interpolating curve reduces to a line segment \([0, 1]\). It is straightforward to verify that the choice \( c_{02} = 1, c_{01} < 1 \) yields no admissible solution.

The assertion of Theorem 2 was based on the equations derived with the help of Lemma 1. However, the numerical evidence on Figure 2 was computed for the general equations (16), without simplifications inherited from the quaternion approach. The results obtained show no difference among general and simplified version of equations. This gives a strong computational support to the following conjecture.

**Conjecture 1.** The only planar true rational cubic PH curves are rationally parametrized circles arcs.

### 8. Numerical solution

It is possible to accompany \( q_6 \) with the rest of the symbolic Gröbner basis but this would make [14] unbearably long, and other ways to the numerical solution have to be worked out. Note that the polynomial case is covered in [13] completely. So we consider the true space problem, and the extended equations set (32) only. Let \( \mathcal{I} \) denote the ideal spanned by (32), and \( \mathcal{V} \) be the corresponding
variety. If a polynomial equations solver is available, it is a straightforward but perhaps not most efficient to find all the solutions of the system (32). The admissible solutions $\mathcal{V}$ are then selected from the whole polynomial equations solution set.

As an alternative, we provide an algorithm that depends on the computation of real roots of a couple of single variable polynomials of degree $\leq 6$ only. A numerical test carried out in [14] suggests that near three such root-finding problems are to be expected. Quite a gain in efficiency compared with the straightforward approach. The basic idea of the procedure is to examine the ideal $\mathcal{I}$ along a binary tree (Fig. 3) that depends mainly on the coefficients $\psi_0$ and $\psi_1$ at the variables $\psi_0$ and $\psi_1$ in $e_1$ respectively,

$$
\psi_0 := \omega_0 \left(c_{02} - \omega_0 \psi_1\right), \quad \psi_1 := \omega_1 - \omega_0 c_{02},
$$

$$
e_1 = \omega_0 \left(\omega_0 + 2 \omega_1\right) c_{01} - (2 \omega_0 \psi_1 + 1) c_{12} + \mu_0 \psi_0 + \mu_1 \psi_1.
$$

The algorithm gathers the possible candidates for $\mathcal{V}$. Any variables computed are filtered to be admissible on the spot. Finally, the quadruples

$$(\mu_{0,\ell}, \mu_{1,\ell}, \psi_{0,\ell}, \psi_{1,\ell})_{\ell=1,2,...}
$$

are inserted in the equations (32), and the actual solutions (either one or three) are selected. The precise derivation of the algorithm and its implementation are left to [14], and a brief outline of the procedure is as follows.

![Figure 3: The numerical algorithm scheme](image-url)
**Case a)** For the particular data case \( c_{12} = c_{01} \) Lemma 6 implies \( \omega_0 = 1 \), and the first equation simplifies to \( e_1 = (\mu_1 - \mu_0) (\omega_1 - c_{02}) \). The cylindrical decomposition shows that \( \omega_1 = c_{02}, \mu_1 \neq \mu_0 \) gives no admissible solution. Thus \( \mu_1 = \mu_0 \). The Gröbner basis of the simplified ideal \( \mathcal{I} \) with respect to the remaining unknowns \( \omega_1 \) and \( \mu_0 \) yields the first basis polynomial

\[
\psi_2 := \mu_0^6 (c_{02} - 1) + 2 \mu_0^4 ((1 - 2c_{02}) c_{01}^2 + (c_{02} - 1) c_{02} + 1) + \cdots + 2 (1 - c_{01}^2).
\]

The equation \( \psi_2 = 0 \) gives an odd number of positive roots. The second Gröbner basis polynomial is linear in \( \mu_1 \), and the solution \( \omega_1 = \sqrt{c_{12}} \) completes the solution quadruples if \( c_{12} = c_{01} \). This covers this case completely, and no other solutions need to be looked for.

**Case b)** From now on we consider the case \( c_{12} \neq c_{01} \), so by Lemma 6 we also have \( \omega_0 \neq 1 \). If both \( \psi_0 \) and \( \psi_1 \) belong to the ideal \( \mathcal{I} \), i.e., \( \psi_0 = 0 \) and \( \psi_1 = 0 \), then it necessarily follows \( c_{02} = 0, \omega_1 = 0, \) and \( \omega_0 = \sqrt{c_{12}/c_{01}} \). Further, from \( e_3 = 0 \) we express \( \mu_0 = 2 (1 - c_{01} c_{12}) / \mu_1 \) too. After substitutions completed, \( e_4 = 0 \) becomes a quadratic equation for the last unknown \( \mu_1 \). Further analysis helps us to establish that the admissible solution exists only for the data on curve segments, implicitly described by all of the conditions

\[
0 < c_{01} < 1, c_{01}^2 + c_{12}^2 < 1, (c_{01} c_{12} - 1) (c_{01}^2 + 6 c_{12} c_{01} + c_{12}^2) + 1 = 0.
\]

For the first \( c_{01} \)-segment, \( 0 < c_{01} < \left( \sqrt{2 - \sqrt{2}} \right) / 2 \), we derive finally

\[
\mu_0 = \frac{4 \sqrt{c_{01} c_{12}} (1 - c_{01} c_{12})}{8 c_{01} c_{12} (c_{01} c_{12} - 1) + 1}, \quad \mu_1 = \frac{\sqrt{8 c_{01} c_{12} (c_{01} c_{12} - 1) + 1} + 1}{2 \sqrt{c_{01} c_{12}}}.
\]

In order to obtain \( \mu_0 \) and \( \mu_1 \) for the values \( \left( \sqrt{2 - \sqrt{2}} \right) / 2 < c_{01} < 1 \), we only have to switch the right-hand sides in (42). We have excluded the curve point \( \left( \sqrt{2 - \sqrt{2}} \right) / 2 = c_{01} \) since then \( c_{12} = c_{01} \).

**Case c)** From this step further on we will not be able to express \( \omega_0 \) in a closed form, depending on the cosine data only. So we compute the possible candidates \( \omega_0 \) from the equation

\[
q_6 = 0.
\]
Let us consider the case when \( \psi_1 \) belongs to the ideal \( I \), so \( \omega_1 = \omega_0 c_{02} \), but \( \psi_0 \),

\[
\psi_0 = \omega_0 (c_{02} - \omega_0 \omega_1) = -\omega_0 \left( \omega_0^2 - 1 \right) c_{02} \neq 0,
\]
does not. From the first equation we may now compute \( \mu_0 \),

\[
\mu_0 = \frac{\omega_0^2 (c_{01} (2c_{02} + 1) - 2c_{02} c_{12}) - c_{12}}{\omega_0 (\omega_0^2 - 1) c_{02}},
\]
and the second, quadratic equation determines the possible \( \mu_1 \) values.

**Case d)** In the last case to be examined we assume \( \psi_1 \not\in I \). So from the equation \( e_1 = 0 \) we express \( \psi_1 \)

\[
\mu_1 = \frac{\omega_0 (-\mu_0 c_{02} + 2 \omega_1 c_{12} - (\omega_0 + 2 \omega_1) c_{01} + \mu_0 \omega_0 \omega_1) + c_{12}}{\omega_1 - \omega_0 c_{02}},
\]
and substitute it in the equations \( e_2, e_3, \ldots, e_6 \). Further, we multiply them so that they regain the polynomial form, and use the same notation \( e_i \) for simplicity. The equations \( e_i, i = 2, 3, \ldots, 6 \), are quadratic in \( \omega_1 \) on general, with leading coefficients \( \psi_3 := (\psi_{3,i})_{i=2}^6 \) being polynomials in \( c_{02}, \omega_0, \) and \( \omega_1 \). If we compute the Gröbner basis of \( \psi_3 \) with respect to \( \psi_1 \), the first polynomial equals

\[
\omega_0^2 (\omega_0^2 - 1)^2 (c_{02} - 1)^3 c_{02}^3,
\]
and it can vanish only if \( c_{02} = 0 \). But then the fifth polynomial simplifies to \(-3\omega_0^2 \omega_1^2\). So at least one coefficient \( \psi_{3,i}, 2 \leq i \leq 6 \), should be distinct from 0 at \( V \) since we have excluded the possibility \( c_{02} = 0, \omega_1 = 0 \) in advance. Suppose that \( \psi_{3,i} \not\in I \). Then we can use the equation \( e_i \), and we eliminate the quadratic \( \mu_0 \) terms from \( e_j, j = 2, 3, \ldots, 6, j \neq i \). The resulting equations, called again \( e_j \), are now linear in \( \mu_0 \). Let us denote the coefficient vectors at powers of \( \mu_0 \) as follows

\[
\mathbf{v}_{r,i} := \left( \text{coefficient at } \mu_0^r \text{ in } e_j \right)_{j \neq i}, \quad i = 2, 3, \ldots, 6.
\]
So the vectors \( \mathbf{v}_{1,i} \), and \( \mathbf{v}_{0,i} \) should be colinear, thus their wedge product \( \mathbf{v}_{1,i} \land \mathbf{v}_{0,i} \) should vanish. Components of this wedge product are given as \( 2 \times 2 \) minors of the matrix \( (\mathbf{v}_{1,i}, \mathbf{v}_{0,i}) \). If we compute polynomial greatest common divisor of the components of \( \mathbf{v}_{1,i} \land \mathbf{v}_{0,i} \), we obtain the product

\[
\eta_i := \text{const}, \psi_1^2 (\omega_0 c_\omega) \psi_4,
\]

20
where \( \text{const}_i = \pm 2^{\pm 1} \) depends on \( i \), but \( \psi_4 = \psi_4 (c_{01}, c_{02}, c_{12}, \omega_0, \omega_1) \),

\[
\psi_4 := 8\omega_0^2 (c_{12} - c_{01}) \omega_1^6 + 4\omega_0 \left( 4c_{01}^3 - 2c_{12}c_{01}^2 + (2 (c_{12}^2 - 1) \omega_0^2 - c_{02} - 3) c_{01} + c_{12} \left( -4c_{12}^2 + c_{02} + 3 \right) \omega_0^2 + 2 \right) \omega_1^5 + \ldots
\]  

(45)
does not. It is straightforward to verify that polynomial components of the vector

\[
\frac{1}{\eta} \mathbf{v}_{1,i} \wedge \mathbf{v}_{0,i}
\]

can vanish simultaneously only in the polynomial case, independently on \( i \). Since \( \psi_1 \neq 0 \), \( \omega_0 c_\omega \neq 0 \) we finally conclude that in the case considered \( \omega_1 \) should be a root of the degree 6 equation \( \psi_4 = 0 \). So (43) and (45) determine admissible pairs \( (\omega_0, \omega_1) \). The corresponding \( \mu_0 \) is computed from the quadratic equation with the largest value of \( |\psi_{3,i}| \), and (44) completes the candidates quadruples.

9. Examples

The setup of examples is as follows. Since \( \delta_0 = \| \Delta \mathbf{P}_0 \| = 0 \) implies the planar data case, we may assume

\[
\mathbf{P}_0 = (0, 0, 0)^T, \quad \mathbf{P}_1 = (1, 0, 0)^T,
\]

without losing generality. Similarly, let the direction \( \mathbf{d}_0 \) lie in the \( x\text{-}y \) plane. So all additional interpolation data needed in (1) are determined by the cosines \( c_{01}, c_{12}, \) and \( c_{02} \),

\[
\mathbf{d}_0 = \begin{pmatrix} \frac{c_{01}}{\sqrt{1 - c_{01}^2}} \\ 0 \end{pmatrix}, \quad \mathbf{d}_1 = \frac{1}{\sqrt{1 - c_{01}^2}} \begin{pmatrix} c_{12} \sqrt{1 - c_{01}^2} \\ c_{02} - c_{01} c_{12} \end{pmatrix} \sqrt{(1 - c_{01}^2)(1 - c_{12}^2) - (c_{02} - c_{01}c_{12})^2}.
\]

The first two examples demonstrate the flexibility of the rational solution compared with the polynomial ones. For each of the examples we leave \( c_{01} \) and \( c_{12} \) fixed, and we vary the values of \( c_{02} \) (Table 1).

Table 1 and Figure 4 show that the first example admits two polynomial solutions for the first two data triples \( c_{01}, c_{12}, c_{02}, \) and none for the others. The unique rational solutions exists for all the data rows. Similarly, there is a single polynomial solution for the first data triple of the second example, and two for the second one. As expected, rational solutions are obtained for all the data rows (Figure 4).
The data of two examples, with \( c_{01} = \frac{3}{4}, c_{12} = \frac{5}{6} \) in the first example, and \( c_{01} = \frac{1}{10}, c_{12} = \frac{1}{13} \) in the second one. The last table columns give the number of polynomial and rational solutions.

The last example shows that the solution path may suffer unexpected kinks extremely close to the boundary \( \partial \mathcal{P} \), and it shows that even the rational solution could not be unique. Suppose that the data cosine triple follows the line segment

\[
c_{01} = \frac{138055009909}{138240000000} \zeta, \quad c_{02} = \frac{19889197}{20000000} \zeta, \quad c_{12} = \frac{138041554309}{138240000000} \zeta, \quad (46)
\]

from \( c_{01} = c_{12} = 0 \) to the boundary \( \partial \mathcal{P} \), i.e.,

\[
0 \leq \zeta \leq \zeta_{\partial \mathcal{P}} := 20736000 \sqrt{\frac{4910936327990}{2111610941312646984855158443}} \approx 1.000000042.
\]

Let \( \varepsilon = 2 \times 10^{-7} \), and let us follow the solution depending on \( \zeta \) in the original variables \( \lambda_0, \lambda_1 \), where we normalize the weights concerned by \( w_0^2 + w_1^2 + w_2^2 = 1 \). Most of the path, the solution changes are moderate, and the solution is unique (Figure 6). Near the boundary, the solution suddenly changes, and the magnification on Figure 7 shows that at a very small interval of length \( < 3\varepsilon \) becomes triple. The polynomial \( q_6 \), introduced in (35) and [14], confirms the solution behaviour (Fig. 8). A closer look at \( q_6 = q_6(\zeta) \) reveals that it has a double root at

\[
\zeta \in \{ \zeta \approx 1 - 6.046\varepsilon, \zeta \approx 1 - 1.735\varepsilon \},
\]

and the solution of the interpolation problem at \( \zeta \in [\zeta, \zeta] \) is triple.
Figure 4: The PH interpolants of the first data example. The polynomial (double) solutions are on the left, and the rational ones are on the right.

Figure 5: The PH interpolants of the second data example. The polynomial solutions are on the left, and the rational ones are on the right.
Figure 6: The unknowns $\lambda_0, \lambda_1$ for the data (46) at $\zeta \in [0, 1 - \varepsilon]$.

Figure 7: The unknowns $\lambda_0, \lambda_1$ for the data (46) at $\zeta \in [1 - 2\varepsilon, 1 + 2\varepsilon]$.

Figure 8: The polynomial $q_6$ for the data (46) at $\zeta \in [1 - 2\varepsilon, 1 + 2\varepsilon]$ on $[0, 0.8]$ (left), and $[0.8, 1]$ (right).


