Interpolation by $G^2$ quintic Pythagorean-hodograph curves in $\mathbb{R}^d$

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Abstract

In this paper a $G^2$ continuous geometric interpolation of Hermite data by Pythagorean-hodograph (PH) quintic curves in $\mathbb{R}^d$ is considered. For two sets of appropriate Hermite data (tangent directions and curvature vectors) at two distinct points a PH quintic which interpolates given data geometrically is sought. The problem reduces to solving a system of nonlinear algebraic equations involving only geometric interpolation parameters as unknowns. Several solutions of the same quality (considering the shape of the resulting interpolants) might exist and a thorough asymptotic analysis is done to establish the existence of an odd number of asymptotic solutions having the best approximation order, i.e. six in this case. This solution for a particular set of data is then traced by homotopy to find an appropriate solution for general data. Numerical examples confirm that the method is efficient in practical computations.

Key words: Pythagorean-hodograph curve, Hermite interpolation, geometric continuity, nonlinear analysis, homotopy

1 Introduction

Pythagorean-hodograph (PH) curves form a special subclass of parametric curves which have several important properties, such as a (piecewise) polynomial arc length and a rational offset. They were first systematically studied in [1]. Since then, a lot of research work was put into studying them in detail. This is particularly true for interpolation and approximation methods using PH curves. Interpolation methods by general polynomial curves are usually based on low degrees
polynomials, i.e., up to 5. Since only odd degree PH curves are regular, this practically reduces interpolation methods by such curves to cubic and quintic cases. For planar cubic PH curves one of the first geometric interpolation methods was given in [2] where \( G^1 \) interpolation of Hermite data (i.e. the positions and the tangent directions) at two given points was analyzed. This work was later generalized in [3] to \( G^2 \) interpolation by the same objects, and in [4], where a thorough analysis of the number of solutions and their properties was done. Its Lagrange counterpart, i.e., geometric interpolation by PH cubics can be found in [5]. For quintic curves, \( C^2 \) continuous interpolating splines were constructed in [6] and Hermite type interpolants in [7].

For spatial curves, \( G^1 \) Hermite interpolation by PH cubics was thoroughly investigated in [8]. Those results were later generalized to some level in [9]. The most general results on this type of interpolation can be found in [10].

Geometric Hermite interpolation by monotone helical quintics, which form a subclass of general spatial PH curves, was done in [11]. In the concluding remarks of the paper the authors exposed an open problem of \( G^2 \) Hermite interpolation by generic spatial PH quintics.

In this paper we follow a recently developed general approach to geometric interpolation by PH curves ([12]) and apply it to the above mentioned open problem of \( G^2 \) Hermite interpolation by PH quintics. For two sets of appropriate Hermite data (tangent directions and curvature vectors) at two distinct points we try to find a PH quintic which interpolates given data in geometric sense. The problem turns out highly nonlinear, since both, the geometric interpolation conditions and PH property, involve unknown parameters in a nonlinear way. The general approach given in [12] is based on the idea that a particular set of parameters, i.e. coefficients of the interpolating quintic PH curve, is eliminated from the original system of equations and the resulting nonlinear system involves only geometric interpolation parameters. However, the analysis of the system is still a difficult task, the fact confirmed by some numerical examples which show that there might exist several solutions, which are qualitatively equivalent (all of them being without a loop e.g.). But a careful and thorough asymptotic analysis reveals that some of the solutions are approximants of a higher approximation order. This gives an idea for a general algorithm: find the solution of a particular problem which has the best approximation order and trace it by homotopy to find a solution for a given set of data. Numerical experiments show that this procedure behaves excellent in practice.

The paper is organized as follows. In Section 2 the interpolation problem is precisely stated and the system of nonlinear algebraic equations is derived. A preliminary numerical example is given in Section 3 to justify the necessity of asymptotic analysis which is done in detail. The existence of several solutions of different approximation order is established and their asymptotic behaviour is analyzed. Section 4 describes a general algorithm for solving the interpolation problem. A homotopy method tracing an appropriate solution of the particular problem to a solution of the problem for general data is explained in detail. Finally, a numerical example is given which confirms theoretical results.
2 Interpolation problem

Suppose that the data
\[ f_0, d_0, P_0, P_1, d_1, f_1 \]
are prescribed in \( \mathbb{R}^d, d \geq 2 \), where the points \( P_i \), the vectors \( d_i \) and \( f_i, i = 0, 1 \), represent the values, the first and the second derivatives of some regular parametric curve, sampled at two distinct parameter values. We will study the following interpolation problem: determine a quintic Pythagorean-hodograph curve \( p : [0, 1] \to \mathbb{R}^d \) that interpolates the prescribed data in a geometric \( G^2 \) sense. The \( G^2 \) interpolation conditions can be written as
\[
    p(u) = P_u, \quad p'(u) = \lambda_u d_u, \quad p''(u) = \lambda^2 u f_u + \mu_u d_u, \quad u = 0, 1, \tag{2}
\]
where
\[
    \lambda_i > 0, \quad \mu_i \in \mathbb{R}, \quad i = 0, 1, \tag{3}
\]
are free parameters. In addition, the interpolating curve \( p \) should satisfy the PH condition, i.e., its parametric speed
\[
    \|p'(t)\|, \quad t \in [0, 1],
\]
should be a piecewise polynomial of degree \( \leq 4 \). Here, \( \|v\| := \sqrt{v^T v} \) denotes the Euclidean norm, implied by the scalar product.

The first simplification of the interpolation problem is straightforward. If the prescribed data are based upon the arc-length parameterization of the underlying curve, we observe
\[
    \|d_i\| = 1, \quad d_i^T f_i = 0, \quad i = 0, 1. \tag{4}
\]
If this is not the case, we may apply the Gram-Schmidt orthogonalization steps
\[
    f_i \to \frac{1}{\|d_i\|^2} \left( f_i - \frac{f_i^T d_i}{\|d_i\|^2} d_i \right), \quad d_i \to \frac{1}{\|d_i\|} d_i, \quad i = 0, 1, \tag{5}
\]
so that the modified data vectors satisfy (4). Note that the interpolation conditions (2) remain of the same form if one properly renames the free parameters. Thus we shall throughout the paper assume that the original data already satisfy (4).

A proper choice of a basis of the polynomial space \( \mathbb{P}_5 \) gives the interpolating curve that satisfies (2) in a simple closed form. With
\[
    \pi_1(t) := (1 - t)^3 t(3t + 1), \quad \pi_2(t) := \frac{1}{2} (1 - t)^3 t^2, \\
    \pi_3(t) := t^3 \left( 6t^2 - 15t + 10 \right), \quad \pi_4(t) := \pi_2(1 - t), \quad \pi_5(t) := -\pi_1(1 - t),
\]
the curve \( p \) reads
\[
    p = f_0 \lambda_0^2 \pi_2 + d_0 (\lambda_0 \pi_1 + \mu_0 \pi_2) + P_0 + \Delta P_0 \pi_3 + d_1 (\lambda_1 \pi_5 + \mu_1 \pi_4) + f_1 \lambda_1^2 \pi_4, \tag{6}
\]
where $\Delta P_0 := P_1 - P_0$. Thus the linear part (2) of the interpolation problem is fulfilled. In order to derive additional equations that correspond to the PH condition, we follow [12, Sec. 3]. Let us define

$$w := \left(\lambda_0 \pi'_2, \lambda_0 \pi'_1 + \mu_0 \pi'_2, \pi'_3, \lambda_1 \pi'_5 + \mu_1 \pi'_4, \lambda_1^2 \pi'_4\right)^T,$$

and

$$V := (v_j)_{j=0}^4 := \left(f_0, d_0, \Delta P_0, d_1, f_1\right) \in \mathbb{R}^{d \times 5}.$$

The hodograph can be expressed as

$$p' = Vw,$$

and the parametric speed is given by

$$\|p'\| = \sqrt{w^T G w}, \quad G := V^T V. \tag{7}$$

The Gram matrix $G$ is a symmetric positive semidefinite matrix that depends on the given data only. The assumption (4) then simplifies the Gram matrix to

$$G = \begin{pmatrix} g_{00} & 0 & g_{02} & g_{03} & g_{04} \\ 0 & 1 & g_{12} & g_{13} & g_{14} \\ g_{02} & g_{12} & g_{22} & g_{23} & g_{24} \\ g_{03} & g_{13} & g_{23} & 1 & 0 \\ g_{04} & g_{14} & g_{24} & 0 & g_{44} \end{pmatrix}.$$

As we know, there are several approaches how to deal with PH curves. For planar curves one would usually use complex representation, for spatial ones quaternion algebra is involved. For curves of higher degree a generalization of these structures might be used leading to Clifford algebras. But there is also a straightforward way to incorporate PH conditions in the interpolation problem, i.e., to write them in a purely algebraic way. Here we found this approach the most promising one, since it led to the simplest set of nonlinear equations independently of the space dimension $d$. In order to apply it, let us recall [12, Rem. 1], with a particular choice of the points, divided differences $[\cdot]$ are based upon. The parametric speed (7) would be a polynomial of degree $\leq 4$ iff the following (scaled) equations

$$e_i := e_i(\lambda_0, \lambda_1, \mu_0, \mu_1) := \frac{1}{c_i} [0, \ldots, 0, 1, \ldots, 1] \|p'(\cdot)\| = 0, \quad i = 0, 1, \ldots, 4, \quad (8)$$

are satisfied, where constants $c_i, \quad i = 0, 1, \ldots, 4$ are equal to 8, 2, 2, 2, 8 respectively. Four equations are independent only, and one may take either the first equation $e_0 = 0$ or the last one $e_4 = 0$ as extraneous. It is straightforward to compute the
polynomials $\lambda_1 e_0, e_1, e_2, e_3, \lambda_0 e_4 \in \mathbb{R}[\lambda_0, \lambda_1, \mu_0, \mu_1]$,

\[
e_1 = \lambda_1 (\lambda_1 g_{44} (7 \lambda_1 - \mu_1) + 60 g_{24}) - \mu_0 (3 \lambda_1 g_{14} + 2 g_{13} - 2)
- \lambda_0^2 (3 \lambda_1 g_{04} + 2 g_{03}) - 8 \lambda_0 (3 \lambda_1 g_{14} + g_{13} - 1),
\]

\[
e_2 = g_{44} \lambda_1^3 - g_{00} \lambda_0^3 - 3 (g_{03} \lambda_0^2 + g_{14} \lambda_1^2) + 24 (\lambda_0 - \lambda_1) (1 - g_{13})
+ 3 ((\mu_0 + \mu_1) (1 - g_{13}) - 20 g_{12} + 20 g_{23}),
\]

\[
e_3 = \lambda_0 (60 g_{02} - \lambda_0 g_{00} (7 \lambda_0 + \mu_0)) + \mu_1 (3 \lambda_0 g_{03} - 2 g_{13} + 2)
+ \lambda_1^2 (3 \lambda_0 g_{04} - 2 g_{14}) - 8 \lambda_1 (3 \lambda_0 g_{03} - g_{13} + 1),
\]

\[
e_4 = 4 g_{03} (3 \mu_0 \mu_1 - 140 \lambda_0 \lambda_1) + \lambda_0^3 \lambda_0 (\lambda_0^2 g_{00} - 169) + 240 \mu_0 g_{02}
+ 2 (\lambda_1^2 g_{04} (37 \lambda_0 + 6 \mu_0) + \lambda_0 (-2 \mu_0^2 g_{00} + 37 \mu_1 g_{03} + 660 g_{02}))
+ 2 \lambda_0^2 g_{00} (3 g_{13} (\mu_1 - 8 \lambda_1) + 60 g_{12} - 25 \mu_0) - 8 \lambda_1 (12 \mu_0 g_{03} + g_{13} - 1)
+ \frac{1}{\lambda_0} \left(6 \lambda_1 (\lambda_1 g_{14} (\lambda_0^3 g_{00} - 24 \lambda_1 g_{14}) + 24 \mu_1 (1 - g_{13}) - 480 (g_{12} g_{13} - g_{23}))
+ 9 (\mu_1^2 (g_{13} g_{13} - 1) + 40 \mu_1 (g_{11} g_{13} - g_{13}) + 400 (g_{12} - g_{22}))
+ 9 \lambda_1^2 (\lambda_1^2 (g_{14}^2 - g_{44}) + 2 \mu_1 g_{13} g_{14} + 64 g_{13}^2 + 40 g_{12} g_{14} - 40 g_{24} - 64)\right),
\]

(9)

and similarly $e_0$. Note that the first three polynomials in (9) are linear in the variables $\mu_0$ and $\mu_1$. If we express them from $e_1$ and $e_3$, we obtain

\[
\mu_0(\lambda_0, \lambda_1) = \frac{(2 - 2 g_{13} + 3 g_{03} \lambda_0) e_1(\lambda_0, \lambda_1, 0, 0) + g_{44} \lambda_1^2 e_3(\lambda_0, \lambda_1, 0, 0)}{D(\lambda_0, \lambda_1)},
\]

\[
\mu_1(\lambda_0, \lambda_1) = \frac{(2 - 2 g_{13} - 3 g_{14} \lambda_1) e_2(\lambda_0, \lambda_1, 0, 0) + g_{00} \lambda_0^2 e_1(\lambda_0, \lambda_1, 0, 0)}{D(\lambda_0, \lambda_1)},
\]

(10)

with the denominator given as

\[
D(\lambda_0, \lambda_1) := g_{00} g_{44} \lambda_1^2 \lambda_0^2 + (3 g_{03} \lambda_0 - 2 g_{13} + 2) (3 g_{14} \lambda_1 + 2 g_{13} - 2).
\]

(11)

Further, if we insert the expressions (10) in the remaining two equations, we end up with a system of two rational equations

\[
e_i(\lambda_0, \lambda_1, \mu_0(\lambda_0, \lambda_1), \mu_1(\lambda_0, \lambda_1)) = 0, \quad i = 2, 4,
\]

(12)

for the unknowns $\lambda_0$ and $\lambda_1$. The polynomial form of (12) is given as

\[
D(\lambda_0, \lambda_1) e_2(\lambda_0, \lambda_1, \mu_0(\lambda_0, \lambda_1), \mu_1(\lambda_0, \lambda_1)) = 0,
\]

\[
D(\lambda_0, \lambda_1)^2 \lambda_0 e_4(\lambda_0, \lambda_1, \mu_0(\lambda_0, \lambda_1), \mu_1(\lambda_0, \lambda_1)) = 0.
\]

(13)

The equations (13) are of the total degree 7 and 14 respectively, quite in reach of nowadays polynomial equations solvers. However, since several solutions are to
be expected, a robust recipe that would usually give the proper solution would be appropriate.

Quite clearly, a change of sign of the data vectors, and the corresponding reparameterization \( t \rightarrow 1 - t \) of the curve \( p \) should lead to a system that evolves from (9). The following remark reflects this.

**Remark 1** Suppose that the elements of the Gram matrix \( G \) are changed as follows,

\[
\begin{align*}
g_{00} & \rightarrow g_{44}, g_{44} \rightarrow g_{00}, & g_{02} & \rightarrow -g_{24}, g_{24} \rightarrow -g_{02}, \\
g_{03} & \rightarrow g_{14}, g_{14} \rightarrow g_{03}, & g_{12} & \rightarrow -g_{23}, g_{23} \rightarrow -g_{12}.
\end{align*}
\]

If we change the unknowns

\[
\lambda_0 \rightarrow -\lambda_1, \lambda_1 \rightarrow -\lambda_0, \quad \mu_0 \rightarrow \mu_1, \mu_1 \rightarrow \mu_0,
\]

too, we obtain the same system of equations (9), since

\[
e_5 \rightarrow e_0, e_0 \rightarrow e_5, \quad e_3 \rightarrow e_1, e_1 \rightarrow e_3.
\]

3 Asymptotic analysis

Not all solutions of the equations (12) that satisfy \( \lambda_0 > 0, \lambda_1 > 0 \) are acceptable from the approximation point of view. One would expect that the interpolation curve \( p \) would be an approximation of order six, at least for moderate data. As an example, let us consider a particular curve \( g : [0, b] \rightarrow \mathbb{R}^3 \),

\[
g(t) := \frac{1}{\sqrt{5}} \begin{pmatrix}
\sqrt{5} \ln(1 + t) \cos t \\
\sqrt{1 + t^2} - 1 + 2 \ln(1 + t) \sin t \\
2 \left( \sqrt{1 + t^2} - 1 \right) - \ln(1 + t) \sin t
\end{pmatrix}, \quad (14)
\]

interpolated at 0 and \( b \).

Fig. 1. Five solutions of a particular \( G^2 \) quintic interpolation problem.

A choice \( b = \frac{1}{4} \) yields five admissible solutions (Fig. 1). One of them is obviously unacceptable, and there is an undesirable kink in the other solution too. The remaining three seem to be similar at a first glance. So we are faced with a question, discussed quite often in the literature (see, e.g., [6,7,13]): which solution should be returned by a computer program as the most sensible one? But the data here are provided by the known curve \( g \), and some further analysis can be carried out. Tab. 1
Table 1
The Hausdorff distance and parameters $\lambda_i$, $\mu_i$, $i = 0, 1$, with $b = \frac{1}{4}$ for all five solutions of a particular $G^2$ quintic interpolation problem.

<table>
<thead>
<tr>
<th>Hausdorff distance</th>
<th>$\frac{1}{b} \lambda_0$</th>
<th>$\frac{1}{b} \lambda_1$</th>
<th>$\frac{1}{b} \mu_0$</th>
<th>$\frac{1}{b} \mu_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.23236 \times 10^{-6}$</td>
<td>1.03695</td>
<td>-0.40493</td>
<td>0.036217</td>
</tr>
<tr>
<td>2</td>
<td>$1.45516 \times 10^{-3}$</td>
<td>1.27367</td>
<td>-11.1875</td>
<td>1.69003</td>
</tr>
<tr>
<td>3</td>
<td>$1.41561 \times 10^{-1}$</td>
<td>2.51591</td>
<td>-3.09427</td>
<td>9.94449</td>
</tr>
<tr>
<td>4</td>
<td>$1.71950 \times 10^{-2}$</td>
<td>3.02391</td>
<td>-13.1529</td>
<td>11.2369</td>
</tr>
<tr>
<td>5</td>
<td>$3.51152 \times 10^{-1}$</td>
<td>2.62773</td>
<td>-13.2049</td>
<td>11.1413</td>
</tr>
</tbody>
</table>

(right), reveals the Hausdorff distance between $g$ and the interpolating curves. Quite clearly, the first solution is the one to be chosen. However, since this error estimate

Fig. 2. Decay rates of the Hausdorff distance between $g$ and the five polynomial interpolants depending on the interval length $b$. The solutions of decay 2 are almost indistinguishable.

selection is not computable in the general case, let us trace the solutions obtained as functions of decreasing $b$. If the Hausdorff distance decays as $\approx \text{const} \ b^{-\alpha}$, the decay rate $\alpha$ may be estimated from Hausdorff distances computed for consecutive $b$-s. The result of the numerical experiment, Fig. 2, indicates that the solutions belong to three entirely different approximation order classes: $\alpha \approx 0, 2, 6$. This gives a motivation for a rigorous asymptotic analysis as well as a hint for a numerical algorithm that gives an appropriate solution of the equations (12): a continuation method [14, e.g.] that begins with a model problem and its particular solution of the approximation order six.

In order to analyse the asymptotic behaviour of the solutions $\lambda_i$, let us assume that the data are sampled from a smooth parametric curve $r : \left[ -\frac{h}{2}, \frac{h}{2} \right] \to \mathbb{R}^d$, $2 \leq d$, parameterized by the arc-length $s$. Although we have to choose $d = 5$ in order to avoid possible irregular PH solution [12, Lemma 2], for the asymptotic analysis it does not really matter which particular dimension $d$ one selects. The analysis will therefore be given for the most natural choice, i.e., $d = 3$. The data to be interpolated are

$$
P_0 = r \left( -\frac{h}{2} \right), \quad d_0 = r' \left( -\frac{h}{2} \right), \quad f_0 = r'' \left( -\frac{h}{2} \right),
$$

$$
P_1 = r \left( \frac{h}{2} \right), \quad d_1 = r' \left( \frac{h}{2} \right), \quad f_1 = r'' \left( \frac{h}{2} \right), \quad (15)
$$

where the interval length $h$ is assumed to be as small as needed. Suppose also that the Frenet frame of $r$, denoted $\mathcal{F}(r)$, is well defined for all $s \in \left[ -\frac{h}{2}, \frac{h}{2} \right]$. Without loss of generality we may choose the coordinate system with the origin at $r(0)$, and the axes determined by the Frenet frame of $r$ at $t = 0$, which somewhat simplifies the expansion of $r$. Suppose that the curvature $\kappa$ and the torsion $\tau$ of the curve $r$ at
it is straightforward to derive the expansion of \(r\) and \(s\) from the Frenet-Serret formulas as

\[
\kappa(s) = \kappa_0 + \frac{\kappa_1}{1!} s + \frac{\kappa_2}{2!} s^2 + \frac{\kappa_3}{3!} s^3 + \mathcal{O}(s^4), \quad \tau(s) = \tau_0 + \frac{\tau_1}{1!} s + \frac{\tau_2}{2!} s^2 + \frac{\tau_3}{3!} s^3 + \mathcal{O}(s^4),
\]

(16)

where the fact \(\kappa_0 > 0\) will be needed throughout this section. Let us recall (16). Since \(r' = (\mathcal{F}(r))_1\), and we have assumed

\[
r(0) = (0, 0, 0)^T, \quad \mathcal{F}(r)(0) = (\delta_{ij})_{i,j=1}^3,
\]

it is straightforward to derive the expansion of \(r\),

\[
r(s) = r(0) + r'(0) \frac{1}{1!} s + r''(0) \frac{1}{2!} s^2 + \ldots,
\]

from the Frenet-Serret formulas as

\[
r(s) = \begin{pmatrix}
    s - \frac{1}{6} \kappa_0^2 s^3 - \frac{1}{5} \kappa_0 \kappa_1 s^4 + \mathcal{O}(s^5) \\
    \frac{1}{2} \kappa_0 s^2 + \frac{1}{4} \kappa_1 s^3 - \frac{1}{24} (\kappa_0^2 + \tau_0^2 \kappa_0 - \kappa_2) s^4 + \mathcal{O}(s^5) \\
    \frac{1}{6} \kappa_0 \tau_0 s^3 + \frac{1}{24} (2 \kappa_1 \tau_0 + \kappa_0 \tau_1) s^4 + \mathcal{O}(s^5)
\end{pmatrix}.
\]

(17)

With a help of (17) we obtain expansions of the asymptotic data in (15). The elements \(g_{ij} := g_{ij}(h)\) of the Gram matrix expand as

\[
g_{00}(h) = \kappa_0^2 - \kappa_0 \kappa_1 h + \frac{1}{4} (\kappa_0 \kappa_2 + \kappa_1^2) h^2 - \frac{1}{24} (\kappa_0 \kappa_3 + 3 \kappa_1 \kappa_2) h^3 + \mathcal{O}(h^4),
\]

\[
g_{22}(h) = h^2 - \frac{1}{12} \kappa_0^2 h^4 + \mathcal{O}(h^6), \quad g_{44}(h) = g_{00}(-h),
\]

and

\[
g_{02}(h) = \frac{1}{2} \kappa_0^2 h^2 - \frac{1}{3} \kappa_0 \kappa_1 h^3 + \mathcal{O}(h^4),
\]

\[
g_{03}(h) = \kappa_0^3 h - \frac{1}{2} \kappa_0 \kappa_1 h^2 - \frac{1}{6} \kappa_0 \left( \kappa_0^2 + \kappa_0 \tau_0^2 - \kappa_2 \right) h^3 + \mathcal{O}(h^4),
\]

\[
g_{04}(h) = \kappa_0^2 + \frac{1}{4} \left( \kappa_0 (\kappa_2 - 2 \kappa_0 (\kappa_0^2 + \tau_0^2)) - \kappa_1^2 \right) h^2 + \mathcal{O}(h^4),
\]

\[
g_{12}(h) = h - \frac{1}{6} \kappa_0^2 h^3 + \mathcal{O}(h^4),
\]

\[
g_{13}(h) = 1 - \frac{1}{2} \kappa_0^2 h^2 + \mathcal{O}(h^4),
\]

\[
g_{14}(h) = g_{03}(-h), \quad g_{23}(h) = -g_{12}(-h), \quad g_{24}(h) = -g_{02}(-h).
\]

(19)

A brief look at Remark 1, and the expansions (18) and (19) prove the following lemma.

**Lemma 1** The asymptotic form of the system (9) has a solution

\[
\lambda_0 = \lambda_0(h), \quad \lambda_1 = \lambda_1(h), \quad \mu_0 = \mu_0(h), \quad \mu_1 = \mu_1(h)
\]
for $h$ small enough if and only if
\[
\begin{align*}
\lambda_0 &= -\lambda_1(-h), \quad \lambda_1 = -\lambda_0(-h), \\
\mu_0 &= \mu_1(h), \quad \mu_1 = \mu_0(-h)
\end{align*}
\]
is a solution too.

As $h \to 0$, the parameters $\lambda_i$ of a regular curve should stay positive and bounded, so we may assume that they behave as
\[
\begin{align*}
\lambda_0 &= \xi_0 h^\alpha + o(h^\alpha), \quad \lambda_1 = \xi_1 h^\beta + o(h^\beta), \\
\xi_0 > 0, \xi_1 > 0, \quad \alpha \geq 0, \beta \geq 0. \quad (20)
\end{align*}
\]

**Lemma 2** Suppose that $\kappa_0 \neq 0$, and $h > 0$ is small enough. Further, let the unknowns $\lambda_i$ expand as (20). The only possible exponents $\alpha, \beta$ are
\[
\alpha = \beta = 0, \quad (21)
\]
or
\[
\alpha = \beta = 1. \quad (22)
\]

**PROOF.** Let us insert the expansions (18) and (19) in the polynomial form of the system (13). The first equation $e_2$ expands as
\[
\begin{align*}
e_2 = & \kappa_0^6 \lambda_0^3 \lambda_1^2 \left(\lambda_1^3 - \lambda_0^3\right) \left(1 + \mathcal{O}(h)\right) + 3 \kappa_0^6 \lambda_0^3 \lambda_1^2 \left(\lambda_1^2 - \lambda_0^2\right) \left(h + \mathcal{O}(h^2)\right) \\
+ & 3 \kappa_0^6 \lambda_0^3 \lambda_1 \left(3 \lambda_0^2 + 4 \lambda_1 \lambda_0 + 3 \lambda_1^2\right) \left(h^2 + \mathcal{O}(h^3)\right) \\
+ & 21 \kappa_0^6 \lambda_0^3 \lambda_1 \left(\lambda_1^2 - \lambda_0^2\right) \left(h^3 + \mathcal{O}(h^4)\right) \\
- & \frac{1}{2} \kappa_0^5 \left(\kappa_0 \lambda_0 - \lambda_1\right) \left(7 \lambda_0^2 + 88 \lambda_1 \lambda_0 + 7 \lambda_1^2\right) \left(h^4 + \mathcal{O}(h^5)\right) \\
- & 18 \kappa_0^6 \left(\lambda_0^2 - \lambda_1^2\right) \left(h^5 + \mathcal{O}(h^6)\right) + 39 \kappa_0^6 \left(\lambda_0 - \lambda_1\right) \left(h^6 + \mathcal{O}(h^7)\right) \\
+ & 5 \kappa_0^5 \left(\kappa_1 + \frac{3}{4} \left(\lambda_0 \lambda_1 \tau_0 \left(\kappa_1 \tau_1 - \kappa_0 \tau_2\right)\right)\right) \left(h^8 + \mathcal{O}(h^9)\right). \quad (23)
\end{align*}
\]

If we insert (20) in (23), we obtain a sum of terms with various exponents of $h$, depending on $\alpha$ and $\beta$. Six of the terms, each of them in a particular subregion of $\{ (\alpha, \beta) \mid \alpha \geq 0, \beta \geq 0 \}$ respectively, become dominant as $h \to 0$ (Table 2). This leaves us with the following possible exponents
\[
\begin{align*}
0 \leq \alpha = \beta & < 2, \quad (24) \\
\alpha + \beta = 2 \wedge \alpha \geq 0 \wedge \beta \geq 0, \quad (25) \\
\alpha = 1 \wedge \beta > 1, \quad \beta = 1 \wedge \alpha > 1, \quad \alpha \geq 2 \wedge \beta \geq 2, \quad (26)
\end{align*}
\]
which could not be excluded by a single dominating term (Fig. 3). The expansion Fig. 3. Exponents of the dominating terms and the corresponding dominance region. The part of the first quadrant $\alpha \geq 0, \beta \geq 0$ that can not be excluded is shown in black.
The dominating terms and the corresponding dominance region.

Table 2
The dominating terms and the corresponding dominance region.

of the second equation $e_4$ is rather long. But the particular case simplification

$$e_4 = -900\kappa_{10}^{10}h^{12} + o\left(h^{12}\right), \quad \alpha > 1 \wedge \beta > 1,$$

rules out the last possibility in (26). It also shortens (24) to $0 \leq \alpha = \beta \leq 1$. In the latter case, if $\alpha \in (0, 1)$, the equations expand as

$$e_2 = -\kappa_0^{10}\xi_0^3\xi_1^3 \left(\xi_0^3 - \xi_1^3\right) h^{7\alpha} + o\left(h^{7\alpha}\right),$$

$$e_4 = -\kappa_0^{10}\xi_0^3\xi_1^3 \left(18\xi_0^5 + 15\xi_1\xi_0^4 + 6\xi_2\xi_0^3 - 8\xi_3\xi_0^2 + 9\xi_1\xi_0\right) + o\left(h^{12\alpha}\right).$$

The leading terms can not both vanish since the leading one of $e_2$ implies $\xi_1 = \xi_0$, but then

$$e_4 = -40\kappa_0^{10}\xi_0^{12} + o\left(h^{12\alpha}\right).$$

This proves that the only possible exponents in the case (24) are given by (21). Let us consider now the case (25). The main part of the expansion reads

$$e_4 = 6\xi_0^7 \left(1 - \xi_0\xi_1\right) \left(2\xi_0\xi_1 - 7\right) h^{7\alpha+5} + o\left(h^{7\alpha+5}\right), \quad 0 \leq \alpha < 1,$$

$$e_4 = -\kappa_0^{10}\xi_1^6 \left(9 - 4\xi_0\xi_1\right)^2 + o\left(h^{18-6\alpha}\right), \quad 1 < \alpha \leq 2. \quad (27)$$

Both leading terms in (27) may vanish. However, these particular cases give

$$\xi_1 = \frac{1}{\xi_0} : e_4 = -25\kappa_0^{10}\xi_0^6 h^{6\alpha+6} + o\left(h^{6\alpha+6}\right), \quad 0 \leq \alpha < 1,$$

$$\xi_1 = \frac{2}{\xi_0} : e_4 = 200\kappa_0^{10}\xi_0^6 h^{6\alpha+6} + o\left(h^{6\alpha+6}\right), \quad 0 \leq \alpha < 1,$$

$$\xi_1 = \frac{9}{4\xi_0} : e_4 = -\frac{178623225\kappa_0^{10}h^{16-4\alpha}}{65536\xi_0^4} + o\left(h^{16-4\alpha}\right), \quad 1 < \alpha \leq 2,$$
which eliminates all the possible \( \alpha \) but \( \alpha = 1 \). Finally, the expansions

\[
e_2 = \frac{1}{2} \kappa_0^6 \xi_1 (\xi_1 (7\xi_1 + 36) - 78) h^2 + o \left( h^7 \right), \quad \alpha > 1, \beta = 1,
\]

\[
e_4 = -9\kappa_0^{10} (3\xi_1 + 1)^2 (\xi_1 (\xi_1 + 4) - 10)^2 + o \left( h^{12} \right), \quad \alpha > 1, \beta = 1,
\]

eliminate the possibility \( \alpha > 1, \beta = 1 \), since the leading terms could not vanish simultaneously. The case \( \alpha = 1, \beta > 1 \) follows similarly. The proof is completed. \( \square \)

**Lemma 3** Let the assumptions of Lemma 2 be fulfilled. If \( \alpha = \beta = 0 \), the unknowns \( \lambda_i \) and \( \mu_i \) expand as

\[
\lambda_0 = \frac{2\sqrt{10}}{\kappa_0} + \frac{2\sqrt{10}\kappa_1}{3\kappa_0^2} h + \mathcal{O}(h^2), \quad \lambda_1 = \frac{2\sqrt{10}}{\kappa_0} - \frac{2\sqrt{10}\kappa_1}{3\kappa_0^2} h + \mathcal{O}(h^2),
\]

\[
\mu_0 = -\frac{8\sqrt{10}}{\kappa_0} + \mathcal{O}(h), \quad \mu_1 = \frac{8\sqrt{10}}{\kappa_0} + \mathcal{O}(h), \quad (28)
\]

**PROOF.** At the limit \( h = 0 \), the unknowns \( \lambda_0 = \xi_0, \lambda_1 = \xi_1 \) should satisfy the equations \( e_2 = 0, e_4 = 0 \) when \( h = 0 \). This gives

\[
e_{2|h=0} = \pi_1(\xi_0, \xi_1) = 0, \quad e_{4|h=0} = \pi_2(\xi_0, \xi_1) = 0,
\]

where

\[
\pi_1(\xi_0, \xi_1) := \kappa_0^6 \xi_0^3 \xi_1^2 \left( \xi_1^3 - \xi_0^3 \right),
\]

\[
\pi_2(\xi_0, \xi_1) := \kappa_0^{10} \xi_1^4 \xi_0^3 \left( \kappa_0^{10} \xi_0^3 \xi_1^6 - 18\xi_0^5 - 15\xi_1^4 - 6\xi_1^2 \xi_0^3 + 8\xi_1^2 \xi_0^3 - 9\xi_1^2 \right).
\]

The first polynomial of the Gröbner basis of the ideal \( \langle \pi_1, \pi_2 \rangle \) with respect to variables \( (\xi_0, \xi_1) \) is determined as

\[
\kappa_0^{10} \xi_0^3 \xi_1^{12} \left( \kappa_0^6 \xi_0^3 \xi_1^2 - 40 \right) \left( \kappa_0^4 \xi_1^4 - 5\kappa_0^2 \xi_1^2 + 25 \right).
\]

This polynomial has precisely one positive root \( \xi_1 = \frac{2\sqrt{3\pi}}{\kappa_0} \), and the rest of the basis vanishes at \( \xi_0 = \frac{2\sqrt{3\pi}}{\kappa_0} \). This gives the limit solutions \( \lambda_i = \xi_i, i = 1, 2 \). The Jacobian at the limit solution is nonsingular, since its determinant equals

\[
-\frac{3145728000000000\sqrt{10}}{\kappa_0}.
\]

But then the Implicit function theorem implies that the unknowns \( \lambda_i \) admit the Taylor series expansion,

\[
\lambda_0 = \frac{2\sqrt{10}}{\kappa_0} + \xi_0 h + \mathcal{O}(h^2), \quad \lambda_1 = \frac{2\sqrt{10}}{\kappa_0} + \xi_1 h + \mathcal{O}(h^2). \quad (29)
\]
If we insert (29) in the expansions of \( e_2 \) and \( e_4 \), the \( \mathcal{O}(h) \) terms determine the constants \( \zeta_0 \) and \( \zeta_1 \), which confirms the first row of (28). The second one follows then from the expansion of (11). 

Let us consider now the second possible exponent choice (22), i.e., \( \alpha = \beta = 1 \). The leading terms of equations turn out rather long. If we compute the Gröbner basis of both with respect to variables \( (\xi_0, \xi_1) \), the first basis function, independent of \( \xi_0 \), turns out as

\[
\kappa_0^{10} (\xi_1 - 1)^9 (\xi_1 - 3) (\xi_1^2 - 3\xi_1 - 1)^2 \pi_3(\xi_1) \pi_4(\xi_1) \pi_5(\xi_1)^2 h^{12},
\]

where

\[
\begin{align*}
\pi_3(\xi_1) & := (\xi_1 + 1)^3 \left( \xi_1^2 + 8\xi_1 + 1 \right)^3 \left( 2\xi_1^2 + 8\xi_1 + 15 \right), \\
\pi_4(\xi_1) & := \xi_1^6 - 28\xi_1^5 + 823\xi_1^4 - 1236\xi_1^3 + 2478\xi_1^2 - 19008\xi_1 + 20736, \\
\pi_5(\xi_1) & := 4\xi_1^{10} + 51\xi_1^9 + 256\xi_1^8 + 618\xi_1^7 + 2506\xi_1^6 + 2025\xi_1^5 + 5612\xi_1^4 - 22578\xi_1^3 - 29560\xi_1^2 - 12003\xi_1 - 1606.
\end{align*}
\]

The polynomial \( \pi_3 \) has obviously no positive roots. The only positive root \( \xi_1 \approx 1.7811 \) of the polynomial \( \pi_5 \) determines \( \xi_0 \approx -0.316622 \) from the rest of equations. The following argument rules out also the positive root \( \xi_1 = \frac{1}{2} \left( 3 + \sqrt{13} \right) \) of the polynomial \( \xi_1^2 - 3\xi_1 - 1 \). The remaining polynomials of the Gröbner basis determine the corresponding \( \xi_0 = \frac{1}{2} \left( 3 + \sqrt{13} \right) \). But the leading term of the denominator (11) expansion

\[
D (\xi_0 h, \xi_1 h) = \kappa_0^4 \left( \xi_0^2 \xi_1^2 - 9\xi_0\xi_1 - 3\xi_1 - 3\xi_0 - 1 \right) h^4 + \mathcal{O}(h^5)
\]

vanishes at

\[
\xi_0 = \xi_1 = \frac{1}{2} \left( 3 + \sqrt{13} \right).
\]

However, the second rational equation in (12) expands at \( \xi_i \) values (30) as

\[
e_4(\xi_0 h, \xi_1 h, \mu_0(\xi_0 h, \xi_1 h), \mu_1(\xi_0 h, \xi_1 h)) = \frac{135}{2} \left( 659 - 183\sqrt{13} \right) \kappa_0^2 h^3 + \mathcal{O}(h^4).
\]

So the solution (30) is extraneous, and it is due to the switch from the rational (12) to the polynomial form (13) of the equations. This yields the possible choices \( \xi_1 = 1, 3, u_0, u_1 \), where \( u_0 \approx 1.27927 \) and \( u_1 \approx 2.55324 \) are positive roots of \( \pi_4 \). The rest of the Gröbner basis gives the corresponding values \( \xi_0 = 1, 3, u_1, u_0 \) respectively. The last three pairs \( (\xi_0, \xi_1) \) produce also a nonsingular Jacobian of the polynomial part of the scaled equations

\[
\begin{align*}
\frac{1}{h^7} e_2 (\xi_0 h, \xi_1 h, \mu_0(\xi_0 h, \xi_1 h), \mu_1(\xi_0 h, \xi_1 h)) + \mathcal{O}(h), \\
\frac{1}{h^{12}} e_4 (\xi_0 h, \xi_1 h, \mu_0(\xi_0 h, \xi_1 h), \mu_1(\xi_0 h, \xi_1 h)) + \mathcal{O}(h),
\end{align*}
\]

(31)
at the limit solution. So one can follow the steps of the proof of Lemma 3 and confirm the following lemma in a similar manner. Of course, some further expansions have to be carried out on the way, but Lemma 1 shortens them significantly: if \( \lambda_0 = \xi_0 h + \zeta_0 h^2 + O(h^3) \), \( \lambda_1 = \xi_1 h + \zeta_1 h^2 + O(h^3) \) is a simple admissible solution pair then if \( \xi_0 \neq \xi_1 \), there exists also an admissible solution pair \( \lambda_0 = \xi_1 h - \zeta_1 h^2 + O(h^3) , \lambda_0 = \xi_0 h - \zeta_0 h^2 + O(h^3) \). If \( \xi_0 = \xi_1 \) then the solution should be of the form \( \lambda_0 = \xi_0 h + \zeta_0 h^2 + O(h^3) , \lambda_1 = \xi_0 h - \zeta_0 h^2 + O(h^3) \).

**Lemma 4** Let the assumptions of Lemma 2 be fulfilled. The system (9) has in the case \( \alpha = \beta = 1 \) three simple asymptotic solutions given by

\[
\lambda_0 = \lambda_0(h) := 3h + \frac{11\kappa_1}{12\kappa_0} h^2 + O(h^3), \quad \lambda_1 = -\lambda_0(-h), \\
\mu_0 = \mu_0(h) := -12h + O(h^2), \quad \mu_1 = \mu_0(-h),
\]

and

\[
\lambda_0 = \lambda_0(h) := u_0 h + \zeta_0 h^2 + O(h^3), \quad \lambda_1 = \lambda_1(h) := u_1 h + \zeta_1 h^2 + O(h^3), \\
\mu_0 = \mu_0(h) := v_0 h + O(h^2), \quad \mu_1 = \mu_1(h) := v_1 h + O(h^2), \\
\lambda_0 = -\lambda_1(-h), \quad \lambda_1 = -\lambda_0(-h), \quad \mu_0 = \mu_1(-h), \quad \mu_1 = \mu_0(-h),
\]

with

\[
\zeta_0 = -\frac{0.653557\kappa_1}{\kappa_0}, \quad \zeta_1 = \frac{0.28438\kappa_1}{\kappa_0}, \quad \nu_0 = -1.79886, \quad \nu_1 = 11.0729.
\]

The case \( \xi_0 = \xi_1 = 1 \) requires some additional attention since at this limit solution the Jacobian of the equations (31) vanishes identically as \( h \to 0 \).

**Lemma 5** Let us suppose again that the assumptions of Lemma 2 are fulfilled. Besides the asymptotic solutions established in Lemma 4, the system (9) has for \( \alpha = \beta = 1 \) in the general case an odd number \( \leq 15 \) of simple asymptotic solutions. All of them are of the form

\[
\lambda_0 = \lambda_0(h) = h + \zeta h^2 + \rho h^3 + \theta h^4 + O(h^5), \quad \lambda_1 = -\lambda_0(-h), \\
\mu_0 = \mu_0(h) = -2\zeta h^2 - 6\rho h^3 + \chi h^4 + O(h^5), \quad \mu_1 = \mu_0(-h).
\]

Here, \( \zeta \) and \( \rho \) are real solutions of a particular system of polynomial equations (36). The term general excludes some exceptional relations between the coefficients of the curvature expansions that may allow multiple solutions.

**PROOF.** Since the expansion of the unknowns \( \lambda_i \) starts with the linear term \( h \), the
expansion should be of the form
\[
\lambda_0 = h + h (\zeta_0 h^\alpha + o(h^\alpha)), \quad \lambda_1 = h + h \left(\zeta_1 h^\beta + o(h^\beta)\right), \quad \zeta_0 \neq 0, \quad \zeta_1 \neq 0,
\]
with \(\alpha, \beta > 0\). The leading terms
\[
e_2 = 15\kappa_0^6 \zeta_0^3 h^{3\alpha+7} + o\left(h^{3\alpha+7}\right), \quad 0 < \alpha < 1 \land \alpha < \beta,
\]
\[
e_2 = -15\kappa_0^6 \zeta_1^3 h^{3\beta+7} + o\left(h^{3\beta+7}\right), \quad 0 < \beta < 1 \land \beta < \alpha,
\]
rule out all the possible exponents in \(\mathcal{D} := \{ (\alpha, \beta) | 0 < \alpha < 1 \lor 0 < \beta < 1 \}\) but \(\alpha = \beta\). In the latter case we obtain
\[
e_2 = 15\kappa_0^6 \zeta_0 - \zeta_1) (\zeta_0 + \zeta_1)^2 h^{3\alpha+7} + o\left(h^{3\alpha+7}\right),
\]
\[
e_4 = 1800\kappa_0^{10} (\zeta_0 + \zeta_1)^2 (4\zeta_0 + \zeta_1) h^{3\alpha+12} + o\left(h^{3\alpha+12}\right).
\]
This additionally implies \(\zeta_1 = -\zeta_0\), but then
\[
e_2 = -20\kappa_0^6 \zeta_1 h^{5\alpha+7} + o\left(h^{5\alpha+7}\right)
\]
rules out \(\mathcal{D}\) completely. So we may assume \(\alpha = \beta = 1\), but we additionally allow that \(\zeta_0, \zeta_1\) might depend on \(h\) as decreasing functions of the argument. Thus we obtain
\[
e_2 = 30\kappa_0^5 \Delta_0^2 (2\kappa_0 \Delta_0 - \kappa_1) h^{10} + \cdots,
\]
\[
e_4 = 1800\kappa_0^9 \Delta_0^2 (20\kappa_0 \Delta_0 + 12\kappa_0 \Delta_1 - 3\kappa_1) h^{15} + \cdots,
\]
with \(\Delta_0 := \frac{\zeta_0 + \zeta_1}{2}, \Delta_1 := \frac{\zeta_0 - \zeta_1}{2}\). A particular linear combination of these equations reveals
\[
\frac{1}{36000\kappa_0^9 h^{15}} \left(e_4 - 1800\kappa_0^9 h^5 e_2\right) = \pi_6(\Delta_0, \Delta_1) + O(h^4),
\]
\[
\pi_6(u, v) := u^3 + \pi_7(u, v) u^2 h + \pi_8(u, v) u h^2 + \pi_9(u, v) h^3.
\]
The coefficients of polynomials \(\pi_i, i = 7, 8, 9\) depend on the data curvature only, but not on \(h\). The expansion (34) clearly implies
\[
\Delta_0 = \rho h + o(h), \quad \rho \in \mathbb{R},
\]
and let also \(\Delta_1 = \zeta + o(1), \zeta \in \mathbb{R}\). These assumptions give two polynomial equations for the unknowns \(\rho\) and \(\zeta\). From (33) we obtain
\[
\frac{1}{h^2} e_2 := \pi_{10}(\zeta, \rho) + o(1), \quad \pi_{10}(\zeta, \rho) = \pi_{12}(\zeta) \rho^2 + \pi_{13}(\zeta) \rho + \pi_{14}(\zeta),
\]
\[
\pi_{10}(\zeta, \rho) = \pi_{12}(\zeta) \rho^2 + \pi_{13}(\zeta) \rho + \pi_{14}(\zeta),
\]
and the relation (34) gives
\[
\frac{1}{h^3} \pi_6 (\rho h + o(h), \zeta + o(1)) =: \pi_{11} (\zeta, \rho) + o(1),
\]
\[
\pi_{11} (\zeta, \rho) = \rho^3 + \pi_{15} (\zeta) \rho^2 + \pi_{16} (\zeta) \rho + \pi_{17} (\zeta).
\]

The coefficient polynomials involved in \( \pi_{10} \) and \( \pi_{11} \) are
\[
\pi_{12} (\zeta) = 4\kappa_0 \zeta - \kappa_1, \quad \pi_{13} (\zeta) = -\frac{2}{3} \kappa_0 \zeta^3 - \frac{3}{2} \kappa_1 \zeta^2 + \ldots,
\]
\[
\pi_{14} (\zeta) = -\frac{2}{3} \kappa_0 \zeta^5 + \kappa_1 \zeta^4 + \ldots,
\]
\[
\pi_{15} (\zeta) = -\frac{7}{10} \zeta^2 - \frac{9}{20} \kappa_1 \zeta + \ldots, \quad \pi_{16} (\zeta) = -\frac{13}{60} \zeta^4 + \frac{23}{40} \kappa_1 \zeta^3 + \ldots,
\]
\[
\pi_{17} (\zeta) = -\frac{43}{360} \zeta^6 + \frac{29 \kappa_1}{120 \kappa_0} \zeta^5 + \ldots.
\]

The system of polynomial equations that should determine \( \zeta \) and \( \rho \) finalizes as
\[
\pi_{10} (\zeta, \rho) = 0, \quad \pi_{11} (\zeta, \rho) = 0.
\]  

(36)

A little of the computer algebra work verifies a reduction
\[
\pi_{12} (\zeta) \pi_{11} (\zeta, \rho) = \pi_{18} (\zeta, \rho) \pi_{10} (\zeta, \rho) + \pi_{19} (\zeta) \rho + \pi_{20} (\zeta).
\]  

(37)

If the unknowns \( \zeta, \rho \) satisfy (36), \( \rho \) could be expressed in terms of \( \zeta \) from (37),
\[
\rho = -\frac{\pi_{20} (\zeta)}{\pi_{19} (\zeta)}.
\]  

(38)

But then the polynomial form of the first equation in (36) factorises as
\[
\pi_{19} (\zeta) \pi_{10} (\zeta, -\pi_{20} (\zeta) / \pi_{19} (\zeta)) = \pi_{12}^2 (\zeta) \pi_{21} (\zeta),
\]
where
\[
\pi_{21} (\zeta) = \frac{700}{243} \kappa_0^3 \zeta^{15} - \frac{875}{81} \kappa_0^2 \kappa_1 \zeta^{14} + \ldots.
\]  

(39)

Thus the equation (39) has an odd number of simple real solutions in general, and from (38) consequently the system (36) too. The exceptional cases are determined by particular varieties of the curvature expansion coefficients. All of them are excluded by assumptions
\[
1 \in \langle \pi_{12}, \pi_{10}, \pi_{11} \rangle \subset \mathbb{R} [\zeta, \rho], \quad 1 \in \langle \pi_{21}, \pi_{21}' \rangle \subset \mathbb{R} [\zeta, \rho].
\]

If \( (\zeta, \rho) \) is a simple solution of the system (36), the unknowns \( \Delta_i \) admit series expansion in \( h \), and so do \( \lambda_i \). By Lemma 1, the latter has to be of the form (32). The expansion of \( \mu_i \) is then straightforward. The proof is concluded. \( \square \)
Theorem 2 Suppose that $\kappa_0 \neq 0$, and $h > 0$ is small enough. The system (9) has asymptotically an $O(1)$ solution, determined in Lemma 3. Further, all the solutions, established in Lemma 4, are of the order $O(h^2)$. At last, there is an odd number of optimal $O(h^6)$ solutions, confirmed in Lemma 5.

PROOF. The existence part of the assertions has already been proved in the lemmas. It remains to prove the asymptotic approximation order only. Let us recall the parametric distance (see [15], e.g.) as a measure of the distance between parametric curves $f : [c, d] \to \mathbb{R}^d$ and $g : [a, b] \to \mathbb{R}^d$, defined as

$$d_{\varphi}(f, g) := \inf_{\varphi} \max_{a \leq t \leq b} \| (f \circ \varphi)(t) - g(t) \|, \quad (40)$$

where the infimum is taken among all diffeomorphisms $\varphi : [a, b] \to [c, d]$, and $\| \cdot \|$ is the usual Euclidean norm. Any particular reparameterization $\varphi$ gives an upper bound on the parametric distance (40). Here, $f = r : [-\frac{h}{2}, \frac{h}{2}] \to \mathbb{R}^d$ is a smooth data curve, and $g = p : [0, 1] \to \mathbb{R}^d$ is the quintic PH interpolating polynomial curve.

Let us consider the asymptotic solutions, determined in Lemma 5, and let us choose the reparameterization $\varphi : [0, 1] \to [-\frac{h}{2}, \frac{h}{2}]$ as a quintic polynomial, determined by conditions

$$\varphi(0) = -\frac{h}{2}, \varphi(1) = \frac{h}{2}, \varphi'(0) = \lambda_0, \varphi'(1) = \lambda_1, \varphi''(0) = \mu_0, \varphi''(1) = \mu_1. \quad (41)$$

The quintic polynomial curve $p$ agrees with $r \circ \varphi$ three-fold at 0 and at 1, respectively. Thus the Newton form of the interpolation error reads

$$(r \circ \varphi)(t) - p(t) = t^3(t - 1)^3 [0, 0, 0, 1, 1, 1, t] (r \circ \varphi)$$

$$= t^3(t - 1)^3 (r \circ \varphi)(6)(\eta), \quad \eta \in [0, 1].$$

The chain rule applied to $r \circ \varphi$ reveals that the error term would be $O(h^6)$ if the derivatives of the reparameterization satisfy $\varphi^{(\ell)} = O(h^\ell), \ell = 1, 2, \ldots, 5$ on $[0, 1]$. Note additionally that $\varphi^{(6)} \equiv 0$. Perhaps the simplest way to confirm the assertion is to compute the divided difference table that determines Newton forms of $\varphi$. The
upper diagonal that gives the first one reads

\[ [0] \varphi = -\frac{h}{2}, \]
\[ [0, 0] \varphi = \lambda_0 = h + \mathcal{O}(h^2), \]
\[ [0, 0, 0] \varphi = \mu_0 = -\zeta h^2 + \mathcal{O}(h^3), \]
\[ [0, 0, 0, 1] \varphi = h - \lambda_0 - \frac{\mu_0}{2} = 2\rho h^3 + \mathcal{O}(h^4), \]
\[ [0, 0, 1, 1] \varphi = -3h + 2\lambda_0 + \lambda_1 + \frac{\mu_0}{2} = \left(\theta + \frac{1}{2} \chi\right) h^4 + \mathcal{O}(h^5), \]
\[ [0, 0, 1, 1, 1] \varphi = 6h - 3\lambda_0 - 3\lambda_1 - \frac{\mu_0}{2} + \frac{\mu_1}{2} = \mathcal{O}(h^5). \]

So

\[ \varphi'(t) = \left( [0] \varphi + [0, 0] \varphi t + [0, 0, 0] \varphi t^2 + \cdots \right)' = h + \mathcal{O}(h^2), \]

and \( \varphi \) is a regular reparametrisation that satisfies \( \varphi^{(\ell)} = \mathcal{O}(h^\ell), \ell = 2, 3, \ldots, 5. \)

This concludes the proof for the \( \mathcal{O}(h^6) \) solutions. The \( \mathcal{O}(1) \) case of Lemma 3 follows from the fact that the interpolating curve is bounded independently of \( h \). For the \( \mathcal{O}(h^2) \) case one can choose a parabola that interpolates the values \(-\frac{h}{2}, \lambda_0, \frac{h}{2}\) at \( 0, 0, 1 \) as a reparametrisation \( \varphi \). The proof is completed. \( \square \)

Quite clearly, one of the \( \mathcal{O}(h^6) \) solutions would reproduce exactly a quintic PH curve, and the others if they exist would be \( \mathcal{O}(h^6) \) and not better. On the other hand, the solution of Lemma 3 could not do better that \( \mathcal{O}(1) \) and \( \mathcal{O}(h^2) \) respectively in general. A constant

\[ \Theta(p) := \min_{-h/2 \leq s \leq h/2} \Theta(p; s), \quad \Theta(p; s) := \left\| r(s) - p \left( \frac{1}{2} \right) \right\|, \]

bounds the Hausdorff distance between the curve \( r \) and the PH interpolant \( p \) from below. The solution from Lemma 3 yields

\[ \Theta(p) = \frac{5}{4\tau_0} + \mathcal{O}(h). \]

For the solutions of Lemma 4 we obtain

\[ \Theta(p; s)^2 = h^2 \pi_{22}^2 \left( \frac{s}{h} \right) + \mathcal{O}(h^3), \quad (42) \]

where \( \pi_{22} \) is a linear polynomial with coefficients independent of \( h \). But \( \Theta(p)^2 = \mathcal{O}(h^4) \), so the first term of \( s/h \) should be a root of \( \pi_{22} \). This gives the values \( s \) that minimize the right-hand side of (42) for the three solutions as

\[ s = \mathcal{O}(h^2), \quad s \approx \pm 0.343966h + \mathcal{O}(h^2), \]
and the lower bounds
\[
\Theta(p) = \frac{\sqrt{\kappa_0^4 + \kappa_1^2}}{8\kappa_0} h^2 + \mathcal{O}(h^3), \quad \Theta(p) \approx \frac{\sqrt{0.00311317\kappa_0^4 + 0.112154\kappa_1^2}}{10\kappa_0} h^2 + \mathcal{O}(h^3)
\]

respectively. Theorem 2, and the previous discussion prove that the Hausdorff distance \( \text{dist}_H (r, p) \) behaves precisely like
\[
\text{dist}_H (r, p) = \text{const} + \mathcal{O}(h), \quad \text{const} > 0, \quad \text{dist}_H (r, p) = \text{const} h^2 + \mathcal{O}(h^3), \quad \text{const} > 0,
\]

for the solutions of Lemmas 3 and 4 respectively. So, for \( h \) small enough, they should differ significantly from the optimal order solutions of Lemma 5. This explains the background of the numerical algorithm developed in the next section.

4 Algorithm and numerical examples

In practical applications there is usually a large number of data to be interpolated. The presented \( G^2 \) PH quintic interpolation scheme constructs polynomial curves which can be joined together to the spline entirely locally. Asymptotic analysis reveals that there might exist many solutions of the interpolation problem considered and one must decide which of them is to be used on each spline segment. Quite clearly, the solution with the best approximation order, \( i.e. \), six in this case, should be chosen. But this can be done only in an asymptotic way, \( i.e. \), when the data is taken from a smooth parametric curve over some small parameter interval. To avoid this limitation we propose an algorithm based on a continuation method that can be used also for the nonasymptotic data.

The idea is to choose a particular set of data for which all the solutions can be computed and distinguished by the order of approximation. Then this particular data is connected with the original data by a smooth homotopy. Starting with the “best solution” of the particular interpolation problem, we trace this solution by a continuation method to reach the original data. In this way the asymptotic results are carried over to the nonasymptotic data.

Particular data
\[
V^* := \left( f_0^*, d_0^*, \Delta P_0^*, d_1^*, f_1^* \right)
\]

are taken from a quintic PH curve
\[
p^*(t) := \left( \frac{1}{48} (-48t^4 + 184t^2 - 43) + \frac{536}{625} \right) \quad \left( \frac{1}{72} (48t^8 - 280t^6 + 435t + 184) - \frac{54992}{28125} \right) \quad \left( \frac{1}{6} (8t^4 - 14t^2 + 3) - \frac{298}{625} \right)
\]
Table 3
The Hausdorff distance and parameters $\lambda_i$, $\mu_i$ for seven solutions of the particular case, where $h$ is the length of the curve $p^*$ on the interval $[t_0, t_1]$.

<table>
<thead>
<tr>
<th>Hausdorff distance</th>
<th>$\lambda_0$</th>
<th>$\lambda_1$</th>
<th>$\mu_0$</th>
<th>$\mu_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.994512h</td>
<td>0.994512h</td>
<td>0.0271798$h^2$</td>
</tr>
<tr>
<td>2</td>
<td>$1.143 \times 10^{-5}$</td>
<td>0.812225h</td>
<td>1.20396h</td>
<td>0.248020$h^2$</td>
</tr>
<tr>
<td>3</td>
<td>$1.143 \times 10^{-5}$</td>
<td>1.20396h</td>
<td>$0.812225h$</td>
<td>$-0.335774h^2$</td>
</tr>
<tr>
<td>4</td>
<td>$3.4566 \times 10^{-3}$</td>
<td>1.22545h</td>
<td>2.52048h</td>
<td>$-10.9048h$</td>
</tr>
<tr>
<td>5</td>
<td>$3.4566 \times 10^{-3}$</td>
<td>2.52048h</td>
<td>1.22545h</td>
<td>$10.9048h$</td>
</tr>
<tr>
<td>6</td>
<td>$4.7791 \times 10^{-2}$</td>
<td>3.03129h</td>
<td>3.03129h</td>
<td>$-12.3336h$</td>
</tr>
<tr>
<td>7</td>
<td>4.2009</td>
<td>24.8279</td>
<td>24.8279</td>
<td>$-110.916$</td>
</tr>
</tbody>
</table>

at the parameter values $-\frac{1}{10}$ and $\frac{1}{10}$. For the numerical part it is convenient to separate geometric properties of the data Gram matrix in the distance and the angle part by

$$\delta_i := +\sqrt{g_{ii}}, \ i = 0, 1, \ldots, 4, \ c_{ij} := \frac{1}{\delta_i \delta_j} g_{ij}, \ i, j = 0, 1, \ldots, 4. \quad (45)$$

The constants (45) that correspond to the particular data (43) will be denoted by $\delta^*_i$, $c^*_{ij}$. Clearly, one of the solutions is the curve $p^*$ itself, but there are six additional ones. The unknown parameters for all of them are given in Table 3. Solutions 2 and 3 are order 6 approximants of $p^*$, solutions 4, 5 and 6 have order two and the last solution is far away from $p^*$. The results in Table 3 match the results from Lemma 3, Lemma 4 and Lemma 5. One can see that parameters $\lambda_i$ and $\mu_i$, $i = 0, 1$, for order six approximants differ significantly from parameters for the other solutions. The same can be observed already from Table 1.

The homotopy algorithm has three steps:

1. Translation, rotation and scaling of the original data.
2. Construction of a homotopy function.
3. Solving the system by a continuation method.

Before we explain each step in detail, let us shortly recall the theory of spatial rotations. As it is known, they are closely connected with the quaternions. Namely, any nonzero quaternion $q = \pm [q_0, (q_1, q_2, q_3)^T] \in \mathbb{H}$, where $\mathbb{H}$ is a field of quaternions, defines a rotation $R = \text{Rot}(q)$, where

$$\text{Rot}(q) = \frac{1}{||q||^2} \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}.$$  

The mapping $\text{Rot} : \mathbb{H} \rightarrow \{R \in \mathbb{R}^{3 \times 3}, R^T R = 1, \det R = 1\}$ is called a kinematical mapping. Conversely, every rotation can be represented by two antipodal unit
quaternions $\pm q \in \mathbb{H}$, $\|q\| = 1$ (see [16]).

**STEP 1:**

The original data $V = \left( f_0, d_0, \Delta P_0, d_1, f_1 \right)$ are translated, rotated and scaled as

$$
P_0 \rightarrow (0, 0, 0)^T, \\
P_1 \rightarrow \frac{\delta_2^*}{\delta_2} R \Delta P_0, \\
d_i \rightarrow R d_i, \quad i = 0, 1, \\
f_i \rightarrow \frac{\delta_2}{\delta_2^*} R \Delta f_i, \quad i = 0, 1,$$

where $R \in \mathbb{R}^{3 \times 3}$ is a rotation matrix that transforms a vector $\Delta P_0$ onto $\frac{\delta_2}{\delta_2^*} \Delta P^*$. More precisely,

$$R = \text{Rot} \left( \left[ \frac{\cos \phi}{2}, \frac{\sin \phi}{2} a \right] \right),$$

where the rotation axis $a$ and the rotation angle $\phi$ are given as

$$a = \frac{1}{\delta_2 \delta_2^*} \Delta P_0 \times \Delta P^*_0, \quad \cos \phi = \frac{1}{\delta_2 \delta_2^*} \Delta P_0 \cdot \Delta P^*_0.$$

Note that the solution of the transformed problem (46) is just a scaled solution of the original one

$$\lambda_i \rightarrow \frac{\delta_2^*}{\delta_2} \lambda_i, \quad \mu_i \rightarrow \frac{\delta_2^*}{\delta_2} \mu_i, \quad i = 0, 1.$$

**STEP 2:** Construction of a homotopy function that connects the particular data $V^*$ with the transformed data (46).

Since $d_i \perp f_i$ and $d_i^* \perp f_i^*$, $i = 0, 1$,

$$Q_i := \left( d_i, \frac{1}{\delta_i} f_i, \frac{1}{\delta_i} d_i \times f_i \right) \quad \text{and} \quad Q_i^* := \left( d_i^*, \frac{1}{\delta_i^*} f_i^*, \frac{1}{\delta_i^*} d_i^* \times f_i^* \right)$$

are the rotation matrices. Let $q_i$ and $q_i^*$ be the unit quaternions, that correspond to $Q_i$ and $Q_i^*$. The sign must be chosen in such a way that $q_i^T q_i^* > 0$, $i = 0, 1$. The quaternion line

$$h_i(t) := (1 - t)q_i^* + t q_i, \quad i = 0, 1,$$

defines a rational spherical motion of degree two. The homotopy function is defined as

$$H : [0, 1] \rightarrow V, \quad H(t) = \left( f_0(t), d_0(t), \Delta P_0(t), d_1(t), f_1(t) \right).$$
where
\[
\Delta P_0(t) := \Delta P_0, \\
f_i(t) := ((1 - t)\delta_i + t \delta_{i+1}) \text{Rot} (h_i(t)) \begin{pmatrix} 0, & 1, & 0 \end{pmatrix}^T, \quad i = 0, 1, \quad (47) \\
d_i(t) := \text{Rot} (h_i(t)) \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}^T, \quad i = 0, 1.
\]

Quite clearly, \( H(0) = V^* \) and \( H(1) = V \), where \( V \) denotes the data after the first step.

**STEP 3: Continuation method.**

Choose one solution from Table 3. Trace this solution by a continuation method along the homotopy path defined by \( H(t) \). The result is the solution of the interpolation problem for data (46).

The proposed algorithm works efficiently for most data configurations. The following problems may occur. The solution that we trace can be lost along the homotopy path if two solutions meet and turn into a pair of complex solutions or if the solution crosses the boundary of the domain \( \{(\lambda_0, \lambda_1, \mu_0, \mu_1) : \lambda_i > 0, \mu_i \neq 0, i = 0, 1\} \). The reason for this problem is usually that the interpolation problem does not have a solution at all. It may happen that the solution exists, but the homotopy path contains parts where it does not. In this case we should use the extended continuation method that enables us to trace the solutions in complex and back.

As an example, let us connect particular data \( V^* = (f^*_0, d^*_0, \Delta P^*_0, d^*_1, f^*_1) \), taken from the quintic PH curve (44) at parameters \(-\frac{1}{10}\) and \(\frac{1}{10}\), with the data \( V = (f_0, d_0, \Delta P_0, d_1, f_1) \), taken from the curve (14) at parameters 0 and \(\frac{1}{4}\). Let us trace the first solution from Table 3 by using homotopy approach considered above. In the first step of the algorithm, the original data \( V \) are transformed to data \( \tilde{V} = (\tilde{f}_0, \tilde{d}_0, \Delta \tilde{P}_0, \tilde{d}_1, \tilde{f}_1) \), as described in (46) (see Fig. 4). Secondly, homotopy path that connects the transformed data to the particular data \( V^* \) has to be constructed as presented in (47) (see Fig. 5). Finally, the selected solution is traced by the continuation method along the obtained homotopy path. As expected, parameters \( \lambda_i, \mu_i, i = 0, 1 \) obtained from the first row in Table 3, after transformation in the first step of the algorithm, transform to the parameters given in the first row in Table 1 (see Fig. 6).

Fig. 4. Original data \( V \) (dashed gray) and the data \( \tilde{V} \) after the first step of the algorithm (black).

Fig. 5. Homotopy from the particular data (gray) to the transformed original data \( \tilde{V} \) (black).
Fig. 6. Transformations of parameters \( \lambda_0 \) (left) and \( \lambda_1 \) (right) along the homotopy path from the best solution of the transformed original data \( \tilde{V} \) to the best solution for the particular data \( V^* \).

As the second example, let us consider an interpolation by a \( G^2 \) quintic PH spline. Let the data be taken from the curve (14) at parameters \( \frac{i}{4}, i = 0, 1, \ldots, 8 \). Using the same approach as in the previous example for each particular segment we obtain the final \( G^2 \) PH spline shown in Fig. 7.

Fig. 7. A \( G^2 \) quintic PH spline interpolating the curve (14) at parameters \( \frac{i}{4}, i = 0, 1, \ldots, 8 \).
References