AREA-PERIMETER RELATIONS FOR TWO-DIMENSIONAL LATTICES

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1. Introduction. Minkowski’s theorem for the two-dimensional unit square lattice states that any convex domain of area greater than four which is symmetric about a lattice point, contains at least one other lattice point. In this note we shall prove that any convex domain whose area is greater than half its perimeter, contains a lattice point. Thus, if $D$ is a convex domain containing no lattice points, we have the inequality

$$A(D) \leq \frac{1}{2} P(D),$$

where $A(D)$ is the area of $D$ and $P(D)$ its perimeter. Inequality (1.1) may be stated in the following equivalent form: The supremum of $A(D)/P(D)$ over all convex domains $D$ containing no lattice points is less than or equal to one half. We shall, in fact, prove a stronger and more general result.

THEOREM. Let $\Lambda = \Lambda(\omega_1, \omega_2)$ be a lattice generated by two noncollinear vectors $\omega_1$ and $\omega_2$ making an angle $\theta$ with one another, where $0 < \theta < \pi$, and let $c(\Lambda)$ be the supremum of $A(D)/P(D)$ over all convex domains $D$ containing no points of $\Lambda$. Then if $|\omega_1| \leq |\omega_2|$ we have the inequalities

$$\frac{1}{2} |\omega_2| \sin \theta \leq c(\Lambda) \leq \frac{1}{2} \max \{|\omega_1|, |\omega_2| \sin \theta\}.$$  

Note that for rectangular lattices the inequalities in (1.2) determine $c(\Lambda)$ exactly and give us $c(\Lambda) = \frac{1}{2} \max \{|\omega_1|, |\omega_2| \sin \theta\}$.

The lower bound in (1.2) is easily derived by considering the parallelogram $D_n = \{(x, y) | 0 < x < n, 0 < y < 1\}$, the coordinates $(x, y)$ being relative to the basis $\omega_1$, $\omega_2$. As $n \to \infty$ it is clear that $A(D_n)/P(D_n) \to \frac{1}{2} |\omega_2| \sin \theta$, and this implies $c(\Lambda) \geq \frac{1}{2} |\omega_2| \sin \theta$.

2. The upper bound for $c(\Lambda)$. The upper bound for $c(\Lambda)$ will be established by reducing the problem to rectangular lattices and symmetric domains.

Let $\omega'_1 = \omega_1$, and let $\omega'_2$ be a vector of length $|\omega_2| \sin \theta$, perpendicular to $\omega_1$. (There are two such vectors, but they differ only in sign.) Let $\Lambda' = \Lambda(\omega'_1, \omega'_2)$ denote the rectangular lattice determined by the basis vectors $\omega'_1$, $\omega'_2$. We shall prove the following lemma.

**Lemma 1.** If $D$ is a convex domain containing no points of $\Lambda$, there exists another convex domain $D'$ containing no points of $\Lambda'$, such that

(a) $P(D') \leq P(D)$, $A(D') = A(D),$

(b) $D'$ is symmetric about the lines $x' = \frac{1}{2}$, $y' = \frac{1}{2}$, the coordinates $x'$ and $y'$ being relative to the basis $\omega'_1$, $\omega'_2$.

**Proof.** Let $D^0$ be the region obtained from $D$ by symmetrization with respect to the line $x' = \frac{1}{2}$. Symmetrization preserves convexity and areas and does not increase perimeters. Therefore $D^0$ is convex, $A(D^0) = A(D)$, and $P(D^0) \leq P(D)$.  

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We shall show now that $D^0$ contains no points of the lattice $\Lambda'$. If $D^0$ contained a lattice point of $\Lambda'$, say the point $m\omega'_1 + n\omega'_2$, then the line $y' = n$ (perpendicular to $\omega_2$) would intersect $D^0$ in a line segment of length greater than 1. The same line would also intersect $D$ in a line segment of the same length and this line segment, in turn, would contain a lattice point of $\Lambda$, contradicting the hypothesis of the lemma. Therefore $D^0$ contains no lattice points of $\Lambda'$.

We can argue now with $D^0$ as we did with $D$, except that we symmetrize $D^0$ with respect to the line $y' = \frac{1}{2}$. The symmetrization of $D^0$ gives us a domain $D'$ with the properties described in the lemma.

In view of Lemma 1, to deduce the upper bound in (1.2) it suffices to prove

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[thick] (-1,0) -- (3,0) -- (3,2) -- (-1,2) -- (-1,0);
\draw[thick] (0,-1) -- (0,2);
\draw[thick] (1,-1) -- (1,2);
\draw[thick] (2,-1) -- (2,2);
\fill[black] (0,0) circle (0.1);\fill[black] (0,1) circle (0.1);\fill[black] (0,2) circle (0.1);
\fill[black] (1,0) circle (0.1);\fill[black] (1,1) circle (0.1);\fill[black] (1,2) circle (0.1);
\fill[black] (2,0) circle (0.1);\fill[black] (2,1) circle (0.1);\fill[black] (2,2) circle (0.1);
\fill[black] (-1,0) circle (0.1);\fill[black] (-1,1) circle (0.1);\fill[black] (-1,2) circle (0.1);
\fill[black] (-0.5,0) circle (0.1);\fill[black] (-0.5,1) circle (0.1);\fill[black] (-0.5,2) circle (0.1);
\fill[black] (-1,-1) circle (0.1);\fill[black] (-1,-0.5) circle (0.1);\fill[black] (-1,0) circle (0.1);
\fill[black] (0,-1) circle (0.1);\fill[black] (0,-0.5) circle (0.1);\fill[black] (0,0) circle (0.1);
\fill[black] (1,-1) circle (0.1);\fill[black] (1,-0.5) circle (0.1);\fill[black] (1,0) circle (0.1);
\fill[black] (2,-1) circle (0.1);\fill[black] (2,-0.5) circle (0.1);\fill[black] (2,0) circle (0.1);
\end{tikzpicture}
\caption{}
\end{figure}

**Lemma 2.** Let $\Lambda = \Lambda(\omega_1, \omega_2)$ be a rectangular lattice. If $D$ is a convex domain, symmetric about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ (coordinates relative to the basis $\omega_1, \omega_2$) and if $D$ contains no points of $\Lambda$, then

\[
\frac{A(D)}{P(D)} \leq \frac{1}{2} \max \{ |\omega_1|, |\omega_2| \}.
\]

**Proof.** Let $a = \sup \{ x : (x, \frac{1}{2}) \in D \}$, and $b = \sup \{ y : (\frac{1}{2}, y) \in D \}$, where the coordinates $x$ and $y$ are relative to the basis $\omega_1, \omega_2$. We consider three cases, depending on the sizes of $a$ and $b$.

**Case 1.** $a \leq \frac{1}{2}$ and $b \leq \frac{3}{2}$. By the isoperimetric inequality we have

\[
(2.1) \quad \frac{A(D)}{P(D)} \leq \sqrt{\frac{A(D)}{4\pi}}.
\]
We now obtain an upper bound for $A(D)$. In this case, $D$ lies within the cross-shaped region $C$ shown in Figure 1. Therefore $A(D) \leq A(C) = 3|\omega_1| |\omega_2|$, and (2.1) becomes
\[
\frac{A(D)}{P(D)} \leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \sqrt{(|\omega_1| |\omega_2|)} < \frac{1}{2} \sqrt{(|\omega_1| |\omega_2|)} \leq \frac{1}{2} \max \{|\omega_1|, |\omega_2|\}.
\]

![Figure 2](image)

**Fig. 2**

**Case 2.** $a > \frac{3}{2}$. We consider $D'$, the quarter of $D$ with $x \geq \frac{3}{2}$ and $y \geq \frac{1}{3}$. We now obtain an upper bound for $A(D')$. Since $D$ is convex, the region $D'$ lies below some line through $(1, 1)$ with nonpositive slope as shown in Figure 2. Therefore $D'$ is a subset of the shaded region $B$ shown in Figure 2. Since $a > \frac{3}{2}$ the region $B$ has an area not exceeding that of the rectangle $PQRS$. Therefore $A(B) \leq \frac{1}{3} |\omega_2| \cdot (a - \frac{1}{3}) |\omega_1|$. Thus
\[
A(D) = 4A(D') \leq 4A(B) \leq 2 |\omega_1| |\omega_2| (a - \frac{1}{3}).
\]

But since $P(D) \geq 4 |\omega_1| (a - \frac{1}{3})$ we have the inequality
\[
\frac{A(D)}{P(D)} \leq \frac{1}{2} |\omega_2|.
\]

This proves Lemma 2 when $a > \frac{3}{2}$.

**Case 3.** $b > \frac{3}{4}$. The argument in this case is similar to Case 2 and we obtain the inequality
\[
\frac{A(D)}{P(D)} \leq \frac{1}{2} |\omega_1|.
\]

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