INTRODUCTION

There has always been a lot of talk about language in mathematics education, the reason being, perhaps, that the phenomenon of mathematics — i.e. that, which appears to our senses — is the foreign, artificial, code-like character of this language.

For some people the language of mathematics is only a wall, which conceals a mysterious garden of unwordly beauty. There is no other way to the garden but through the wall, but, once a gate is found, one can get in and enjoy the wonders. Teaching mathematics is trying to convince the students that there is, indeed, the garden behind the wall and helping them find the key to it.

For other people, the wall is mathematics. They are curious about the construction, the structure, the use of the language. They believe that mathematics is a "discourse" and that teaching mathematics is initiating students into this discourse.

I adhere to the former point of view. For me mathematics is mostly tacit and contemplative knowledge; it is about seeing and feeling things, relations, connections among ideas, moving those things around in one’s imagination. Terminology, notations, algorithms help in doing that by clustering ideas into manageable chunks and alleviating memory load, as when part of the work can be done by mechanical processing of material representations. The material representations are also necessary in communicating ideas and in having control over the coherence and validity of the ideas. And it is true that the work of representing and communicating is an important source of new ideas in mathematics. Therefore I do not deny the role of language in mathematics. However, I do not agree with equating mathematics with a discourse, and mathematical thinking with communicating as it is done in the so-called "discursive approach" in mathematics education.

This talk will be a polemic with this approach. More specifically, it will be a polemic with views expressed mainly in the ESM Special Issue on "Bridging the individual and the social:

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1 This talk is based on a paper submitted for publication in a book edited by J. Kilpatrick, C. Hoyles and O. Skovsmose (for the BACOMET group), "Meaning and Communication", to be published in Kluwer some time in the future.
Discursive approaches to mathematics education" (2001, ESM 46) and (Sfard et al., 1998). The two sources will be labeled as ESM46 and FLM18.1 in the sequel.

The need to take issue with this position has been with me for several years now, but at each attempt I felt defeated by the enormity of the task. Finally, this fall, I pulled myself together and attempted to compose an argument. It may not convince those who adhere to the "discursive paradigm" in mathematics education, but it may provoke them to refine their views, word their assumptions in more careful terms and sharpen their interpretation of empirical data used to illustrate their theories.

It was not easy for me to undertake this polemic also for personal reasons: some of the advocates of the discursive approach are my colleagues and friends. But let me make this clear: this is a critique of texts, not of people who have written them at some point in their lives and may have changed their views since then. Moreover, this is a critique of my interpretation of these texts, which need not be the same as the authors’ intended meaning. Opening up my interpretation to public scrutiny offers an opportunity to the authors to clarify their intentions, and to me — to better understand their positions. Hopefully, this will lead to progress in the development of theory in mathematics education.

The talk is composed of two parts. In the first I present my interpretation of the discursive approach as a program in mathematics education. In the second I’ll offer a commentary on selected parts of the program.

1. DISCURSIVE APPROACH AS A PROGRAM IN MATHEMATICS EDUCATION

Discursive approach to mathematics education, as presented in FLM18.1 and ESM46 (denoted by "DA" in the sequel) could be seen as a "program" in mathematics education in the sense of (Sierpinska, 1996). It promotes a certain ideology and encourages a particular didactic action, while developing a theory and conducting experimental research guided by this theory. I describe DA by identifying these three facets of the program.

1.1 The ideology of DA

A discursive worldview. DA views the world through the eyes of a semiotician. The basic assumption is that everything that matters is a sign, and, conversely, to make something matter, it must be turned into a sign (see, e.g. Lotman, 1990: 5). DA stresses the power of discourses to create phenomena and to change people’s perception and experience of phenomena: "objects of thought, discourse and social manipulation are semiotic and thus culturally constituted entities" (Parmentier, 1985: 376; see also Bourdieu, 1982/2001; 1994). Discourses affect how people
perceive themselves; in their own eyes, the meaning of what they say and do is determined by their position (their "voice") within a society and this is also how they interpret the behavior of others.

_A discursive philosophy of mathematics._ Mathematics is seen as a historically developed socio-cultural practice (ESM46: 66, 72); it is a "historically developed practice, dealing with certain types of objects, tools and rules" (ibid.). Mathematics was seen as a "well-defined type of discourse" in (FLM18.1: 50), but this radical view was replaced by a more moderate one in the more recent writing. DA is admitting the existence of not one "well-defined" discourse of mathematics but of a large variety of mathematical discourses among mathematicians, in various professions, and in schools. It is acknowledged that school produces its own school mathematical discourses that need not be, and most of the time are not, identical with mathematical practices outside of school (ESM46: 100-101).

_A model of the learner of mathematics._ DA endorses the view that "all our thinking, with mathematical thinking being no exception, is essentially discursive" (FLM18.1: 50). The learner of mathematics is thus an _apprentice_ of mathematical discourse, represented by the teacher and textbooks. DA is aware of the difficulty this approach has in accounting for individual creativity and change of cultures (ESM46: 93). DA conceives of an individual as a "collection of multiple subjectivities, through the many overlapping and separate identities of gender, ethnicity, class, size, age, etc." (ESM46:105). The learner is therefore not conceived of as a person or a psychological subject with his or her idiosyncratic cognitive and emotional functioning, but as a member of a social group, a community with a background culture and history. It does not matter what the learner thinks to himself². The learner’s behavior is interesting only insofar as it is interpreted by other members of the group as a sign, i.e. as having some meaning for them.

_A model of the teacher of mathematics._ The teacher of mathematics is a _representative participant_ of a mathematical culture, i.e. of a discourse and praxis of mathematics, who is supposed to create conditions for the initiation of students into this culture (ESM46: 74). The teacher acts as a participant in his own right in the classroom discourse, but also directs, "canalizes" the discussion towards the relevant mathematical topics and voices students’ formulations in the appropriate mathematical "speech genre" (ESM46: 75).

The teacher acts as a participant in his own right in the classroom discourse but also directs, channels the discussion towards relevant mathematical topics.

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² The masculine forms of pronouns are used to alleviate the reading of the text.
A model of the mathematics classroom. The class is a community, namely a community of mathematical discourse (not only a community of mathematical activity, since it includes reflection and debate about mathematical activity) (ESM46: 71). Students work on solving common problems, agreeing on approaches and techniques through social interaction, conversation, discussion and other forms of communication. DA calls for replacing the dialogue among pupils in the classroom by "a polylogue" in a mathematical community or a "polyphonic discourse among all possible voices that helped to create the history of that community of practice" (ESM46: 74).

A model of a curriculum. The program does not specify the preferred content of teaching. It only stresses that the content should be relevant from the historical and cultural point of view (it should be "historically rooted", ESM46: 72). The most important objective of teaching mathematics is to initiate students into historically developed ways of doing and talking and get them involved in the mathematics "speech genre" (ibid.).

1.2 Didactic action

DA is not satisfied with the present trend of classroom conversation for the sake of conversation alone. It requires that the art of mathematical communication be an object of teaching in its own right: "communication skills cannot be taken for granted" (FLM18.1: 51). The implicit "metadiscursive" rules of the mathematical "genre" need to be inculcated in the learners through practice so that they become a habitus, a kind of second nature (ESM46: 29-30, with reference to Bourdieu, 1980).

According to (ESM46: 74), didactic action that could be considered as representing the discursive approach can be found in the recent work of Cobb and his collaborators (e.g. Cobb, Boufi, McClain, and Whitenack, 1997; Cobb in FLM18.1), and in the Russian "dialogue of cultures" schools. I could perhaps add (with caution, since this reference does not appear in the DA texts) Boero and his collaborators’ experiments of introducing children into the practice of theoretical knowledge, based on the Vygotskian notion of scientific concepts and the Bakhtinian notion of "voice" (Boero, Pedemonte, Robotti, 1997; Boero, Pedemonte, Robotti, Chiappini, 1998).

1.3 Theoretical foundations

In the discursive approach, language, communication, discourse and thought do not constitute separate objects of theoretical reflection. All are included in communication: the genetic roots and purpose of language is communication; discourse is any specific instance of communicating and thinking is a kind of communication, namely communicating with oneself. This perspective is
dictated by the assumption that the subject of acts of communication, discourse and thinking is not a psychological individual or a person, but a social group of participants in a common culture or a collection of multiple subjectivities.

1.3.1 The meaning of "discourse"

According to Webster’s dictionary, "discourse" can mean "the capacity of orderly thought; rationality", or "verbal interchange of ideas, especially conversation", as well as, "a formal orderly and extended expression of thought on a subject", and "connected speech or writing". Interestingly, "discursive" speech or writing can mean both "moving from topic to topic without order" and "proceeding coherently from topic to topic" or "marked by analytical reasoning". Thus, "discourse" in ordinary use covers all kinds of talk or writing, from casual conversation to scholarly argumentation. One thing that seems to be always required of it, however, is that it refers to something else than itself: it is talk or writing about something. Reciting the alphabet would not be an instance of discourse, because it does not refer to anything outside the language.

In DA "discourse" is defined as "any specific instance of communicating, whether diachronic or synchronic, whether with others or oneself, whether predominantly verbal or with the help of any other symbolic system" (ESM46: 28).

1.3.2 Communication is a socio-cultural practice

DA rejects the classical sender-receiver communication model (ESM46: 66). It focuses not on transmission of information from one individual to another but on the participation in an activity of "sharing communalities and constructively dealing with the meanings people seem to have in common" (ESM46: 67).

1.3.3 Language is not just as code

The classical model of communication was based on the notion of language as a code; the sender would encode his thought in some symbolic form and the receiver would decode it into his own thought. According to Lotman, one of the many critics of this model,

[T]he term ‘code’ refers to an idea of a structure just created, artificial and introduced based on an agreement made at some point in the process of communication. A code has no history. On the other hand, the term ‘language’ unconsciously evokes in us the thought about the continuity of being. Language is a code plus its history. (Lotman, 1999: 31-2, my translation from Polish).

The classical model of communication took notice of neither the history of language nor of the context of communication. Conversations apparently took place in a historical, social and cultural void. In DA communication is part of historically developed socio-cultural practices of a community.
1.3.4 Thinking is communication

DA endorses a social view of mental functions. The idea of conceptualizing thinking as communicating (ESM46:25) could have been inspired by Wertsch’s interpretation of Vygotsky’s and Bakhtin’s position, as in the following citation:

[C]ertain aspects of human mental functioning are fundamentally tied to communicative processes. The use of the term ’voice’ provides a constant reminder that even psychological processes carried out by an individual in isolation are viewed as involving processes of a communicative nature. (Wertsch, 1991: 13)

DA also backs up its position by reference to Davydov’s writings.

[M]ental functions are essentially seen as not rooted in the individual, but in the communication between individuals, in their relationships between each other and in their relationships with the objects created by people. (Davydov, 1997/1991, cited in ESM46: 67)

For Davydov, the subject of thinking is a "collective", i.e. a group of people participating in a common task. The collective thinking, memory and other so-called higher mental functions are assumed not only as social in a synchronic and contextual way, but also as historical and cultural. Davydov’s position is that of the dialectical materialism orthodoxy, based on the writings of Marx, Engels and Lenin (Davydov, 1990). It looks at the society from a far range; individual members of the "human society" are not distinguishable the way they are in a "civil society". Sometimes DA endorses this point of view, and sometimes it is more in tune with Vygotsky’s position, which is closer to the ideology of "civil society", looking at the society from the perspective of its members and speaking about the mental functioning of an individual.

Vygotsky was less concerned with applying dialectical materialism in developing his theory than Davydov. Vygotsky still focused on the individual. He was looking at how the intermental functioning advances the intramental functioning, a perspective which led him to the notion of the zone of proximal development (Wertsch, 1991: 28). He saw culture and society not as "factors" that influence the development of the individual but as an environment with which the individual interacts. This environment acts on the individual as much as the individual acts on the environment (see citation from Vygotsky in ESM46: 67). Thus he appeared to view society as an emergent of the many interactions between participants and not, as in the dialectical materialist philosophy endorsed by Davydov, a super-entity, which "confers the historically developed forms of activity" on the individual (Davydov, 1990: 232).

Vygotsky’s "fundamentally a-social theory of psychological tools" (as deemed by Clot, 1999, cited in ESM46: 67) has been unsatisfactory for DA which then turned its attention to Bakhtin’s notion of "genre" as more "inherently social" (Clot, ibid., p. 174). The importance of being initiated into a speech genre for effective communication is highlighted in the following citation:
If speech genres did not exist and we had not mastered them, if we had to originate them during the speech process and construct each utterance at will for the very first time, speech communication would be almost impossible" (ESM46: 69).

This duality of perspectives (Davydov and Bakhtin on the one hand, and Vygotsky on the other, or "human society" and "civil society" perspectives) leads DA to interpreting the idea of "thinking is communicating" both as a methodological principle and an epistemological and ontological stance. If one takes the "human society" point of view, one does not need the notion of thinking at all. What is studied and what matters for the socio-cultural development of knowledge is not what this or that individual thinks to him or herself but what the social group decides to do and how it decides to formulate its conclusions and solutions in a process of communication. This is how the assumption, "thinking is communicating" could be understood.

However, this methodological interpretation is not quite explicit in DA, where references to Davydov and Marx co-exist with speaking of thinking as communicating as an actual activity of the psychological subject. With reference to Vygotsky, DA claims that all human activity has social origins, and communicational public speech developmentally precedes inner private speech. This assumption is then claimed to imply (similarly as in Wertsch, cited above) that thinking is a case of communication, namely communication with oneself (ESM46: 26). This is not meant to say that all thinking is verbal, though: "the word ‘communication’ is used here in a very broad sense and is not confined to interactions mediated by language" (ibid.). How broad this sense is assumed to be is not easy to grasp. It seems that the only characteristic retained is the communicational intent with the aim of being effective (ESM46: 27, 28, 32):

[W]hen one is looking at cognition as a form of communication, an individual becomes automatically a nexus in the web of social relations. This is true whether this individual is in real-time interaction with others or acts alone. Further, from the proposed vision of cognition it follows that thinking is subordinated to, and informed by, the demand of making communication effective. (ESM46: 27; my emphasis)

1.3.5 Learning is initiation into a discourse

DA assumes that the main motive for learning (i.e. changing one’s previous conceptions) is "to adjust one’s discursive uses of words to those of other people", especially if these other people are viewed as superior in a social or intellectual sense (ESM46: 49). Thus, "discursive conflicts" (ibid., p. 48) are what drives learning and not cognitive conflicts as assumed in the Piagetian theory of equilibration of cognitive structures.

"Learning mathematics [is] defined as an initiation [in]to mathematical discourse" (ESM46: 28). The learner must learn two things: the means of communication and the metadiscursive rules. The latter regulate the flow of communication; they are considered as akin to
concepts such as Wittgenstein’s *language games*, Goffman’s *frames*, Bruner’s *formats* and Bourdieu’s *habitus*.

"Means of communication" are described as "shaping the content of the discourse" (ibid.). A natural question is: means of communication of what? Normally, we would say, "means of communication of thoughts", but, in this theory, thinking itself is communication and speaking about expressing a thought whose existence would be independent of the means of communication is considered meaningless (ESM46: 27, 29). If so, then perhaps one shouldn’t speak about the content of communication at all, but only about the discursive means of communication, the patterns of communication, and the genre of the discourse.

1.3.6 A participation model of teaching

DA promotes more an ideal of a mathematics teacher than it proposes a theory of the profession of teaching mathematics. The "participation model" of teaching presented in (ESM46: 117) seems the closest to a conceptualization of teaching, but it is still put forward as an ideal model to follow (an alternative to the so-called "acquisition model", and a model recommended by a reform movement). This conceptualization of teaching also captures the teacher-in-act in the classroom but is silent about the work of the teacher when preparing for the next class and making choices regarding tasks and didactic actions, based on the teacher’s knowledge of mathematics, interpretation of the curriculum and evaluation of the happenings and progress in the previous classes. It would be a real methodological challenge to develop a "discursive approach" to investigating teacher’s thinking and action between classes. For the time being teacher’s work in all its phases has been an object of investigations from much broader perspectives (e.g. Chevallard, 1999; Coulange, 2001; Salin, 2002).

Concerning ways in which a teacher can initiate students into the practice of mathematics, DA asserts, based on empirical evidence, that indirect methods ("co-constructive creation of mathematical models", ESM46: 81) are more effective than teaching ready made models.

2. COMMENTARY

DA appears to be part of an epidemic of "discursive approaches" in social sciences. The label can be found in psychology and psychiatry (e.g., Edwards & Potter, 1992; Harr & Gillet, 1995; Gillet, 1999), sociology (Bingham, 1994), political science (Dryzek, 1990). However, the meanings of this expression vary from domain to domain and they have changed over the years. Twenty years ago "discursive approach" in education may have meant using essay writing, discussion and debate forums as forms of communication in a whole range of subjects at school.
(Chilver, 1982). In political science, "discursive democracy" has been defined as an alternative to "instrumental democracy" (Dryzek, 1990). In sociology, "discursive approach" may refer to critical approaches that are concerned with the problems of shaping social, cultural and political realities by certain discourses; not only analyzing the discourses but also proposing political action to change the realities for culturally discriminated groups.

2.1 Cultural conformism

DA defines itself in opposition to "cognitivist" or "individualistic" approaches in psychology and in opposition to the "acquisition" metaphor in teaching methodology. But it is not a "critical theory", in the sense that it does not challenge using mathematics as a selection tool in the education system (Bourdieu, 1994: 49)\(^3\), and it does not question the existing school mathematical discourses to the point of wanting to transform them in directions that would allow the "economically disadvantaged groups" to perform as well as others. The statement that these groups do not perform as well as others seems to be taken as a fact (ESM46: 94). It would be interesting to investigate what percentage of mathematicians who made significant contributions to their field came from economically disadvantaged groups and to what they owed their success. Was it indeed their exceptional facility to become initiated into the existing mathematical discourses or was it, rather, their ability to grasp the main mathematical ideas and build new ones from them, constructing novel mathematical discourses? Srinivasa Ramanujan and Stefan Banach, to name only these two famous mathematicians raised in poverty, would fall into the latter category.

2.2 "Discursive conflict" and "socio-cognitive conflict": The social motives of learning

DA proposes to replace "cognitive conflict" by "discursive conflict" as a motive for learning. The notion of "discursive conflict" appears close to the notion of "socio-cognitive conflict" introduced over twenty years ago in social psychology (Perret-Clermont, 1979; Doise & Mugny, 1981), also in reaction to the limitations of the notion of cognitive conflict. The socio-cognitive conflict can be seen as a discursive conflict; it was understood as the co-occurrence of contradictory statements in a situation of social interaction (Blaye, 1989: 186). The notion of socio-cognitive conflict had been thoroughly researched and studied from several theoretical points of view (the cognitive point of view of Piaget, the socio-cultural perspective of Vygotsky, and theories of social learning, e.g. Bandura, 1980). Results of this research could caution DA researchers against

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\(^3\) “[C]’est souvent avec une grande brutalité psychologique que l’institution scolaire impose ses jugements totaux et ses verdicts sans appel qui rangent tous les élèves dans une hiérarchie unique des formes d’excellence — domine aujourd’hui par une discipline, les mathématiques” (Bourdieu, 1994: 49)
some of their conjectures. The conditions of progress through socio-cognitive conflicts are not obvious and they are complex, not reducible to the "model effect" or wanting to adjust one’s use of words to that of someone considered more knowledgeable (see, for example, Mugny, Doise & Perret-Clermont, 1975-76). It was found that interactions, which could be regulated otherwise than by a co-elaboration of a solution (for example, by the social or other authority of one of the partners), were not efficient in generating progress. Moreover, it was not possible to confirm experimentally a greater efficiency of inter-individual conflicts over intra-individual conflicts (children were doing better in a social conflict situations only if they did not have the possibility of verifying their hypotheses experimentally, Blaye, 1989: 190).

2.3 "The regulating effects of discourses" and "institutional contracts": The impact of the didactic contract on what is learned in school

The hypotheses proposed in social psychology have inspired mathematics educators to study the social dimensions of learning mathematics in more natural situations of ordinary classrooms. Extensive research in this direction was conducted using a variety of theoretical frameworks (interactionist, socio-cultural, theory of didactic situations, anthropological approach; see, for example, Venturini, Amade-Escot & Terrisse, 2002). While interactionist approaches stressed the verbal interaction patterns among students and the teacher in a classroom, researchers starting from what is now called an "anthropological approach" in France, looked more at the implicit institutional contracts regulating the mutual positioning and behavior of the participants. Already in 1986, Balacheff was explaining the difficulty of obtaining the engagement of students in mathematical proving by the social character of the classroom situation. In this situation, most of the time, the student acts as a practical person, not as a theoretician; he or she aims at producing a solution (a text) that would be acceptable for the teacher, not at producing knowledge (Balacheff, 1986; see also accompanying comments in Sierpinska, 1994: 18-19).

DA assumes that the social situation of the classroom, the relations of the assumed epistemological authority of the teacher, teacher’s responsibility for the students’ learning are factors that allow students to learn "what counts in the community’s speech genre" (ESM46: 75). They may indeed learn all that, but they may also learn that it is the teacher’s and not their responsibility that they learn and that what they claim is mathematically valid. Teacher’s feeling of epistemological responsibility may result in the students’ feeling of dependence, and this may happen even under the conditions of an apparently a-didactic situation, with students supposed to be working in small groups on a relatively open problem and even in computer labs (Laborde, 1992: 1-4; Arsac, Balacheff & Mante, 1992; Bellemain & Capponi, 1992). Students’ dependence
on the teacher for checking the validity of their solutions is part of the didactic contract, a
specificity of the school institution. In these conditions it is not realistic to expect that students
will be "initiated" into the discourse of working mathematicians; they can only be initiated into
the discourse of school mathematics, where proofs are texts written in a distinct genre, having
little to do with the truth of statements they are supposed to prove and even less with
communication of a result of an investigation.

2.4 Taking into account the social, cultural and historical sources of knowledge in
analyses of communication episodes

This is not to say that school mathematics is bad mathematics and that it should be replaced by
"genuine" mathematics. Mathematics at school will always be school mathematics, as long as
schools are schools and not research institutes or other workplaces. On the contrary, school
mathematics must be accepted and taken seriously. For research in mathematics education, this
implies that, whenever an "episode" of group or classroom communication is analyzed, the
history and culture of the classroom in which it took place, as well as the experimental contract
between the observed students and the researchers must be included not only as "background
information" for the reader but as important data to be analyzed. Part of this history is the design
of the tasks given to the students. These tasks were designed relative to some goals and socio-
cultural constraints. It is important to factor these data in an interpretation of the episode. It would
be natural for DA, with its focus on the socio-cultural and historical roots of knowledge to apply
this principle to its own research. It was rather unexpected, therefore, that the presentations of
episodes provided in ESM46 were extremely detailed on the level of tape-recorded and
transcribed verbal exchanges, but very cursory at the level of the socio-cultural context of these
exchanges, the histories of the participants and information about the design of the tasks. I could
only agree with Hoyles’ commentary on the ESM46 texts: "What I missed was any discussion of
the design of the activities and the design or choice of the tools or sign systems that were
introduced to foster mathematics learning" (Hoyles, 2001).

2.5 Collectivization of thought

The slogan in the 60s was: individualization of teaching. Now the slogan is: collectivization of
teaching. And even: collectivization of thought. Nobody can expect this author, who had the
opportunity to live through the realization of this ideology in socialist Poland, to take the idea of
collective thought seriously. The official political discourse was full of statements such as "the
collective of [name of factory] expressed their full support for the leading role of the party". This usually meant that the attempt of the workers to go on strike was unsuccessful.

According to Wertsch, Vygotsky’s and other Soviet psychologists’ theories were an attempt to apply Marx’s thesis that "human’s psychological nature represents the aggregate of internalized social relations that have become functions for the individual and form the individual’s structure" (Wertsch, 1991: 26). Translating this ideological assumption of social and cultural roots of human thinking, voluntary attention and other "higher mental functions" into a psychological theory and supporting it with evidence from empirical studies was affording political activists a scientific justification for decisions that were not so scientific. Reality had to be made to fit the theory and if it didn’t, so much the worse for the reality. Children had to become participants of the socio-cultural community and initiated into the model discourse at an early age; they were sent to a nursery and then to a kindergarten, where they learned the politically correct discourses of the time. At work, adults were part of a collective, which then collectively made decisions, formulated resolutions and generally never expressed any individual original thought that would go "against the will of the socialist society". Anybody expressing such individual original thought was immediately hailed an "enemy of the people" and weeded out from the "healthy body of the socialist society".

2.6 Smooth but meaningless communication or difficult but meaningful communication?

Language — genre = code

Bakhtin was saying that without a mastery of speech genres, "speech communication would be almost impossible" (cited in ESM46: 69). Almost impossible but not completely impossible: this is supported by the possibility of communication between people coming from different cultural backgrounds and speaking to each other in a language which is foreign for at least one of them. If someone has learned a foreign language late in life, it is a code for that person; a language without history, without a genre. A dialogue with this person can be very difficult but not impossible, and all the more significant. If the interlocutors use the same familiar genre, communication is smooth but trivial: they have nothing to say to each other (Lotman, 1999, p. 32). High level of information obtains in cases of difficult translation between the utterances of the interlocutors (ibid., p. 33)\(^4\). It may well be that progress in science owes more to difficulties in

\(^4\) A similar idea had been expressed in Lotman (1990: 80-81): "We should not forget that not only understanding, but also misunderstanding is a necessary and useful condition in communication. A text that is absolutely comprehensible is at the same time a text that is absolutely useless. An absolutely understandable and understanding partner would be convenient but unnecessary, since he or she would be a mechanical copy of my 'I' and our conversation would provide us with no increase in information."
communication than to efficient communication. One may conjecture that research done by a multilingual team is more likely to be innovative than if the team shares the same "genre" (and jargon). Can the same be conjectured about multilingual mathematics classes? (see Adler, 2001; Sierpńska, 2002).

2.7 Problems with defining communication by intent and effectiveness

In view of DA’s rejection of the classical model of communication, its focus on intent and effectiveness is rather unexpected. The classical model also defined communication by intent and effectiveness. This invited many theoretical difficulties (see, e.g., Lotman, 1999: 31; Bruner, 1974: 262). How could one establish whether the interpretation of a message was indeed the one consciously intended by the author of the message? If this is impossible, then how can we tell whether communication was effective or not? How can we even tell whether an utterance was an instance of communication or not (did the speaker intend to communicate)?

[It]ent in communication is difficult to deal with for a variety of reasons, not the least demanding of which is the morass into which it leads when one tries to establish whether something was really, or consciously intended. To obviate such difficulties, it has become customary to speak of the functions that communication or language serve and to determine how they do so. This has the virtue, at least, of postponing ultimate questions about 'reality' and 'consciousness' in the hope that they may become more manageable. (Bruner, 1974: 262)

A solution was proposed by Roman Jakobson who modified the classical sender-receiver model of communication to include the "context" while shifting the focus from the "vertices" of the structure (i.e. sender, receiver, message, context) to the relations amongst them. These relations were conceptualized as functions of language in communication: emotive/expressive, poetic, conative, phatic, metalinguistic and referential (Jakobson, 1960; see also a discussion and an application of this model to language acquisition in infants in Bruner, 1974). These functions of language were assumed to be all there in any act of communication; different acts of communication differed only by a hierarchical order of these functions. Jakobson’s categories can be quite useful in modeling the use of language in the teaching and learning of mathematics, as I hope to demonstrate in a future publication.

2.8 Relations between discourse, communication and thought

Based on Vygotsky’s assumption of social origins of higher mental functions, DA conceptualizes thinking as communicating, namely as communicating with oneself, and claims that all our thinking is discursive. This conceptualization does mathematical thinking no justice and it doesn’t seem useful for the purposes of mathematics education. Arguments against it, however, cannot be based on the abundant empirical evidence that thinking, especially mathematical thinking, is not
all verbal, because DA claims that communication (and discourse) is not necessarily verbal and
may be based on other signs, e.g. images. The following arguments may apply, however:

1) Communication is a voluntary act; not all thinking is voluntary.
2) Mathematical thinking requires a well-developed spatial visualization (not only in geometry
but also in dealing with numbers and algebra); spatial visualization is seeing things and moving
spatial configurations around in one’s mind; "seeing" is not intentional pointing things to oneself
and therefore cannot be regarded as an act of communication.
3) While "higher mental functions" are often analytic, i.e. mediated by conventional sign systems
and therefore have social origins, they are based on brain activity that has biological origins
preceding the use of conventional sign systems both in individual development and in the
evolution of the species.
4) Not all instances of communication are discursive.
5) Discursive thinking is not necessarily communicational.

Let me elaborate on these points.

2.8.1 Involuntary thinking

We seem to be thinking along on several distinct planes simultaneously, which may correspond to
activity in different parts of the brain. There is certainly a plane where thoughts just happen to us
whether we want it or not: it is the plane of involuntary (but not necessarily unconscious)
thinking. This thinking may contain words or images, but neither these words nor these images
are intended for communication. This involuntary thinking can be responsible for the moments of
"illumination", where the solution of a problem is suddenly revealed. Accounts of this
phenomenon in mathematicians can be found in Hadamard (1945).

2.8.2 Spatial visualization in mathematics

Dynamic spatial visualization plays an important role not only in geometric thinking but also in
algebra. Let me explain it on an example.

Example of thinking as "seeing" in algebra

Suppose I am asked to prove that, if A is any real n x m matrix then the vector space $\mathbb{R}^m$ can be
decomposed into a direct sum of the nullspace of A and the range of the transpose of A:

$$\mathbb{R}^m = N(A) \oplus R(A^T)$$

Suppose I see the situation represented by this equation as being about transformations: A
transforms $\mathbb{R}^m$ into $\mathbb{R}^n$, and the transpose of A transforms $\mathbb{R}^n$ into $\mathbb{R}^m$. The nullspace of A is the
part of $\mathbb{R}^m$ that is transformed into the zero vector of $\mathbb{R}^n$ (I see it shrinking to a point). The range
of A$^T$ is the part of $\mathbb{R}^n$ composed of all vectors obtainable as A$^T$y with y in $\mathbb{R}^n$. As I am saying all
this, my mind works as if looking from left to right and back. I keep some imaginary places for
$\mathbb{R}^m$ (on the left) and $\mathbb{R}^n$ (on the right) and I move from one to the other. I see the task in front of
me as understanding why the space on the left can be seen as generated from these two parts. My prototypical image of a direct sum being something like a space spanned on two perpendicular axes, I am immediately seeing these two parts as being in a horizontal/vertical position. Hence the idea of proving that the orthogonal complement of \( N(A) \) is \( R(A^\top) \) (since \( V = W \oplus W^2 \) in general, this does the proof). Thinking this way, I am moving from left to right, going from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) by means of multiplication by \( A \). But this turns out to be difficult. When I write what is the orthogonal complement of \( N(A) \) (written \( N(A)^\perp \)), by definition, \( N(A)^\perp = \{ y \in \mathbb{R}^m; \text{if } Ax = 0 \text{ then } \langle y, x \rangle = 0 \} \), I don’t see at all why this should be equal to the range of \( A^\top \), i.e. the space generated by the columns of \( A^\top \) or the rows of \( A \). When thinking about that, my mind constantly moves between viewing the matrix \( A \) vertically, as a set of columns to viewing it horizontally, as a set of rows, and then transposing the configuration to see which is which in \( A^\top \). All these gymnastics lead nowhere, till I suddenly change the point of view and start looking from right to left, from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) using \( A^\top \), and from below to the top, from \( R(A^\top) \) to \( N(A) \). Instead of trying to show that \( N(A)^\perp = R(A^\top) \), why not try to show that \( R(A^\top)^\perp = N(A) \)? Realizing that two transpositions return the matrix to its original position (again some mental gymnastics in moving from vertical to horizontal positions), I see that it is enough to prove that \( R(A)^\perp = N(A^\top) \), and then apply it to the transpose of \( A \). Equality may be a reflexive relation, but, as I am thinking about this proof, I go from the left to the right of the equalities I write, and I am thinking of \( R(A)^\perp = N(A^\top) \) as saying that the orthogonal complement of \( R(A) \) is the same as the nullspace of the transpose of \( A \), and not that the nullspace of the transpose of \( A \) is equal to the orthogonal complement of the range of \( A \). The former is the first thing one notices, the latter is not so obvious.

I stop my story here. I hope it conveys the importance of the "muscular" mental effort involved in a very simple mathematical reasoning about matrices.

Spatial visualization expresses itself also in the sense of rhyme and rhythm, responsible for mathematicians' proverbial "perfect pitch" for patterns. There are innumerable instances of this phenomenon, at any level of mathematical sophistication. For example, our preference to write polynomials in one variable so that the powers of the variables are arranged in increasing or decreasing order, rather than in random order can be seen as a source of some mathematical notions; for example, the isomorphism between the vector space of polynomials of degree \( \leq n \) and \( \mathbb{R}^{n+1} \). Also, indexing the coefficients of a general quadratic form so that they match the variables brings forth the idea of the matrix representation of a quadratic form (Figure 1).

\[
\begin{align*}
a_0 + a_1 x + \ldots + a_n x^n & \quad (a_0, a_1, \ldots, a_n) \\
a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 &= \begin{bmatrix} x_1 & x_2 & x_3 \\ a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}
\end{align*}
\]

Figure 1. Notational rhythms and rhymes

Given a concrete set of linear equations, one can always solve it by isolating variables from the simpler equations and substituting them into the more complicated ones. But this way of working...
is messy and specific to each concrete case (Figure 2). It is by trying to give some rhythm to the text of the solution that we arrive at the matrix representation and a general method of solution.

\[
\begin{align*}
x_1 &= 5 + 6x_2, \quad x_4 = 4x_3 - x_s \\
x_2 &= 4x_4 + 5x_3, \quad x_3 = 5x_4 - 6x_2 + 3
\end{align*}
\]

It is this sense of rhyme and rhythm that could have led Cayley and Sylvester to the development of matrix theory, and could have supported their study of algebraic invariants and linear transformations, which, at that time, were linear substitutions of variables (see, e.g. Bell, 1956).

Sylvester’s sense of the kinship of mathematics to the finer arts found frequent expression in his writings. Thus, in a paper on Newton’s rule for the discovery of imaginary roots of algebraic equations, he asks in a footnote 'May not Music be described as the Mathematical of sense, Mathematic as the Music of reason? Thus the musician feels Mathematics, the Mathematician thinks Music — Music the dream, Mathematic the working life-each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss — a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz!' (Bell, 1956: 364).

2.8.3 Biological roots of mathematical thinking

According to Vygotsky, thought and language develop separately till the age of two, when their paths start intertwining. At a very early age, 0-2 months, infants start noticing things, recognizing faces and they start babbling. These are the beginnings of thinking and speech, but neither has some hidden communicative intent. Babbling fulfills the same function for the tongue and vocal cords as moving arms and legs do for the development of muscular strength. Infants babble and move without any intention of communication; they do it just because they are "alive and kicking". However, the caretakers react to the babble as an invitation to engage in communication and they "talk back" to the infant. It seems that this is how children learn that the noises they are making can be used for making and maintaining contact with others. While certainly the use of language to maintain social contact with others is quite important in children’s and adult’s lives, the original biological roots of language reappear in its use as an activity in itself. Reflection about language and other means of communication is specific to humans. Thus, while at some early (but not too early) stages of language development, "the purpose of language is communication" (Bruner, 1974: 276), later language may be used as material in the hands of an artist or a scientist.

Objects of scientific knowledge owe their socio-cultural existence to linguistic forms of communication, but these linguistic forms do not completely determine their nature, because the non-discursive modes of thinking may twist and turn them around in discursively unpredictable
ways. The role of non-discursive modes of thinking is difficult to identify in research because of their implicit character and it is not easy to prove their evolutionary precedence over discursive thought. Such attempts have been made, however. For example - in Corballis’ (1997) study of brain functioning during mental rotation of shapes, which a mental activity considered to be "a paradigmatic example" of "a higher-order process that is non-symbolic and analog as opposed to propositional". The author observed that, while this activity engages one hemisphere (the right one) slightly more than the other, this bias is nothing compared to the concentration of the brain activity in the left hemisphere during language processing. This observation led the author to formulating the following hypothesis:

[T]he characteristically symbolic mode of the left hemisphere evolved relatively late and achieved the quality of recursive generativity only in the late stages of hominid evolution. This forced an increasingly right-hemispheric bias into analog processes like mental rotation. Such processes nevertheless remain important and integral even to those processes we think of as highly symbolic, such as language and mathematics. (Corballis, 1997)

Evidence for biological roots of mathematical thinking which stresses the importance of non-discursive modes of thinking is also provided by the phenomenon of Williams syndrome. Williams syndrome individuals are very sociable, communicating with great ease and producing grammatically sophisticated discourse in natural language. However, their visual processing is severely impaired and their mathematical skills remain at the level of a 7-year-old child (Rossen et al., 1996).

Williams syndrome adolescents are characteristically unable to perceive gross distinctions in orientation or to draw or copy simple stick figures. There is selective attention to details of a configuration at the expense of the whole. (Rossen et al., ibid., p. 375)

Given a shape made of identical small shapes, for example, a big letter D made of little Y’s, Williams syndrome adolescents would perhaps notice the Y’s but not that they are arranged into a big letter D. Asked to reproduce the figure, they would draw some Y’s, perhaps in two rows, or vertically. Although WMS adolescents use lexically rich language and correct grammar, they do not score as well as normal subjects on tasks requiring definitions of words. This supports the hypothesis that the biologically more primitive spatial sense is necessary even in such paradigmatically discursive tasks as defining.

2.8.4 Not all instances of communication are discursive

Following Benveniste (1966; 1974), Duval (1995) insisted on the condition that the use of language in discourse refers to something else than language itself (a condition implicit in the ordinary understanding of the word). When humans use some semiotic system of representation (a natural or a formal language, a picture, a diagram, a geometric figure, a graph, a musical
sound, see Duval, 1995, p. 27) to say something "about the world" (i.e. not just to produce the
sounds of a language, as in reciting an alphabet, or to play the scales or to display the colors in a
palette), in a way which is shared by those who use the system to communicate, then we could
say that they produce a discourse.

In this sense, when people use language in its *phatic* function, i.e. for the sake
maintaining contact with an interlocutor (Jakobson, 1960), they are not producing discourse.
Expressions such as "right?" at the end of an addresser’s statement and the "mmm" noises of the
addressee, in response to them are not saying anything "about the world". In some instances of
communication it doesn’t really matter what one says; it only matters that one is there.

Language in phatic function is not something specific to human language, but it is
something specific to communication. In fact, one could define communication as *the property of
any system, made of individual elements endowed with the ability to act independently, which
allows the coordination of these individual actions so that the system remains a system and does
not disintegrate*. One could speak of communication among the cells of an organism, as well as of
communication in social groups of animals or humans. Unlike the DA definition (see section
1.3.2) the proposed definition focuses on the coordination of *differences* between individual
participants rather than on "sharing *communalities*", which, as mentioned earlier, could lead to
trivial exchanges.

While DA focuses on communication in its theory and its ideology, it does not seem to
attach much importance to the phatic function of language in analyzing instances of
communication in the practice of research. In social situations, which force people to maintain
communication (which is the case if the conversation is being audio- or video-recorded for the
purposes of a research), one or both interlocutors may use language mainly in its phatic function.
What they say may have little to do with what they think, and the pressure to speak may even
prevent them from thinking the way they would if they didn’t have to maintain a conversation.
Some students simply cannot communicate and think at the same time because their thinking may
be visual or tactile, or other non-symbolic. They are then declared a failure in mathematics as
well as in communication. Students, who are sufficiently self-confident to not care about the
censure, may manage to speak out loud while in fact, thinking to themselves. They may produce
expected solutions but not a clear and coherent discourse. They may be using wrong terms for
what they think about, since they are not necessarily thinking in words. These students are then
blamed for not having the necessary communicational skills and it is recommended that they be
taught the relevant meta-discursive rules.
Other examples of communication without discourse could perhaps include children’s play with language, like in this inscription produced by a 6 year old:

\[ 1 = 8 + 0 - 2 \quad 2 = 1 \times 7 \quad 0 = \emptyset + 0 \]

Figure 3. A "mathematical sentence" produced by a 6 years old child

This child was trying to imitate inscriptions including number sentences that she saw in the colorful booklets addressed to children her age and in her computer games. She was thus communicating that she "could write". But the ideas of addition and equality used in playing games and sharing food and toys were at that time separate in her mind from the symbols "+" and "=". It is perhaps the same intention to communicate "I can write mathematics" that underlies the behavior of some of our undergraduate students who produce meaningless strings of set theoretic symbols by way of "proofs". This phenomenon has been analyzed as the obstacle of formalism in linear algebra students (Dorier, Robert, Robinet, Rogalski, 2000; Sierpinska, 2000).

2.8.5 Non-communicational functions of discourse

Children as young as 2 are found to engage in linguistic activities outside of a directly communicative context (Clark, 1978: 32). They spontaneously correct and comment upon "their own [and others] pronunciations, word forms, word order, and even choice of language in case of bilinguals"; they make "judgments of linguistic structure and function, deciding what utterances mean, whether they are appropriate or polite, whether they are grammatical" (Clark, ibid.). Children play with language, they "play with different linguistic units, segmenting words into syllables and sounds, making up etymologies, rhyming and punning" (ibid.). For Clark, the growing sophistication of meta-linguistic use of language is closely linked with the growth of meta-cognitive skills, such as: monitoring one’s ongoing utterances, checking the result of an utterance, testing for reality, deliberately trying to learn, predicting the consequences of using inflections, words, phrases or sentences, reflecting on the product of an utterance (ibid.: 34). The last skill includes such uses of language as, "providing definitions" and "explaining why certain sentences are possible and how they should be interpreted". These activities are basic in the construction of any theory, also a mathematical theory.

According to Duval, communication is one of the three functions of use of language, which are not specific to language but are common to all semiotic systems of representation.
(called "meta-discursive functions"). The other two are processing and objectivation. Formalized processing is specific to mathematics, logic and computer science. Objectivation is the use of language in the aim of obtaining some control over one’s activity and over one’s experience, whether physical or mental. Objectivation organizes and re-organizes one’s activity and experience and makes it the object of a conscious evaluation and decision. It is not a mere explicitation or expression of a thought. (Duval, 1995, p. 90)

The work of ’writing up’, the literary creation and finding words in the frame of an analysis come under this function of objectivation first of all. But objectivation is not specifically tied with language; it can also be realized with figural semiotic systems, such as drawing, for example. The function of objectivation is irreducible to a social function of communication. Trying to understand a discourse produced for objectivation purposes as if it was a discourse produced for communication purposes creates not only a misapprehension about what is being said but it also breaks the communication with the author of the discourse. (Duval, 1995: 90, with reference to Lacan, 1966; my translation from French, my emphasis).

DA does not distinguish among objectivation, processing and communication. This leads to occasional misinterpretation of discourses of students, who are using language for objectivation or even processing purposes, as symptomatic of their lack of communicational skills.

Apart from the above meta-discursive functions of language, Duval distinguished functions that are specific to the use of language ("discursive functions"). These functions are called referential (naming objects), apophantic (making statements about the named objects), discourse expansion (linking different statements into a coherent whole) and reflexive (marking the value, the mode or the status of the expressions used). Each function can be fulfilled by means of different "discursive operations" (reference to Grize, 1982). Discursive operations are used by the user of language to schematize or organize the discourse: describing, explaining, reasoning by rhetorical argumentation or by logical deduction, are examples of operations used in the discourse expansion function. The idea of discursive operations may partly overlap with DA’s "meta-discursive rules".

The distribution of the meta-discursive and discursive uses of language as well as of the discursive operations may be characteristic of the domain of reference of the discourse. Mathematics, for example, favors the meta-discursive function of processing, the referential discursive function, and the recurrent use of the operation of description in apophantic uses of language (Duval, 1995, p. 95).

Let me illustrate this on an example of a typical sentence in a linear algebra text: Let $T$ be a linear operator on a vector space $V$ of finite dimension over the field $K$. This sentence draws the reader’s attention to an object and gives it a name, $T$, for further reference. The operation of description is used 6 times. $T$ is a name of an operator described as linear, and described as defined on a space, described as vector space, further described as having its scalars in the field
named K and further *described* as being of dimension *described* as finite. We could represent the recurrent use of the operation of description by means of the following diagram:

\[ T \text{ is [linear [operator]] on a [[finite [dimensional]] [[vector [space]] over a field K]]} \]

The high number of descriptions per sentence and their use in a recurrent manner is quite specific to mathematics. (Other kinds of measures that distinguish mathematical discourse from literary discourses are presented in Duval, 1995, p. 108ff). In oral communication with students and colleagues we often simplify the discourse and omit the descriptors. We just say, "take an operator T" and the rest is understood from the context. But even if we use precise language in our lectures, students ignore the descriptors in an attempt to reduce the complexity of the discourse. For example, a statement such as,

\[
\text{If } U_1 = \text{Span } \{(1,0,1), (0,1,0)\}, \ U_2 = \text{Span } \{(1,0,-1), (0,1,-1)\}, \\
\text{then } U_1 + U_2 = \text{rowspace}( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} ) = \text{rowspace}( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ) = \mathbb{R}^3. 
\]

often becomes, in students’ rendition, something like,

\[
\text{basis } U_1 = (1,0,1), (0,1,0), \ U_2 = (1,0,-1), (0,1,-1), \\
U_1 + U_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim (\text{rref}) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{R}^3.
\]

For some students this is just shorthand, a result of a metonymy, and their concepts are correct even if their writing is rather sloppy. For other students this could be a literal substitution of concepts: the subspaces have each only two vectors and their sum is a matrix made of these vectors; \( \mathbb{R}^3 \) obtains whenever an identity 3x3 submatrix is reached in row reduction. Indeed, for some students the only object of linear algebra is the matrix and the only operation is row reduction. However, instead of lamenting over the students’ lack of conceptual understanding, we could look at their activity in a positive way. The students’ writing could be understood as a *non-discursive representation* — a schema or a script — of a *synoptic apprehension* of the expanded mathematical discourse, and only a first step towards a detailed understanding of the discourse (Duval, 1995, p. 354). This is a healthy and effective approach to the study of mathematics.
3. CONCLUSIONS

My first conclusion is,

Be aware of the political consequences of the ideology your program is promoting.

The second is related to fashion.

In her 1982 thesis Colette Laborde was defining language as a code. This did not prevent her to conduct a thorough study of discursive practices in mathematical textbooks and classrooms as well as to propose didactic situations in which communication was not just a part of a didactic or experimental contract but constituted the very condition of completion of the mathematical task (Laborde, 1982).

Jakobson’s theory of language functions has been forgotten because he was a structuralist and structuralism was criticized and rejected by authoritative opinions of socio-linguists such as Bourdieu. But even if Jakobson’s analyses were aimed at identifying the internal structure of semiotic systems (poetic texts, cookbooks, architecture, music), his research was deeply informed by a vast knowledge of the historical and cultural contexts in which these systems were born. I hope to have hinted at ways in which Jakobson’s theory can inspire us in interpreting instances of communication in our research.

Looking at language from the perspective of theories of representation systems may fail to take into account the historical, social and cultural contexts of uses of these systems, but this study may still lead to valuable insights into the specificities of mathematical discourses, as found, for example, in Duval’s work.

"Cognitivist" and "mentalist’ approaches may be similarly criticized for their socio-cultural blindness and inability to explain the processes of teaching and learning in actual schools and classrooms. But they are also focused on the specificity of mathematics and therefore can be useful in helping teachers to plan what they are going to discourse about in their classrooms, and prepare them to better understand and capitalize on students’ often awkwardly worded contributions to this discourse.

Beware of fashionable discourses.

Use of words such as "code" or "structuralism" does not necessarily discredit the results of a research.

My third conclusion addresses the issue of narrow specialisation.

In his often-cited paper, Thurston (1994) mentions so many different characteristics of understanding and communicating mathematics that mathematics educators of all origins could find arguments to support their theses. DA could even find arguments to support its claim that
Mathematical thinking is a collective enterprise. The point is, however, that for Thurston, mathematical thinking is both a collective and an individual endeavor, which is obviously the only reasonable position one can take in mathematics education.

Mathematics education research can afford narrow theories only at the expense of its scientific relevance.

The last conclusion is related to the fascination of mathematics education by theories from neighboring disciplines. Mathematics education has learned a lot from other disciplines. But in the fervor of conducting learned discourses about theories in psychology, sociology, linguistics, philosophy etc. we risk discoursing mathematics away from our own discipline.

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