ON NONABELIAN TENSOR ANALOGUES OF 2-ENGEL CONDITIONS

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Abstract. Tensor analogues of right 2-Engel elements in groups were introduced by D. P. Biddle and L.-C. Kappe. We investigate the properties of right 2-Engel tensor elements and introduce the concept of $2\otimes$-Engel margin. With the help of these results we describe the structure of $2\otimes$-Engel groups. In particular, we prove a tensor version of Levi’s theorem for 2-Engel groups and determine tensor squares of two-generator $2\otimes$-Engel $p$-groups.

1. Introduction

For any group $G$, the nonabelian tensor square $G \otimes G$ is a group generated by the symbols $g \otimes h$, subject to the relations

$$gg' \otimes h = (g' \otimes h^g)(g' \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h')(g^h \otimes h'),$$

where $g, g', h, h' \in G$ and $g^h = h^{-1}gh$. The more general concept of nonabelian tensor product of groups acting on each other in certain compatible way was introduced by R. Brown and J.-L. Loday in [5], following the ideas of R. K. Dennis [6]. This construction has its origins in algebraic K-theory as well as in homotopy theory, yet it has become interesting from a purely group-theoretical point of view since the paper of R. Brown, D. L. Johnson and E. F. Robertson [4]. Since then, many authors have been concerned with explicit computations of nonabelian tensor squares; see the paper of L.-C. Kappe [9] for a comprehensive survey of these results.

The main topic of [3] is consideration of tensor analogues of the center and centralizers in groups. More precisely, for a given group $G$ the subgroup $Z^\otimes(G)$ consisting of all $a \in G$ with $a \otimes x = 1_\otimes$ for every $x \in G$ is called the tensor center. This concept was introduced by G. J. Ellis [7]. Moreover, for a group $G$ and a non-empty subset $X$, the subgroup $C^\otimes_G(X) = \{a \in G : a \otimes x = 1_\otimes \text{ for all } x \in X\}$ is said to be the tensor annihilator of $X$ in $G$. Also, tensor analogues of right $n$-Engel elements have been defined. Recall that the set of right $n$-Engel elements of a group $G$ is defined by $R_n(G) = \{a \in G : [a, n, x] = 1 \text{ for all } x \in G\}$. Here $[a, n, x]$ stands for the commutator $[\cdots [[a, x], x], \cdots]$ with $n$ copies of $x$. It is well-known that $R_1(G) = Z(G)$ and that $R_2(G)$ is a subgroup of $G$ [13]. In contrast
with this, it was shown that for $n \geq 3$ the set $R_n(G)$ is not necessarily a subgroup [14]. The set of right $n_\otimes$-Engel elements of a group $G$ is then defined as

$$R_n^\otimes(G) = \{a \in G : [a, n^{-1}x] \otimes x = 1_\otimes \text{ for all } x \in G\}.$$ 

One of the results of [3] shows that $R_2^\otimes(G)$ is always a characteristic subgroup of $G$ containing $Z(G)$ and contained in $R_2(G)$. It is also shown by an example that these inclusions may be proper.

The purpose of this paper is to further investigate tensor analogues of 2-Engel structure in groups. In the first part of the paper we determine some further information about $R_2^\otimes(G)$ and provide some new characterizations of this subgroup. In particular, we define the tensor analogue of 2-Engel margin and show that there is a striking resemblance between the results about 2-Engel margin and the results about its tensor analogue. We use these results to obtain the structure of $2_\otimes$-Engel groups. Here the group $G$ is said to be $n_\otimes$-Engel when $[x, n^{-1}y] \otimes y = 1_\otimes$ for any $x, y \in G$. It is straightforward to see that every $2_\otimes$-Engel group is also 2-Engel. A well-known result of F. W. Levi (see [15, pp. 45–46]) states that every 2-Engel group $G$ is metabelian and nilpotent of class $\leq 3$ and the exponent of $\gamma_3(G)$ divides 3. Therefore it is hardly surprising that the following result is obtained: If $G$ is a $2_\otimes$-Engel group, then $G \otimes G$ is abelian, $\gamma_3(G) \leq Z^\otimes(G)$ and $([x, y] \otimes z)^3 = 1_\otimes$ for every $x, y, z \in G$. As a consequence, we obtain several characterizations of $2_\otimes$-Engel groups, once again indicating the strong correspondence between 2-Engel groups and $2_\otimes$-Engel groups.

Let $\mathfrak{G}$ be a group-theoretic property. A group $G$ is said to have a finite covering by $\mathfrak{G}$-subgroups if $G$ equals, as a set, to the union of finite family of $\mathfrak{G}$-subgroups. The finite coverings of groups by their 2-Engel subgroups were studied by L.-C. Kappe [10]. It is proved in that paper that a group $G$ has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. The situation is similar in the context of $2_\otimes$-Engel groups. We prove that a group $G$ can be covered by a finite family of $2_\otimes$-Engel subgroups if and only if $|G : R_2^\otimes(G)| < \infty$. Another result of [10] in this direction is that $G$ has a finite covering by 2-Engel normal subgroups if and only if $G$ is 3-Engel and $|G : R_2(G)| < \infty$. It is to be expected that there is a tensor analogue of this result, but we leave it for future consideration. It is not difficult to see that if $G$ has a finite covering by $2_\otimes$-Engel normal subgroups, then $G$ is $3_\otimes$-Engel and $|G : R_2^\otimes(G)| < \infty$. For the reverse conclusion one would probably need the characterization of $3_\otimes$-Engel groups by their normal closures analogous to [12].

Since every $2_\otimes$-Engel group has an abelian tensor square, there is a good chance to compute tensor squares of $2_\otimes$-Engel groups explicitly. We reduce these computations to consideration of tensor squares of groups of class $\leq 2$. 
With the help of this we compute tensor squares of two-generator $2$-Engel $p$-groups, using the results of [1] and [11]. It is worth mentioning that there is a minor error in the classification of two-generator $p$-groups of class 2 given by [1], so we give the correct result here. We also compute the kernel of the commutator map $\kappa : G \otimes G \to G'$ given by $g \otimes h \mapsto [g, h]$ for any nonabelian two-generator $2$-Engel $p$-group $G$. The group $\ker \kappa$ is of interest as it is isomorphic to the third homotopy group of the space $SK(G, 1)$ [5]. Beside that, we compute the Schur multiplier of $G$.

2. Preliminary results

In this section we summarize without proofs some basic results regarding computations in tensor squares and the results concerning 2-Engel groups which will be used throughout the paper without any further reference. The first lemma gives the right action version of [5, Proposition 3].

**Lemma 1** ([5]). Let $g, g', h, h' \in G$. The following relations hold in $G \otimes G$:

(a) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$.
(b) $(g' \otimes h')^g \circ h = (g' \otimes h')^g \circ h$.
(c) $[g, h] \otimes g' = (g \otimes h)^{-1} (g \otimes h) g'$.
(d) $g' \otimes [g, h] = (g \otimes h)^{-g'} (g \otimes h)$.
(e) $[g, h] \otimes [g', h'] = [g \otimes h, g' \otimes h']$.

Note here that $G$ acts on $G \otimes G$ by $(g \otimes h) g' = g g' \otimes h$. The next result is crucial in studying the analogy between commutators and tensors.

**Proposition 1** ([4]). For a given group $G$ there exists a homomorphism $\kappa : G \otimes G \to G'$ such that $\kappa : g \otimes h \mapsto [g, h]$. Moreover, $\ker \kappa \leq Z(G \otimes G)$ and $G$ acts trivially on $\ker \kappa$.

An element $a$ of a group $G$ is called a right 2-Engel element of $G$ if $[a, x, x] = 1$ for each $x \in G$. In a similar fashion, an element $a$ is said to be a left 2-Engel element of $G$ if $[x, a, a] = 1$ for each $x \in G$. The sets of right 2-Engel elements and left 2-Engel elements of $G$ are denoted by $R_2(G)$ and $L_2(G)$, respectively. For the properties of right 2-Engel elements we refer to [15, Theorem 7.13] and [16, Lemma 2.2, Theorem 2.3]. We list here some of them, especially those which turn out to have tensor analogues.

**Proposition 2** ([15], [16]). Let $G$ be a group, $a \in R_2(G)$ and $x, y, z \in G$.

(a) $a$ is also a left 2-Engel element and $a^G$ is abelian.
(b) $[a, x]^r = [a^r, x^s]$ for all $r, s \in \mathbb{Z}$.
(c) $[a, x, y] = [a, y, x]^{-1}$.
(d) $[a, [x, y]] = [a, x, y]^2$.
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With the help of this we compute tensor squares of two-generator $2$-Engel $p$-groups, using the results of [1] and [11]. It is worth mentioning that there is a minor error in the classification of two-generator $p$-groups of class 2 given by [1], so we give the correct result here. We also compute the kernel of the commutator map $\kappa : G \otimes G \to G'$ given by $g \otimes h \mapsto [g, h]$ for any nonabelian two-generator $2$-Engel $p$-group $G$. The group $\ker \kappa$ is of interest as it is isomorphic to the third homotopy group of the space $SK(G, 1)$ [5]. Beside that, we compute the Schur multiplier of $G$. 

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(e) $a^2 \in Z_3(G)$.
(f) $[a, [x, y], z] = 1$. 

Here $a^G$ denotes the normal closure of $a$ in $G$. This result is the main ingredient of the proof of Levi's theorem [15, pp. 45–46] that every 2-Engel group $G$ is nilpotent of class $\leq 3$ and the exponent of $\gamma_3(G)$ divides 3. We also list some characterizations of 2-Engel groups which will serve as a model for $2_\otimes$-Engel groups.

**Proposition 3 ([15]).** For a group $G$ the following assertions are equivalent:

(a) $G$ is a 2-Engel group.

(b) $C_G(x)$ is a normal subgroup of $G$ for every $x \in G$.

(c) $[x,[y,z]] = [x,y,z]^2$ for any $x,y,z \in G$.

(d) $[x,z,y]^{-1} = [x,y,z]$ for any $x,y,z \in G$.

(e) $x^G$ is abelian for every $x \in G$.

3. Right $2_\otimes$-Engel elements of groups

The main object of this section is the study of tensor analogues of right (left) 2-Engel elements of a given group. More precisely, for an arbitrary group $G$ we define the sets of right (left) $2_\otimes$-Engel elements of $G$ by $R^\otimes_2(G) = \{a \in G : [a,x] \otimes x = 1_\otimes \text{ for all } x \in G\}$ and $L^\otimes_2(G) = \{a \in G : [x,a] \otimes a = 1_\otimes \text{ for all } x \in G\}$, respectively. At the beginning we formulate some elementary properties of these two sets.

**Lemma 2.** Let $G$ be any group. We have:

(a) $R^\otimes_2(G) \subseteq R_2(G)$, $L^\otimes_2(G) \subseteq L_2(G)$.

(b) Every right $2_\otimes$-Engel element of $G$ also belongs to $L^\otimes_2(G)$.

(c) $L^\otimes_2(G) = \{a \in G : a^x \otimes a^y = a \otimes a \text{ for all } x,y \in G\}$.

**Proof.** Let $\kappa : G \otimes G \to G'$ be the commutator map. Let $a \in R^\otimes_2(G)$ and $x \in G$. Then we get $1 = \kappa([a,x] \otimes x) = [a,x,x]$, hence $a \in R_2(G)$. The inclusion $L^\otimes_2(G) \subseteq L_2(G)$ is proved in a similar way, therefore (a) is proved. To prove (b), pick $a \in R^\otimes_2(G)$ and $x \in G$. Then we have $1_\otimes = [a,ax] \otimes ax = [a,x] \otimes ax = ([a,x] \otimes a)^2 = ([x,a] \otimes a)^{-[a,x]x}$, hence $[x,a] \otimes a = 1_\otimes$ and therefore $a \in L^\otimes_2(G)$. So we are left with the proof of (c). Let $S = \{a \in G : a^x \otimes a^y = a \otimes a \text{ for all } x,y \in G\}$. For $a \in S$ and $x \in G$ we have $[a,x] \otimes a = a^{-1}a^x \otimes a = (a^{-1} \otimes a)a^x(a^x \otimes a) = 1_\otimes$, hence $a \in L^\otimes_2(G)$. Conversely, let $a \in L^\otimes_2(G)$ and $x,y \in G$. Then we obtain $a^x \otimes a^y = (a^x)^{y^{-1}} \otimes a)^y = a[a,xy^{-1}] \otimes a)^y = (a \otimes a)^{[a,xy^{-1}]y}(a,xy^{-1} \otimes a)^y$. Since $G$ acts trivially on ker $\kappa$, we have $(a \otimes a)^{[a,xy^{-1}]y} = a \otimes a$, whereas $[a,xy^{-1}] \otimes a = 1_\otimes$ by (b). This proves the assertion.

The following theorem is already proved in [3]:

**Theorem 1 ([3]).** For any group $G$, the set of all right $2_\otimes$-Engel elements of $G$ is a characteristic subgroup of $G$. 

The computations with tensors involving right \(2_\otimes\)-Engel elements are facilitated by the following result which has roots in corresponding rules for computation with 2-Engel elements [15, Theorem 7.13]. Before formulating the result, note that

\[
Z_n^\otimes(G) = \{ a \in G : [a, x_1, \ldots, x_{n-1}] \otimes x_n = 1_\otimes \text{ for all } x_1, \ldots, x_n \in G \}
\]

is a characteristic subgroup of \(G\) contained in the \(n\)-th center \(Z_n(G)\). This subgroup is called the \(n\)-th tensor center of \(G\) [3].

**Proposition 4.** Let \(G\) be a group, \(x, y, z \in G\) and \(a \in R_2^\otimes(G)\).

(a) \([a, x] \otimes y = ([a, y] \otimes x)^{-1}\).

(b) \([a, x] \in C_G^\otimes(x^G)\).

(c) \([a, x]^n \otimes y = ([a, x] \otimes y)^n\) for any \(n \in \mathbb{Z}\).

(d) \(a \otimes x^n = (a \otimes x)^n\) for any \(n \in \mathbb{Z}\).

(e) \([a, x] \otimes [y, z] = 1_\otimes\).

(f) \([x, y] \otimes a = ([x, a] \otimes y)^2\) and \(a \otimes [x, y] = ([a, x] \otimes y)^2\).

(g) \(a^2 \in Z_3^\otimes(G)\).

**Proof.** The identities (a) and (b) are already proved in [3, Lemma 5.1 and Lemma 5.2]. To prove (c), it suffices to assume that \(n > 0\). Now observe that \([a, x]^n \otimes y = ([a, x] \otimes y)((a, x)^{n-1} \otimes y)\), hence (c) follows by an induction on \(n\).

Before we proceed, note first that (a) implies that the elements of the form \(b \otimes z\), where \(b \in a^G\) and \(z \in G\), commute with each other. Expanding \(a \otimes xy\) and \(xy \otimes a\) using the tensor product rules, we have

\[
(1) \quad a \otimes xy = (a \otimes x)(a \otimes y)([a, x] \otimes y)
\]

and

\[
(2) \quad xy \otimes a = (x \otimes a)(y \otimes a)([x, a] \otimes y).
\]

The first equation yields

\[
a \otimes [x, y] = a \otimes (yx)^{-1}(xy) = (a \otimes xy)(a \otimes yx)^{-1}([a, (yx)^{-1}] \otimes xy)
\]

by [3, Lemma 5.1]. Since \(xy\) is a conjugate of \(yx\), we have \([a, (yx)^{-1}] \otimes xy = 1_\otimes\) by (b), hence \(a \otimes [x, y] = ([a, x] \otimes y)^2\). Similarly we prove \(a \otimes [x, y] = ([a, x] \otimes y)^2\). It is also clear that the equation (1) also implies (d).

It remains to prove that \([a, x] \otimes [y, z] = 1_\otimes\) and \(a^2 \in Z_3^\otimes(G)\). Expanding the identity \([a, x] \otimes yz = ([a, yz] \otimes x)^{-1}\), we obtain \(([a, x] \otimes z)([a, x] \otimes y)^z = ([a, z] \otimes x)^{-1}([a, y] \otimes z^{-1}, x^{-1})x^{-z}\). Since \([a, z, x] \otimes [a^2, y^n] = 1_\otimes\), it follows that \([a, y]^z\) acts trivially on \([a, z] \otimes x\). Thus we obtain, after cancellation and relabeling, \(1_\otimes = [a, y] \otimes [x, z] = ([a, [x, z]] \otimes y)^{-1} = ([a, x, z]^2 \otimes y)^{-1}\), hence \([a^2, x, y] \otimes z = 1_\otimes\).

The immediate consequence of Proposition 4 is the following characterization of \(R_2^\otimes(G)\).
Corollary 1. For any group $G$ we have $R_2^\odot(G) = \{a \in G : [a,x] \in C_G^\odot(x^G) \text{ for all } x \in G\}$.

It is known that $a \in R_2(G)$ implies that $a^G$ is abelian. The following corollary gives the corresponding result for right $2\otimes$-Engel elements.

Corollary 2. Let $a \in R_2^\otimes(G)$. Then the normal closure $(a \otimes x)^{G \otimes G}$ is an abelian group for any $x \in G$.

Proof. Let $a \in R_2^\otimes(G)$ and $\tau \in G \otimes G$. As usual, denote with $\kappa$ the commutator map $G \otimes G \to G'$. Then we have $[(a \otimes x), (a \otimes x)^\tau] = [a \otimes x, (a \otimes x)^{\kappa(\tau)}] = [a, x] \otimes [a^{\kappa(\tau)}, x^{\kappa(\tau)}] = 1_\otimes$ by Proposition 4. It follows by conjugation that every two elements of $(a \otimes x)^{G \otimes G}$ commute, as required.

Let $\phi(x_1, \ldots, x_n)$ be any word in the variables $x_1, \ldots, x_n$. For a group $G$ the associated marginal subgroup $\phi^*(G)$ (also called the $\phi$-margin of $G$) consists of all $a \in G$ such that $\phi(g_1, \ldots, ag_i, \ldots, g_n) = \phi(g_1, \ldots, g_i, \ldots, g_n)$ for every $g_i \in G$ and $1 \leq i \leq n$. It is clear that $\phi^*(G)$ is always a characteristic subgroup of $G$. Margins were first introduced by P. Hall [8]. In particular, marginal subgroups for the 2-Engel word $\phi(x,y) = [x, y, y]$ were studied by T. K. Teague [16]. Let $E_1(G) = \{a \in G : [ax, y, y] = [x, y, y] \text{ for all } x, y \in G\} = R_2(G)$ and $E_2(G) = \{a \in G : [x, ay, ay] = [x, y, y] \text{ for all } x, y \in G\}$. Then the 2-Engel margin of $G$ is $E(G) = E_1(G) \cap E_2(G)$. Now, the tensor analogues of these subgroups can be defined as

$$E_1^\otimes(G) = \{a \in G : [ax, y] \otimes y = [x, y] \otimes y \text{ for all } x, y \in G\},$$
$$E_2^\otimes(G) = \{a \in G : [x, ay] \otimes ay = [x, y] \otimes y \text{ for all } x, y \in G\},$$
and let $E^\otimes(G) = E_1^\otimes(G) \cap E_2^\otimes(G)$. It is not difficult to see that these sets are characteristic subgroups of $G$. Using Proposition 4, we also conclude that $E_1^\otimes(G) = R_2^\otimes(G)$.

In [16, Theorem 2.4] it is proved that $E(G) = \{a \in G : [x,a,y][x,y,a] = 1 \text{ for all } x, y \in G\}$. The following result is therefore hardly surprising:

Theorem 2. For any group $G$ we have

$$E^\otimes(G) = \{a \in G : ([x,a] \otimes y)([x,y] \otimes a) = 1_\otimes \text{ for all } x, y \in G\}.$$

Proof. Let $S = \{a \in G : ([x,a] \otimes y)([x,y] \otimes a) = 1_\otimes \text{ for all } x, y \in G\}$, let $a \in S$ and $x, y \in G$. It is clear that $a \in R_2^\otimes(G) = E_1^\otimes(G)$. Using Proposition 4, we have $[x, ay] \otimes ay = [x, y][x, a]^y \otimes ay = ([x, y][x, a]^y \otimes y)[x, y][x, a]^y \otimes a)^y = ([x, y] \otimes y)([x,a]^y)([x, a] \otimes y)^y([x, a] \otimes a)^y([x, a] \otimes a)^y = ([x, y] \otimes y)([x,a]^y)([x, a] \otimes a)^y. \text{ Observe that } ([x, a] \otimes a)^y = (a^{-1}y^a \otimes a)^y = (a \otimes a)^{-1}(a \otimes a) = 1_\otimes \text{ by Lemma 2, hence we only have to prove that } [x, a]^y \text{ acts trivially on } [x,y] \otimes y. \text{ To see this, we first note that } y^{[x,a]^y} = [y, [x,a]]y, \text{ hence } ([x, y] \otimes y)^{[x,a]^y} = [x, y] \otimes [y, [x,a]]y. \text{ As } [x,a] \in R_2^\otimes(G), \text{ we get } [x,a]y \otimes [x,y] = (|[x,a],[x,y]| \otimes y)^{-1} = 1_\otimes \text{ by Proposition 4, thus the}$$
inclusion $S \subseteq E^\otimes(G)$ is proved. Conversely, every $a \in E^\otimes(G)$ also belongs to $R^\otimes_2(G)$. Reversing the above arguments, we obtain $a \in S$, as required.

Let us mention an important consequence of this theorem.

**Corollary 3.** Let $G$ be a group, $x, y \in G$ and $a \in E^\otimes(G)$. Then $([a, x] \otimes y)^3 = [a^3, x] \otimes y = 1_\otimes$.

**Proof.** For $a \in E^\otimes(G)$ we get $1_\otimes = ([x, y] \otimes a)([x, a] \otimes y) = ([x, a] \otimes y)^3$ by Proposition 4, hence also $[a^3, x] \otimes y = 1_\otimes$. □

It is proved in [16] that $Z_2(G) \leq E(G) \leq Z_3(G)$ for any group $G$. Similar arguments show the following.

**Proposition 5.** For any group $G$ we have $Z^\otimes_2(G) \leq E^\otimes(G) \leq Z^\otimes_3(G)$.

**Proof.** It is clear that $Z^\otimes_2(G) \leq E^\otimes(G)$. Now, if $a \in E^\otimes(G)$, then $a^3 \in Z^\otimes_2(G) \leq Z^\otimes_3(G)$. On the other hand, we have $a^2 \in Z^\otimes_3(G)$ by Proposition 4, hence $a \in Z^\otimes_3(G)$. □

4. $2_\otimes$-Engel groups

A group $G$ is said to be $2_\otimes$-Engel when $[x, y] \otimes y = 1_\otimes$ for any $x, y \in G$. It is worth noting that $G$ is $2_\otimes$-Engel precisely when $R^\otimes_2(G) = G$, which is equivalent to $L^\otimes_2(G) = G$ and is also equivalent to $E^\otimes(G) = G$. Using the commutator map argument, it becomes clear that every $2_\otimes$-Engel group is also $2$-Engel. The structure of $2_\otimes$-Engel groups is described in the next result which corresponds to the well-known Levi’s theorem about $2$-Engel groups [15, pp. 45–46]:

**Theorem 3.** Let $G$ be a $2_\otimes$-Engel group. Then we have:

(a) $G \otimes G$ is abelian group.
(b) $\gamma_3(G) \leq Z^\otimes(G)$.
(c) $([x, y] \otimes z)^3 = 1_\otimes$ for any $x, y, z \in G$.

**Proof.** It follows directly from Proposition 4 that $G \otimes G$ is abelian. From the same proposition we obtain $([x, y, z] \otimes v)^2 = [x, y, z]^2 \otimes v = [x, [y, z]] \otimes v = ([x, v] \otimes [y, z])^{-1} = 1_\otimes$. Furthermore, since $E^\otimes(G) = G$, we get (b) and (c) by Corollary 3.

In contrast with this result, there exists a $2$-Engel group $G$ such that $\text{cl}(G \otimes G) = 2$ [2]. The following is a tensor analogue of Proposition 3:

**Corollary 4.** The following statements for a group $G$ are equivalent:

(a) $G$ is $2_\otimes$-Engel.
(b) $[x, y] \otimes z = ([x, z] \otimes y)^{-1}$ for any $x, y, z \in G$.
(c) $x \otimes [y, z] = ([x, y] \otimes z)^2$ for any $x, y, z \in G$.
(d) $x^y \otimes x^z = x \otimes x$ for any $x, y, z \in G$. 
Additionally, if $G$ is a $2_\otimes$-Engel group, then $C_{G}\otimes\gamma_G(g) \triangleleft G$ for any $g \in G$.

**Proof.** By Proposition 4, (a), (b) and (c) are equivalent. The equivalence between (a) and (d) is established in Lemma 2, (c). Now let $G$ be a $2_\otimes$-Engel group, let $g, y \in G$ and let $x \in C_{G}\otimes\gamma_G(g) \leq C_G(g)$. Then we have $x^y \otimes g = x[x, y] \otimes g = [x, y] \otimes g = ([x, g] \otimes y)^{-1} = 1_\otimes$, thus $x^y \in C_{G}\otimes\gamma_G(g)$. This proves the corollary.

It is evident that the condition $C_{G}\otimes\gamma_G(g) \triangleleft G$ for any $g \in G$ may fail to imply that $G$ is $2_\otimes$-Engel, as $C_{G}(g)$ does not necessarily contain $g$.

Turning our attention to finite coverings by $2_\otimes$-Engel subgroups, we mention here a related result of L.-C. Kappe [10] which states that a group $G$ has a finite covering by 2-Engel subgroups if and only if $|G : R_2(G)| < \infty$. Our proof of the tensor analogue follows the lines of Kappe’s proof.

**Theorem 4.** A group $G$ has a finite covering by $2_\otimes$-Engel subgroups if and only if $|G : R_2^{\otimes}(G)| < \infty$.

**Proof.** Suppose that $G = \bigcup_{i=1}^{n} H_i$, where $H_i$ are $2_\otimes$-Engel subgroups of $G$. The standard reduction step, due to B. H. Neumann (see [10]), shows that we may assume that $|G : H_i| < \infty$ for every $i$. Hence the subgroup $D = \bigcap_{i=1}^{n} H_i$ has a finite index in $G$. It is clear that $D \leq R_2^{\otimes}(G)$, hence $|G : R_2^{\otimes}(G)| < \infty$.

Assume now $|G : R_2^{\otimes}(G)| < \infty$. Let $\{g_1, \ldots, g_n\}$ be a transversal of $R_2^{\otimes}(G)$ in $G$ and let $H_i = \langle g_i \rangle R_2^{\otimes}(G)$. We have $G = \bigcup_{i=1}^{n} H_i$, hence it suffices to prove that each $H_i$ is $2_\otimes$-Engel. Let $y = g^i a$ and $x = g^j b$ be arbitrary elements of $\langle g \rangle R_2^{\otimes}(G)$, where $i, j \in \mathbb{Z}$ and $a, b \in R_2^{\otimes}(G)$. Since $R_2^{\otimes}(G) = E_2^{\otimes}(G)$, we obtain, using Proposition 4, $[x, y] \otimes a = [g^j(a) \otimes g^i a] = ([g^j, a] \otimes g^i a)^{ij} = 1_\otimes$, as required.

**Remark.** Suppose that a group $G$ has a finite covering by $2_\otimes$-Engel normal subgroups $N_1, \ldots, N_n$. Again we may assume that $|G : N_i| < \infty$ and by Theorem 4 we also have $|G : R_2^{\otimes}(G)| < \infty$. Since for every $x \in G$ we have $x^G \leq N_i$ for some $i$, we conclude that every normal closure of an element of $G$ is $2_\otimes$-Engel. In particular, we have $1_\otimes = [x^{-y}, x] \otimes x = ([y, x, x] \otimes x)^{-1}$, hence $G$ is $3_\otimes$-Engel. In view of [10] it is likely that a $3_\otimes$-Engel group $G$ with $|G : R_2^{\otimes}(G)| < \infty$ has a finite normal covering by $2_\otimes$-Engel subgroups, but we have not been able to (dis)prove this, since there are no known tensor analogues of results regarding 3-Engel groups [12].

5. Tensor squares of $2_\otimes$-Engel groups

We have proved in the previous section that $2_\otimes$-Engel groups have abelian tensor squares. Moreover, if $G$ is a $2_\otimes$-Engel group, then $\gamma_3(G) \leq Z(G)$ by Theorem 3. Using a result of G. J. Ellis [7], we see that $G \otimes G \cong G/\gamma_3(G) \otimes G/\gamma_3(G)$, hence the calculations of tensor squares reduce to the
calculations of tensor squares of class 2 groups (of course, the situation becomes even better when $G$ is abelian).

Let $G$ be a nonabelian two-generator $2_\oslash$-Engel $p$-group. The group $G/\gamma_3(G)$ is a two-generator $2_\oslash$-Engel $p$-group of class 2. From [1] and [11] we obtain the complete classification of two-generator $p$-groups of class 2, hence we only have to check which of these groups are $2_\oslash$-Engel. The following lemma provides a useful criterion for this task:

**Lemma 3.** Let $G$ be a two-generator group of class two. Then $G$ is $2_\oslash$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$.

**Proof.** Let $G = \langle a, b \rangle$ be a group of class two and let $x, y \in G$. Then $x = a^ib^j[a, b]^k$ and $y = a^{i'}b^{j'}[a, b]^{k'}$ for some $i, i', j, j', k, k' \in \mathbb{Z}$. By means of linear expansion we obtain $[x,y] = [a,b]^{i'i''+j'j''}$, hence $[x,y] \otimes [a,b] = (a \otimes [a,b])^{i'i''+j'j''}$. Therefore $G$ is $2_\oslash$-Engel if and only if $a \otimes [a,b] = b \otimes [a,b] = 1_\oslash$, which is equivalent to $x \otimes [y,z] = 1_\oslash$ for all $x,y,z \in G$. By [9, Theorem 3], $G$ is $2_\oslash$-Engel if and only if $G \otimes G \cong G^{ab} \otimes G^{ab}$. □

The recipe for computing tensor squares of two-generator $2_\oslash$-Engel $p$-groups therefore consists of looking for those two-generator $p$-groups $G$ of class two which satisfy the condition $G \otimes G \cong G^{ab} \otimes G^{ab}$. Note also that if $G^{ab} \cong \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$, then $G^{ab} \otimes G^{ab}$ is isomorphic to the direct product of all $\mathbb{Z}_{gcd(a_i,a_j)}$, where $i,j = 1, \ldots, r$.

First assume $p$ is odd. Then we have the following cases [1]:

**Case 1.** $G \cong \langle (c) \times (a) \rangle \times (b)$, where $[a,b] = c, [a,c] = [b,c] = 1, |a| = p^\alpha, |b| = p^\beta, |c| = p^\gamma$ and $\alpha \geq \beta \geq \gamma \geq 1$. Here we have $G \otimes G \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\gamma}$, hence $G \otimes G \not\cong G^{ab} \otimes G^{ab}$.

**Case 2.** $G \cong \langle (a) \times (b) \rangle$, where $[a,b] = a^{p^\alpha-\gamma}, |a| = p^\alpha, |b| = p^\beta, |[a,b]| = p^\gamma$ and $\beta \geq \gamma \geq 1, \alpha \geq 2\gamma$; by a closer inspection of the proof of [1, Theorem 2.4] it becomes clear that the extra condition $\alpha \geq \beta$ given there is irrelevant. By [1, Theorem 4.2] we have $G \otimes G \cong \langle (a) \otimes (a) \rangle \times \langle (b) \otimes (b) \rangle \times \langle (b) \otimes (a) \rangle$, where $|a \otimes a| = p^{\alpha-\gamma}, |b \otimes b| = p^\beta, |(b \otimes a)(a \otimes b)| = p^{\min\{\alpha-\gamma, \beta\}}$ and $|b \otimes a| = n$, where $n = \gcd(p^\alpha, \sum_{k=0}^{p^\beta-1}(p^\alpha-p^{\alpha-\gamma}+1)^k)$. Applying [1, Lemma 4.1], we immediately obtain $n = p^{\min\{\alpha, \beta\}}$, hence $G \otimes G$ is isomorphic to $\mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\alpha-\gamma} \times \mathbb{Z}_{p^{\min\{\alpha, \beta\}}} \times \mathbb{Z}_{p^{\min\{\alpha-\gamma, \beta\}}}$. Since $G^{ab} \cong \mathbb{Z}_{p^\alpha-\gamma} \times \mathbb{Z}_{p^\beta}$, we get $G^{ab} \otimes G^{ab} \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\alpha-\gamma} \times \mathbb{Z}_{p^{\min\{\alpha-\gamma, \beta\}}}^2$. This yields that $G$ is $2_\oslash$-Engel if and only if $\min\{\alpha - \gamma, \beta\} = \min\{\alpha, \beta\}$ which is equivalent to $\alpha \geq \beta + \gamma$.

**Case 3.** $G \cong \langle (c) \times (a) \rangle \times (b)$, where $[a,b] = a^{p^\alpha-\gamma}c, [c,b] = a^{-p^{2(\alpha-\gamma)}-c^{-p^{\alpha-\gamma}}}$, $|a| = p^\alpha, |b| = p^\beta, |[a,b]| = p^\gamma, |c| = p^n$, $\alpha \geq \beta \geq \gamma > \sigma \geq 1$ and $\alpha + \sigma \geq 2\gamma$. Let $\delta = \min\{\alpha - \gamma, \beta\}$ and $\tau = \min\{\alpha - \gamma, \sigma\}$. Then we have $G \otimes G \cong \mathbb{Z}_{p^\alpha-\gamma} \times \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^\gamma}^2$, hence it is not isomorphic to $G^{ab} \otimes G^{ab}$.

For $p = 2$ the situation is more complicated [11]:

**Case 4.** $G \cong \langle (c) \times (a) \rangle \times (b)$, where $[a,b] = c, [a,c] = [b,c] = 1, |a| = 2^n,$
\(|b| = 2^\beta, |c| = 2^\gamma\) and \(\alpha \geq \beta \geq \gamma \geq 1\). Here we have

\[
G \otimes G \cong \begin{cases} 
\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\gamma} & : \beta > \gamma, \\
\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\gamma+1} \times \mathbb{Z}_{2^\gamma-1} \times \mathbb{Z}_{2^\min(a-1,\gamma)} & : \beta = \gamma.
\end{cases}
\]

It follows from here that \(G \otimes G \not\cong G^{ab} \otimes G^{ab}\).

(Case 5.) \(G \cong \langle a \rangle \times \langle b \rangle\), where \([a, b] = a^{2\alpha-\gamma}, |a| = 2^\alpha, |b| = 2^\beta, [[a, b]] = 2^\gamma\) and \(\alpha, \beta, \gamma \in \mathbb{N}, \alpha \geq 2\gamma, \beta \geq \gamma\) and \(\alpha + \beta > 3\). In this particular case, \(G \otimes G\) is isomorphic to \(\mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{2^\alpha-\gamma+1} \times \mathbb{Z}_{2^\min(a-\gamma,\beta)} \times \mathbb{Z}_{2^\min(a,\beta)}\). It is straightforward to verify that \(G \otimes G \not\cong G^{ab} \otimes G^{ab}\).

(Case 6.) \(G \cong \langle (c) \times (a) \rangle \times \langle b \rangle\), where \([a, b] = a^{2\alpha-\gamma}c, |a| = a^{-2\alpha(a-\gamma)}c^{-2\alpha-\gamma}, |a| = 2^\alpha, |b| = 2^\beta, [[a, b]] = 2^\gamma, |c| = 2^\sigma\) with \(\alpha, \beta, \gamma, \sigma \in \mathbb{N}, \alpha + \sigma \geq 2\gamma\) and \(\beta \geq \gamma > \sigma\). Let \(\rho = \min\{\alpha - \gamma + \sigma, \beta\}\). Then we have

\[
G \otimes G \cong \begin{cases} 
\mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^\gamma+1} \times \mathbb{Z}_{2^\gamma-1} & : \alpha = \gamma + 1, \beta = \gamma, \\
\mathbb{Z}_{2^\alpha-\gamma+1} \times \mathbb{Z}_{2^\beta} \times \mathbb{Z}_{2^\min(a,\beta)} \times \mathbb{Z}_{2^\rho} & : \alpha \geq \gamma + 2 \text{ or } \beta \geq \gamma + 1.
\end{cases}
\]

It is clear that \(G \otimes G\) is not isomorphic to \(G^{ab} \otimes G^{ab}\).

We summarize our conclusions in the following theorem:

**Theorem 5.** Let \(G\) be a nonabelian two-generator \(2\supset\)Engel \(p\)-group. Then \(p \neq 2\) and \(G/\gamma_2(G) \cong \langle a \rangle \times \langle b \rangle\), where \([a, b] = a^{2\rho-\gamma}, |a| = p^\rho, |b| = p^\beta, [[a, b]] = p^\gamma\) with \(\alpha \geq \beta \geq \gamma \geq 1, \alpha \geq 2\gamma\) and \(\alpha \geq \beta + \gamma\). We have \(G \otimes G \cong \langle (a \otimes a) \times (b \otimes b) \times (\langle b \otimes a \rangle(a \otimes b)) \times (b \otimes a) \rangle \cong \mathbb{Z}_{2^\rho} \times \mathbb{Z}_{p^\rho-\gamma}\).

Our considerations also show the following.

**Corollary 5.** Every \(2\supset\)Engel \(2\)-group is abelian.

More generally, if \(G\) is a \(2\supset\)Engel group without elements of order 3, then \(G' \lesssim Z^2(G)\) by Theorem 3. This, together with the result of Ellis [7], implies \(G \otimes G \cong G^{ab} \otimes G^{ab}\).

Let \(G\) be a group. From a topological point of view, the third homotopy group \(\pi_3SK(G,1)\) of the suspension of \(K(G,1)\) is of some interest. A combinatorial description of \(\pi_nSK(G,1)\) has been given by J. Wu [17]. Observing the formula \(\pi_3SK(G,1) \cong \ker \kappa [5]\), one can use a different approach when \(G \otimes G\) is explicitly computed. Applying Theorem 5, we describe \(\pi_3SK(G,1)\) for any nonabelian two-generator \(2\supset\)Engel \(p\)-group \(G\). We also determine the Schur multiplier \(H_2(G)\) of \(G\).

**Corollary 6.** Let \(G\) be a nonabelian two-generator \(2\supset\)Engel \(p\)-group, let \(\kappa : G \otimes G \to G'\) be the commutator map and let \(a, b, \alpha, \beta, \gamma\) be as in Theorem 5. Then \(\pi_3SK(G,1) \cong \ker \kappa \cong \langle (a \otimes a) \times (b \otimes b) \times (\langle b \otimes a \rangle(a \otimes b)) \times (b \otimes a) \rangle \cong \mathbb{Z}_{2^\rho} \times \mathbb{Z}_{p^\rho-\gamma} \times \mathbb{Z}_{p^\rho-\gamma} \) and \(H_2(G) \cong \mathbb{Z}_{p^\rho-\gamma}\).

**Proof.** As \(\kappa(a \otimes a) = \kappa(b \otimes b) = \kappa((b \otimes a)(a \otimes b)) = \kappa((b \otimes a)^{p\gamma}) = 1, \) Theorem 5 gives \(\ker \kappa \cong \langle (a \otimes a) \times (b \otimes b) \times (\langle b \otimes a \rangle(a \otimes b)) \times (b \otimes a) \rangle \cong \mathbb{Z}_{2^\rho} \times \mathbb{Z}_{p^\rho-\gamma} \times \mathbb{Z}_{p^\rho-\gamma},\) as required. To compute the Schur multiplier of \(G\), note for instance that the exactness of rows and columns in commutative diagram (1) in [4] implies
$H_2(G) \cong \ker \kappa / \Delta(G)$, where $\Delta(G) = \langle x \otimes x : x \in G \rangle$. Now, every $x \in \langle a, b \rangle$ can be written in the form $x = a^m b^n [a, b]^k$, where $m, n, k \in \mathbb{Z}$. Expanding $x \otimes x$ linearly, we obtain $x \otimes x = (a \otimes a)^m (b \otimes b)^n ((b \otimes a)(a \otimes b))^{mn}$. This yields $\Delta(G) \cong \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \cong \mathbb{Z}_{p^\beta} \times \mathbb{Z}_{p^{\alpha - \gamma}}$, hence the result. \hfill $\square$

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References


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