FINITE GROUPS IN WHICH SOME PROPERTY OF TWO-GENERATOR SUBGROUPS IS TRANSITIVE

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Abstract. Finite groups in which a given property of two-generator subgroups is a transitive relation are investigated. We obtain a description of such groups and prove in particular that every finite soluble-transitive group is soluble. A classification of finite nilpotent-transitive groups is also obtained.

1. Introduction

Let $\mathfrak{X}$ be a group theoretical class. A group $G$ is said to be $\mathfrak{X}$-transitive (or an $\mathfrak{X}$T-group) if for all $x, y, z \in G \setminus \{1\}$ the relations $\langle x, y \rangle \in \mathfrak{X}$ and $\langle y, z \rangle \in \mathfrak{X}$ imply $\langle x, z \rangle \in \mathfrak{X}$. In graph theoretical terms, let $\Gamma_{\mathfrak{X}}(G)$ be the simple graph whose vertices are the nontrivial elements of $G$, and $a$ and $b$ are connected by an edge if and only if $\langle a, b \rangle \in \mathfrak{X}$. Then $G$ is an $\mathfrak{X}$T-group precisely when all the connected components of $\Gamma_{\mathfrak{X}}(G)$ are complete graphs. Several authors have studied $\mathfrak{X}$T-groups for some special classes $\mathfrak{X}$. When $\mathfrak{X}$ is the class of all abelian groups, these groups are also known as commutative-transitive groups or CT-groups. Weisner [10] has shown that finite CT-groups are either soluble or simple. Finite nonabelian simple CT-groups have been classified by Suzuki [6]. These are precisely $\text{PSL}(2, 2^f)$, where $f > 1$. A characterization of finite soluble CT-groups has been given by Wu [11] who has also obtained information on locally finite CT-groups and polycyclic CT-groups. When $\mathfrak{X} = \mathfrak{A}_c$, the class of all groups which are nilpotent of class $\leq c$, similar results have been obtained in [1].

The purpose of this note is to obtain a description of finite $\mathfrak{X}$T-groups for the group theoretical classes $\mathfrak{X}$ having the following properties:

$(\star)$ $\mathfrak{X}$ is subgroup closed, it contains all finite abelian groups and is bigenetic in the class of all finite groups.

Here a class $\mathfrak{X}$ is said to be bigenetic (a terminology due to Lennox [4]) in the class of all finite groups when a finite group $G$ is in $\mathfrak{X}$ if and only if all its two-generator subgroups are. Examples of classes satisfying $(\star)$ are the class of all abelian groups, all nilpotent groups, all supersoluble groups and all soluble groups. First we show that if $\mathfrak{X}$ is a class satisfying $(\star)$, then every finite $\mathfrak{X}$T-group

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which does not belong to \( \mathfrak{X} \) is either a Frobenius group with kernel and complement belonging to \( \mathfrak{X} \), or it has no normal \( \mathfrak{X} \)-subgroups, i.e., it is \( \mathfrak{X} \)-semisimple as defined in [5]. We also show that in several cases, e.g., in the soluble or super-soluble case, the second possibility does not occur. As a consequence we obtain that a finite group is soluble if and only if it is soluble-transitive. In the case when \( \mathfrak{X} = \mathfrak{N} \), the class of all nilpotent groups, there exist simple \( \mathfrak{NT} \)-groups. We obtain a complete classification of finite \( \mathfrak{NT} \)-groups which generalizes some results of [11].

2. Results

Given a group theoretical class \( \mathfrak{X} \), let \( R_{\mathfrak{X}}(G) \) be the product of all normal \( \mathfrak{X} \)-subgroups of \( G \) (the \( \mathfrak{X} \)-radical of \( G \)). In general \( R_{\mathfrak{X}}(G) \) does not belong to \( \mathfrak{X} \). Our first result shows that this is however true within the class of all finite \( \mathfrak{X} \)-transitive groups when \( \mathfrak{X} \) satisfies the properties \((\ast)\).

**Lemma 2.1.** Let \( \mathfrak{X} \) be a class of groups satisfying \((\ast)\) and let \( G \) be a finite \( \mathfrak{XT} \)-group. Then \( R_{\mathfrak{X}}(G) \) is an \( \mathfrak{X} \)-group.

**Proof.** Let \( M \) and \( N \) be normal \( \mathfrak{X} \)-subgroups of \( G \). It suffices to show that \( MN \) also belongs to \( \mathfrak{X} \). Suppose first that \( M \cap N \neq 1 \) and let \( x \in M \cap N \setminus \{1\} \).

First note that for any \( m \in M \setminus \{1\} \) and \( n \in N \setminus \{1\} \) we have that \( \langle m, x \rangle \) and \( \langle x, n \rangle \) belong to \( \mathfrak{X} \). As \( G \) is an \( \mathfrak{XT} \)-group, we conclude that \( \langle m, n \rangle \) is an \( \mathfrak{X} \)-group. Now let \( m_1, m_2 \in M \setminus \{1\} \) and \( n \in N \setminus \{1\} \). We may suppose that \( m_1 \neq 1 \). Then \( \langle m_1 n, m_1 \rangle = \langle m_1, n \rangle \) is in \( \mathfrak{X} \) and \( \langle m_1, m_2 \rangle \) is in \( \mathfrak{X} \). Thus it follows that \( \langle m_1 n, m_2 \rangle \) also belongs to \( \mathfrak{X} \). Similarly we can prove that \( \langle mn_1, n_2 \rangle \) is in \( \mathfrak{X} \) for every \( m \in M \setminus \{1\} \) and \( n_1, n_2 \in N \setminus \{1\} \). Now take \( m_1, m_2 \in M \setminus \{1\} \) and \( n_1, n_2 \in N \setminus \{1\} \) and suppose that \( m_1 n_1 \neq 1 \), \( m_2 n_2 \neq 1 \). Then \( \langle m_1 n_1, m_2 \rangle \) and \( \langle m_2, m_2 n_2 \rangle \) in \( \mathfrak{X} \), whence \( \langle m_1 n_1, m_2 n_2 \rangle \) belongs to \( \mathfrak{X} \). This shows that every two-generator subgroup of \( MN \) belongs to \( \mathfrak{X} \). Since \( \mathfrak{X} \) is bigeneric in the class of all finite groups, we get that \( MN \) is an \( \mathfrak{X} \)-group, as required.

Suppose now that \( M \cap N = 1 \). Then \([M, N] = 1 \). As above it suffices to prove that every two-generator subgroup of \( MN \) is in \( \mathfrak{X} \). At first let \( m_1, m_2 \in M \setminus \{1\} \) and \( n \in N \setminus \{1\} \). Then the groups \( \langle m_1 n, n \rangle = \langle m_1, n \rangle \) and \( \langle n, m_2 \rangle \) are abelian, hence they belong to \( \mathfrak{X} \). By the transitivity we have that \( \langle m_1 n, m_2 \rangle \) belongs to \( \mathfrak{X} \). Similar argument shows that \( \langle mn_1, n_2 \rangle \) is in \( \mathfrak{X} \) for every \( m \in M \setminus \{1\} \) and \( n_1, n_2 \in N \setminus \{1\} \). From this it follows that if \( m_1, m_2 \in M \setminus \{1\} \) and \( n_1, n_2 \in N \setminus \{1\} \), then \( \langle m_1 n_1, m_2 \rangle \) and \( \langle m_2, m_2 n_2 \rangle \) are in \( \mathfrak{X} \), hence \( \langle m_1 n_1, m_2 n_2 \rangle \) is also in \( \mathfrak{X} \). This concludes the proof. \( \square \)

**Theorem 2.2.** Let \( \mathfrak{X} \) be a class of groups satisfying \((\ast)\). Let \( G \) be a finite \( \mathfrak{XT} \)-group. Then one of the following holds.

(i) \( G \) belongs to \( \mathfrak{X} \).

(ii) \( G \) is \( \mathfrak{X} \)-semisimple.

(iii) \( G \) is a Frobenius group with kernel and complement both belonging to \( \mathfrak{X} \).
Lemma 2.3. Let \( \mathfrak{X} \) satisfy (\( \ast \)). Let \( G \) be a finite \( \mathfrak{X} \)-group and \( H \) an \( \mathfrak{X} \)-subgroup of \( G \). Then

\[
C_G^\mathfrak{X}(H) = \{ x \in G : \langle x, h \rangle \in \mathfrak{X} \text{ for some } h \in H \setminus \{1\} \}
\]

is an \( \mathfrak{X} \)-subgroup of \( G \) containing \( H \).

Proof. Clearly \( C_G^\mathfrak{X}(H) \) contains \( H \). Let \( x, y \in C_G^\mathfrak{X}(H) \setminus \{1\} \). Then there exist \( h, k \in H \setminus \{1\} \) such that \( \langle x, h \rangle \in \mathfrak{X} \) and \( \langle y, k \rangle \in \mathfrak{X} \). Since \( \langle h, k \rangle \in \mathfrak{X} \), we get that \( \langle x, y \rangle \) also belongs to \( \mathfrak{X} \). If \( xy \neq 1 \), then \( \langle xy, y \rangle = \langle x, y \rangle \) belongs to \( \mathfrak{X} \), hence also \( \langle xy, k \rangle \in \mathfrak{X} \). Thus \( xy \in C_G^\mathfrak{X}(H) \). Note also that every two-generator subgroup of \( C_G^\mathfrak{X}(H) \) is in \( \mathfrak{X} \), hence \( C_G^\mathfrak{X}(H) \) also belongs to \( \mathfrak{X} \). \( \square \)
Proposition 2.4. Let $\mathcal{X}$ be a group theoretical class satisfying $(\ast)$. Let $G$ be a Frobenius group with kernel $F$ and complement $H$. Then $G$ is an $\mathcal{X}$T-group if and only if $C^\mathcal{X}_G(F)$ and $C^\mathcal{X}_G(H)$ are $\mathcal{X}$-groups.

Proof. Let $\mathcal{X}$ and $G$ be as above. If $G$ is an $\mathcal{X}$T-group, then it follows from Theorem 2.2 that $F$ and $H$ belong to $\mathcal{X}$. Consequently $C^\mathcal{X}_G(F)$ and $C^\mathcal{X}_G(H)$ are also $\mathcal{X}$-groups by Lemma 2.3. Conversely, suppose that $C^\mathcal{X}_G(F)$ and $C^\mathcal{X}_G(H)$ are $\mathcal{X}$-groups. Let $x, y, z \in G \setminus \{1\}$ and suppose that $\langle x, y \rangle \in \mathcal{X}$ and $\langle y, z \rangle \in \mathcal{X}$. Assume first that $y \in F$. Then $x, z \in C^\mathcal{X}_G(F)$ and consequently $\langle x, z \rangle \in \mathcal{X}$. If $y \notin F$, then $y \in H^g$ for some $g \in G$. But then $x, z \in C^\mathcal{X}_G(H^g) = (C^\mathcal{X}_G(H))^g$, whence $\langle x, z \rangle$ belongs to $\mathcal{X}$. Thus $G$ is an $\mathcal{X}$T-group. □

When $\mathcal{X}$ is the class of all abelian groups, then all three possibilities of Theorem 2.2 can occur [6, 11]. In some cases, however, we can exclude the existence of $\mathcal{X}$-semisimple $\mathcal{X}$T-groups.

Theorem 2.5. Let $\mathcal{X}$ be a class of groups satisfying $(\ast)$, and suppose that $\mathcal{X}$ contains all finite dihedral groups and that every finite $\mathcal{X}$-group is soluble. If $G$ is a finite $\mathcal{X}$T-group which is not in $\mathcal{X}$, then $G$ is a Frobenius group with complement belonging to $\mathcal{X}$. In particular, $G$ is soluble.

Before proving this result we mention here the well known Thompson’s classification of minimal simple groups, i.e., finite nonabelian simple groups all whose proper subgroups are soluble. It turns out [9] that every such group is isomorphic to one of the following groups.

(i) $\text{PSL}(2, p)$, where $p$ is a prime, $p > 3$ and $p^2 - 1 \neq 0 \mod 5$.
(ii) $\text{PSL}(2, 2^f)$, where $f$ is a prime.
(iii) $\text{PSL}(2, 3^f)$, where $f$ is an odd prime.
(iv) $\text{PSL}(3, 3)$.
(v) $\text{Sz}(q)$, where $q = 2^{2n+1}$ and $2n + 1$ is a prime.

If $G = \text{PSL}(2, F)$ where $F$ is a Galois field of odd characteristic and $|F| > 5$, then $G$ can be generated by an involution and an element of even order. This can be easily seen as follows. Let $q = |F|$. By Dickson’s theorem [2], $G$ contains elements $a$ and $b$ with $|a| = (q - 1)/2$ and $|b| = (q+1)/2$. Note that precisely one of $|a|, |b|$ is even, without loss of generality we may assume that this is true for $|a|$. Then $N_G(a) = D_{q-1}$ and this is the only maximal subgroup of $G$ containing $a$; this follows from the proof of Dickson’s theorem [2]. So if we choose any involution $u$ from $G \setminus N_G(a)$, we have $\langle a, u \rangle = G$, as required. A similar result holds true for $\text{PSL}(3, 3)$ and $\text{Sz}(q)$. In the first case note that $\text{PSL}(3, 3)$ can be generated by the canonical projections of matrices

\[
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix},
\]
which are of orders 2 and 8 in PSL(3,3), respectively. For the Suzuki groups Sz(q) it follows from [8] that they can always be generated by an involution and an element of order 4. We summarize this in the following lemma.

**Lemma 2.6.** Let $G$ be one of the following groups: PSL(2, $F$) where $F$ is a Galois field of odd characteristic and $|F| > 5$, PSL(3,3) or Sz(q). Then $G$ can be generated by an involution and an element of even order.

Note that for the groups PSL(2, $2^f$) the conclusion of the above lemma does not hold. In this case we have the following result that can be verified by straightforward calculation.

**Lemma 2.7.** Let $G = PSL(2, 2^f)$, $f > 1$. Denote by $\zeta$ a generator of $GF(2^f)$ and let $a$, $b$ and $c$ be the elements of $G$ which are projections of

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \zeta \\
\zeta^{-1} & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & \zeta \\
\zeta^{-1} & \zeta
\end{pmatrix},
\]

respectively. Then $\langle a, b \rangle$ and $\langle b, c \rangle$ are dihedral groups and $\langle a, c \rangle = G$.

**Proof of Theorem 2.5.** We may suppose that $G$ does not belong to $\mathcal{X}$, hence $R_{\mathcal{X}}(G) \neq G$. If we prove that $G$ is soluble, then $R_{\mathcal{X}}(G) \neq 1$ and our claim follows from Theorem 2.2. So suppose that there exist finite insoluble $\mathcal{X}$T-groups, and let $G$ be a counterexample of minimal order. Then every proper subgroup of $G$ is soluble. By Theorem 2.2 we have that $R_{\mathcal{X}}(G) = 1$. Let $R$ be the soluble radical of $G$. Since $\mathcal{X}$ contains all finite abelian groups, we have that $R = 1$. It is now easy to see that $G$ has to be simple. By Thompson’s classification of minimal simple groups [9], $G$ is isomorphic to one of the groups in the above mentioned list. By Lemma 2.7, $G$ is not isomorphic to any of PSL(2, $2^f$), where $f$ is a prime. If $G$ is one of the groups of Lemma 2.6, then $G = \langle a, b \rangle$, where $|a| = 2$ and $|b| = 2k$, $k > 1$. We have that $\langle a, b^k \rangle$ is a dihedral group and $\langle b^k, b \rangle$ is a cyclic group, hence $G$ is in $\mathcal{X}$ by the $\mathcal{X}$T-property, a contradiction. This concludes the proof. \qed

Using Theorem 2.5, we obtain a rather surprising characterization of finite soluble groups.

**Corollary 2.8.** Every finite soluble-transitive group is soluble.

Note that the class of all supersoluble groups also satisfies all the assumptions of Theorem 2.5. Thus we have the following.

**Corollary 2.9.** Let $G$ be a finite supersoluble-transitive group. If $G$ is not supersoluble, then $G$ is a Frobenius group with supersoluble complement. In particular, $G$ is always soluble.

In view of Corollary 2.8 we may ask if every finite supersoluble-transitive group is supersoluble. This is not true however, as the group $A_4$ shows. It is also not difficult to find an example of a Frobenius group with supersoluble complement which is not supersoluble-transitive. This example also shows that Proposition
2.4 is in a certain sense best possible. Indeed, it is not possible to replace \( C_X^G(F) \) and \( C_X^G(H) \) by \( F \) and \( H \), respectively.

**Example 2.10.** Let \( A = \langle x \rangle \oplus \langle y \rangle \) be an elementary group of order 9 and let \( \alpha \) be the automorphism of \( A \) given by the matrix
\[
\begin{pmatrix}
2 & 2 \\
2 & 1
\end{pmatrix}.
\]

Then \( \langle \alpha \rangle \) acts fixed-point-freely on \( A \). Let \( G = A \rtimes \langle \alpha \rangle \) be a group of order 36 which is not supersoluble-transitive. To see this, note that \( \langle \alpha^2, (\alpha y)^2 \rangle \) is a dihedral group, \( \langle (\alpha y)^2, \alpha y \rangle \) is cyclic, whereas \( \langle \alpha^2, \alpha y \rangle = G \) is not supersoluble. Denoting by \( \mathcal{S} \) the class of all supersoluble groups, note that \( C_{G}^{\mathcal{S}}(\langle \alpha \rangle) \) has 20 elements and it is thus not a subgroup of \( G \). On the other hand, \( C_{G}^{\mathcal{S}}(A) \) is a subgroup of index 2 in \( G \).

Theorem 2.5 cannot be applied in the case of \( \mathfrak{NT} \)-groups, where \( \mathfrak{N} \) denotes the class of all nilpotent groups. Thus it is to be expected that there exist finite insoluble \( \mathfrak{NT} \)-groups. This is confirmed by the following characterization of finite \( \mathfrak{NT} \)-groups which is essentially contained in [1]. We include a proof for the sake of completeness.

**Theorem 2.11.** Let \( G \) be a finite \( \mathfrak{NT} \)-group. Then one of the following holds.

(i) \( G \) is nilpotent.

(ii) \( G \) is a Frobenius group with nilpotent complement.

(iii) \( G \cong \text{PSL}(2,2^f) \) for some \( f > 1 \).

(iv) \( G \cong \text{Sz}(q) \) with \( q = 2^{2n+1} > 2 \).

Conversely, every finite group under (i)–(iv) is an \( \mathfrak{NT} \)-group.

**Proof.** If \( G \) is soluble and not nilpotent, then the Fitting subgroup \( F \) of \( G \) is a proper nontrivial subgroup of \( G \). By Theorem 2.2, \( G \) is a Frobenius group with nilpotent complement. So suppose that \( G \) is not soluble. It is easy to see that in every finite \( \mathfrak{NT} \)-group \( G \) the centralizers of nontrivial elements are nilpotent, i.e., \( G \) is an \( \text{CN} \)-group. By a result of Suzuki [7, Part I, Theorem 4], the centralizer of any involution in \( G \) is a 2-group. Let \( P \) and \( Q \) be any Sylow \( p \)-subgroups of \( G \) and suppose that \( P \cap Q \neq 1 \). Since \( P \) and \( Q \) are nilpotent and \( G \) is a \( \mathfrak{NT} \)-group, we conclude that \( \langle P, Q \rangle \) is nilpotent. This shows that the Sylow subgroups of \( G \) are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [7], we conclude that \( G \) has to be simple. Additionally, it follows from [8] that \( G \) is isomorphic either to \( \text{PSL}(2,2^f) \), where \( f > 1 \), or to \( \text{Sz}(q) \) with \( q = 2^{2n+1} > 2 \).

Let \( G \) be a finite Frobenius group with the kernel \( N \) and a complement \( H \) and suppose that \( H \) nilpotent. Let \( x, y, z \in G \setminus \{1\} \) and let the groups \( \langle x, y \rangle \) and \( \langle y, z \rangle \) be nilpotent. Let \( c \) be the nilpotency class of \( \langle x, y \rangle \). First suppose that \( x \in N \) and \( y \notin N \). Then there exists an integer \( c \) such that \( [x, c, y] = 1 \), which implies \( [x, c-1, y] = 1 \), since \( H \) acts fixed-point-freely on \( N \). By the same argument we get \( x = 1 \), which is not possible. This shows that if \( x \in N \) then \( y \notin N \) and similarly
also \( z \in N \). But in this case \( \langle x, z \rangle \) is clearly nilpotent, since \( N \) is nilpotent. Thus we may assume that \( x, y, z \notin N \). Let \( x \in H^g \) and \( y \in H^k \) for some \( g, k \in G \) and suppose \( H^g \neq H^k \). We clearly have \( C_G(x) \leq H^g \) and \( C_G(y) \leq H^k \). Let \( \omega \) be any commutator of weight \( c \) with entries in \( \{x, y\} \). Then \( \omega \in C_G(x) \cap C_G(y) = 1 \) implies that \( \langle x, y \rangle \) is nilpotent of class \( \leq c - 1 \), a contradiction. Hence we conclude that \( \langle x, y \rangle \leq H^g \) and similarly also \( \langle y, z \rangle \leq H^g \). Therefore we have \( \langle x, z \rangle \leq H^g \). But \( H^g \) is nilpotent, hence the group \( \langle x, z \rangle \) is also nilpotent. This shows that the groups under (ii) are \( \mathfrak{NT} \)-groups.

It remains to prove that the groups under (iii) and (iv) are \( \mathfrak{NT} \)-groups. If \( G = \text{PSL}(2, 2^f) \), \( f > 1 \), then every centralizer of a nontrivial element of \( G \) is abelian by [6]. It follows from here that \( G \) is an \( \mathfrak{NT} \)-group. Now let \( G = \text{Sz}(q) \) where \( q = 2^{2n+1} > 2 \). By Theorem 3.10 (c) in [3], \( G \) has a nontrivial partition \( (G_i)_{i \in I} \), where for every \( i \in I \) the group \( G_i \) is nilpotent and contains the centralizers of all of its nontrivial elements. Let \( x, y, z \in G \setminus \{1\} \) and suppose that the groups \( \langle x, y \rangle \) and \( \langle y, z \rangle \) are nilpotent. Let \( a \) and \( b \) be nontrivial elements in \( Z(\langle x, y \rangle) \) and \( Z(\langle y, z \rangle) \), respectively, and suppose that \( a \in G_i \) and \( b \in G_j \) for some \( i, j \in I \). Then \( y \in C_G(a) \cap C_G(b) \leq G_i \cap G_j \), hence \( i = j \). But now we get \( x, z \in G_i \) and since \( G_i \) is nilpotent, the same is true for the group \( \langle x, z \rangle \). Hence \( G \) is an \( \mathfrak{NT} \)-group. \( \square \)

References
