THE EXPONENTS OF NONABELIAN TENSOR PRODUCTS OF GROUPS

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Abstract. We prove that the exponent of the nonabelian tensor product of two locally finite groups can be bounded in terms of exponents of given groups. Several estimates for the exponents of nonabelian tensor squares are obtained. In particular, if the group $G$ is nilpotent of class $\leq 3$ and of finite exponent, then the exponent of its nonabelian tensor square divides the exponent of $G$.

1. Introduction

Let $M$ and $N$ be groups acting upon each other in a compatible way, that is,

\[ ^mm' \otimes n = m (n (m^{-1} m')) \quad \text{and} \quad ^n n' = n (n^{-1} n'), \]

for $m, m' \in M$ and $n, n' \in N,$ and acting upon themselves by conjugation. The nonabelian tensor product $M \otimes N$ of $M$ and $N$ is a group generated by the symbols $m \otimes n$, where $m \in M$ and $n \in N$, with defining relations

\[ mm' \otimes n = (m m' \otimes m) (m \otimes n) \quad \text{and} \quad m \otimes nn' = (m \otimes n) (m m' \otimes n'), \]

where $m, m' \in M$ and $n, n' \in N$. When $M = N$ and all actions are conjugations, then $M \otimes M$ is called the nonabelian tensor square of $M$. The concept of the nonabelian tensor product of groups was introduced by Brown and Loday in [7], following the ideas of Dennis [10]. This construction has its origins in the algebraic K-theory and in homotopy theory. Group theoretical aspects of nonabelian tensor products have been studied extensively by several authors, starting with the paper of Brown, Johnson and Robertson [8]. We also mention here a survey paper by Kappe [22] for a rather thorough overview of known results and literature.

One of the main themes of the group theoretical part of research on the nonabelian tensor products of groups has been to determine which group theoretical properties are closed with respect to forming the nonabelian tensor product. Ellis [12] showed that the nonabelian tensor product of two finite groups is again a finite group. Nakaoka [27] and Visscher [30] had studied solvability and nilpotency of the nonabelian tensor product. Blyth and Morse [6] showed that the nonabelian tensor square of a polycyclic group is again polycyclic. A similar result was proved to hold true for arbitrary nonabelian tensor product of polycyclic groups [25].

In this paper we investigate the nonabelian tensor product of locally finite groups of finite exponent. Using Zel’manov’s solution of the Restricted Burnside Problem [29] and the above mentioned result of Ellis [12], it is not difficult to conclude that if $M$ and $N$ are finite groups acting compatibly on each other, then the exponent of $M \otimes N$ can be bounded by a function depending on $\exp M$, $\exp N$, and the number

\[ \exp (m m' \otimes m) (m \otimes n) \quad \text{and} \quad m \otimes nn' = (m \otimes n) (m m' \otimes n'), \]

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of generators of $M$ and $N$. There are at least two questions arising here: Can we bound $\exp(M \otimes N)$ in terms of $\exp M$ and $\exp N$ only? What happens in the locally finite case? It is proved in [24] that if $G$ is a locally finite group of exponent $n$, then the exponent of the Schur multiplier $\mathcal{M}(G)$ of $G$ can be bounded by a function depending only on $n$. This result, together with the commutative diagram (1) in [8], shows that if $G$ is locally finite of exponent $n$, then $G \otimes G$ is locally finite, and the exponent of $G \otimes G$ can be bounded in terms of $n$ only. One of the main results of this paper shows that a similar result holds true in a more general situation, when dealing with arbitrary nonabelian tensor products. We have the following.

**Theorem 1.** Let the groups $M$ and $N$ be locally finite, acting compatibly upon each other. Then the group $M \otimes N$ is locally finite. If furthermore $M$ and $N$ have finite exponents that are $\pi$-numbers, then $\exp(M \otimes N)$ is also a $\pi$-number and can be bounded by a function depending only on $\exp M$ and $\exp N$.

The proof of Theorem 1 is given in Section 2. In principle one could obtain an explicit bound for $\exp(M \otimes N)$ from the proof, but it would probably be far from being best possible. The main obstacle in determining exact bounds for the exponent of the nonabelian tensor product is the fact that there is very little known about the exponent of the Schur multiplier and related homological functors such as the Schur multiplier of a pair groups [13, 15] (see also Section 2) that are embedded in the product. Some of the recent results in this direction can be found in [16, 24, 26]. In Section 3 we deal with the nonabelian tensor square. We first obtain bounds for the exponent of the nonabelian exterior square of $G$ defined by $G \wedge G = G \otimes G/\nabla(G)$, where $\nabla(G) = \langle x \otimes x : x \in G \rangle$. These estimates already provide bounds for $\exp(G \otimes G)$. Namely, since $G$ acts trivially on $\nabla(G)$ which is central in $G \otimes G$, we have that $\exp \nabla(G)$ divides $\exp G$, whence $\exp(G \otimes G)$ divides $\exp(G \wedge G)$ and $\exp(G \otimes G)$. We show that this bound can be substantially improved for groups that are nilpotent of class $\leq 3$. We prove the following result.

**Theorem 2.** Let $G$ be a finite nilpotent group of class $\leq 3$. Then $\exp(G \otimes G)$ divides $\exp G$.

This generalizes Theorem 12 of [26] where we prove a related result for Schur multipliers of groups of class $\leq 3$. It is shown by an example that Theorem 2 cannot be extended to groups of class 4 or more. Furthermore, Theorem 2 does not carry over to metabelian groups, even if we restrict ourselves to the nonabelian exterior square. On the other hand, we show that if $G$ is a finite metabelian group, then $\exp(G \wedge G)$ divides $\exp G \wedge G$ divides $\exp G \otimes G$, and consequently $\exp(G \otimes G)$ divides $(\exp G)^3 \exp[G, G]$.

Computation of exponents of nonabelian tensor squares is closely related to the Schur conjecture stating that for any finite group $G$, the exponent of $\mathcal{M}(G)$ divides $\exp G$. This conjecture was proven to be false [3] (see also [24, 26]), but so far the only counterexamples that have been constructed are 2-groups. In the case of nonabelian tensor squares, there is a wealth of examples of groups $G$ such that $\exp(G \otimes G)$, and even $\exp(G \wedge G)$, do not divide $\exp G$. We construct, for each prime $p > 3$, a finite group $G$ of exponent $p$ such that $\exp(G \wedge G) > p$. As for the 3-groups, such an example does not exist by Theorem 2, since the groups of exponent 3 are nilpotent of class $\leq 3$ by Levi’s theorem, cf. [29, Theorem 5.2.1]. On the other hand, there exists a finite group of exponent 9 such that the exponent of its nonabelian exterior square is greater than 9.
Finally we mention a potential application of the above mentioned results. In [14], Ellis listed methods for computing various homological and homotopical functors. Some of the algorithms presented therein require, at some point, an estimate for the exponent of a certain group related to the nonabelian tensor product of groups. Good bounds for the exponent are therefore of vital importance for effective computations, especially in the case of c-nilpotent multipliers $M^{(c)}(G)$ defined by Baer [2] (see also Section 2). A direct consequence of Theorem 1 is that if $G$ is a locally finite group of exponent $n$ and $c$ is a positive integer, then the exponent of the c-nilpotent multiplier $M^{(c)}(G)$ of $G$ can be bounded by a function depending only on $c$ and $n$. This result ensures that, at least when considering a class of finite groups of fixed exponent, one can find a bound for $\exp M^{(c)}(G)$ depending only on $c$. This may speed up computations to a certain extent, at least for large groups. On the other hand, computation of c-nilpotent multipliers is generally a difficult problem, and determining precise bounds seems to be out of reach at the moment.

2. EXPONENT OF THE NONABELIAN TENSOR PRODUCT

At first we mention some properties of nonabelian tensor products of groups, cf. [7, 8] for details. Whenever the groups $M$ and $N$ act trivially on each other, then their tensor product $M \otimes N$ is isomorphic to the ‘usual’ tensor product $M^{ab} \otimes_{\mathbb{Z}} N^{ab}$. If $M$ and $N$ act compatibly on each other, then these actions induce an action of the free product $M \ast N$ on $M \otimes N$ given by $p(m \otimes n) = p(m) \otimes p(n)$ for all $m \in M$, $n \in N$ and $p \in M \ast N$.

Let $G$ and $H$ be groups. A crossed module is a group homomorphism $\mu : G \rightarrow H$ together with an action of $H$ on $G$ satisfying $\mu(hg) = h\mu(g)h^{-1}$ and $\mu(g)g' = gg'g^{-1}$ for all $g, g' \in G$ and $h \in H$. The following result is an immediate consequence of the above definition.

**Lemma 3.** Let $\mu : G \rightarrow H$ be a crossed module. Then $\text{im}\, \mu$ is a normal subgroup of $H$ and $\ker\, \mu$ is central in $G$.

In [24] it is proved that if $G$ is a locally finite group of finite exponent, then the exponent of $\mathbb{M}(G)$ can be bounded in terms of $\exp G$ only. A similar argument shows the following generalization of this result.

**Lemma 4.** Let $G$ be a locally finite group and suppose that $G/Z(G)$ has exponent $n$. Then the exponent of $[G, G]$ can be bounded in terms of $n$ only. If $n$ is a $\pi$-number, then $\exp [G, G]$ is also a $\pi$-number.

**Proof.** Let $G$ be as above. From [23] it follows that there exists $m = f(n) > 1$ divisible by $n$ such that the map $x \mapsto x^m$ is an endomorphism of $G$. Let $x, y \in G$. Then $[x, y]^m = (x^{y}x^{-1})^m = x^m(y^{x}x^{-1})^m = x^m y(x^{-m}) = [x^m, y] = 1$. It follows from here that $\exp(G, G)$ divides $m$. Now we show that $\exp(G, G)$ is a $\pi$-number. For any $\omega \in [G, G]$ there exists a finitely generated subgroup $H$ of $G$ such that $\omega \in [H, H]$. Note that $H$ is finite and $\exp H/Z(H)$ divides $n$. Thus we may assume without loss of generality that $G$ is finite. By a result of Ganea [19], we have an exact sequence

$$Z(G) \otimes_{\mathbb{Z}} G^{ab} \rightarrow \mathbb{M}(G) \rightarrow \mathbb{M}(G/Z(G)) \rightarrow [G, G] \cap Z(G) \rightarrow 0,$$

hence $\exp([G, G] \cap Z(G))$ divides $\exp \mathbb{M}(G/Z(G))$. Using [5, Proposition I.6.9], we conclude that $\exp \mathbb{M}(G/Z(G))$ is a $\pi$-number, hence also $\exp([G, G] \cap Z(G))$ is a $\pi$-number. Since $\exp([G, G])/([G, G] \cap Z(G))$ divides $n$, this concludes the proof. □
Before embarking on the proof of Theorem 1, we recall a notion from homological algebra. By a pair of groups \((G, N)\) we mean a group \(G\) with normal subgroup \(N\). A morphism of pairs \((G, N) \to (G', N')\) is a group homomorphism \(G \to G'\) that sends \(N\) into \(N'\). This determines the category of pairs of groups. In [13, 15], Ellis introduced the notion of the relative Schur multiplier \(\mathcal{M}(G, N)\) of a pair of groups \((G, N)\) as the third homology of the relative classifying space \(B(G, N)\). Note that the group \(\mathcal{M}(G, G)\) is isomorphic to the ordinary Schur multiplier \(\mathcal{M}(G)\). It is therefore not surprising that the Schur multiplier of a pair of groups shares several properties with the classical Schur multiplier of a group, see [15] for further details.

In particular, we have a notion of a covering pair of the pair \((G, N)\). Recall [15] that a crossed module \(\partial : E \to G\) is a relative central extension if \(\partial(E) = N\) and \(G\) acts trivially on \(\ker \partial\). A relative central extension \(\partial : N^* \to G\) of the pair \((G, N)\) is called a covering pair if there exists a subgroup \(A\) of \(N^*\) such that \(A \leq Z_G(N^*) \cap [N^*, G]\), \(A \cong \mathcal{M}(G, N)\) and \(N^*/A \cong N\). Here we have used the notation \(Z_G(N^*) = \langle x \in N^* : gx = x \text{ for all } g \in G \rangle\) and \([N^*, G] = \langle x^g x^{-1} : x \in N^* \rangle \in G\). Note that a covering pair \(\partial : G^* \to G\) of the pair \((G, G)\) coincides with the usual notion of a covering group \(G^*\) [5]. By [15, Theorem 5.4] any pair of finite groups admits at least one covering pair. A straightforward argument shows that this implies that any pair of locally finite groups admits a covering pair.

**Proof of Theorem 1.** Let \(M\) and \(N\) be locally finite groups acting compatibly on each other, and suppose that \(\exp M = e\) and \(\exp N = f\). For the sake of convenience we allow \(e\) or \(f\) to be \(\infty\) when the exponents of \(M\) or \(N\), respectively, are not bounded. Let \(\pi\) be the set of all prime divisors of \(e\) and \(f\); when either \(e = \infty\) or \(f = \infty\), let \(\pi\) be the set of all primes. Let \(G\) be the Peiffer product [20, 31] of \(M\) and \(N\). To be more precise, let \(G = M \ast N/\langle I \rangle\), where \(I\) an \(J\) are the normal closures in \(M \ast N\) of \((^{m}mn^{-1}n^{-1} : m \in M, n \in N)\) and \((^{m}mn^{-1}m^{-1} : m \in M, n \in N)\), respectively. Note that \(G\) is an image of \(M \times N\), hence \(G\) is locally finite and \(\exp G\) divides \(ef\). Let \(\mu : M \to G\) and \(\nu : N \to G\) be the natural maps and denote \(M = \mu(M)\) and \(N = \nu(N)\). Then \(M\) and \(N\) are normal subgroups of \(G\) and \(G = M \ast N\). As \(\mu : M \to G\) and \(\nu : N \to G\) are crossed modules, it follows that \(\ker \mu\) is central in \(M\) and \(\ker \nu\) is central in \(N\). A similar argument as in the proof of [8, Proposition 9] shows that we have an exact sequence

\[
(M \otimes \ker \nu) \times (\ker \mu \otimes N) \xrightarrow{\iota} M \otimes N \xrightarrow{\pi} M \otimes N \to 1,
\]

where \(\iota\) is induced by \((m \otimes n', m' \otimes n) \mapsto (m \otimes n')(m' \otimes n)\). It can be seen that \(\text{im } \iota\) is a central subgroup of \(M \otimes N\), and since \(\nu m = \nu(m)\) and \(\nu n = \nu(n)n\) for all \(m \in M, n \in N\), it follows that \(\ker \mu\) and \(\ker \nu\) act trivially on \(N\) and \(M\), respectively. We thus have that the groups \(M \otimes \ker \nu\) and \(\ker \mu \otimes N\) are abelian. For \(m \in M\) and \(n' \in \ker \nu\) we obtain \((m \otimes n')^l = m \otimes (n')^l = 1\), hence \(\exp(M \otimes \ker \nu)\) divides \(f\). Similarly, \(\exp(\ker \mu \otimes N)\) divides \(e\), hence \(\exp \text{im } \iota\) divides \(ef\). Besides we have that \(\text{im } \iota\) is a locally finite group. As \(M \otimes N \cong (M \otimes N)/\langle I \rangle\), it suffices to show that \(M \otimes N\) is locally finite, and that \(\exp(M \otimes N)\) is a \(\pi\)-number bounded in terms of \(e\) and \(f\). Thus from now on we may assume without loss of generality that \(M\) and \(N\) are normal subgroups of \(G\), \(G = MN\), and all actions are induced by conjugation in \(G\).

At first consider the case when \(N \leq M\), that is, \(G = M\). Define

\[
N \wedge G = N \otimes G/\langle x \otimes x : x \in N \rangle
\]
and let $K$ be the kernel of the commutator map $N \wedge G \to G$. By [7, Corollary 4.6], $K$ is a part of an exact sequence

$$H_3(G/N) \to K \to H_2(G) \to,$$

hence $K$ is locally finite. From here it follows that $N \wedge G$ is also locally finite. We have that $K \cong M(G, N)$ by [15, Theorem 3.1]. Let $\partial : N^* \to G$ be a covering pair of the pair $(G, N)$ and let $A$ be a subgroup of $N^*$ as above. Note that $N^*$ is locally finite. Let $\{t_1, \ldots, t_r\}$ be a transversal of $N$ in $G$ and let $T = \langle t_1, \ldots, t_r \rangle$. Clearly we have $G = TN$. As $T$ acts on $N^*$, we can form the semidirect product $S = N^* \rtimes T$. The group $S$ is locally finite. Note that $T$ acts trivially on $A$, so $A$ is a normal subgroup of $S$. We then have an exact sequence

$$1 \to N^*/A \to S/A \to T \to 1,$$

whence $\exp S/A$ divides $(\exp G)^2$. As $G$ acts trivially on $\ker \partial \cong A$, we have that $A \leq Z(S) \cap [S, S]$. Thus $\exp S/Z(S)$ also divides $(\exp G)^2$. By Lemma 4 we conclude that $\exp [S, S]$, and hence also $\exp M(G, N)$, are $\pi$-numbers and can be bounded by a function depending only on $\exp G$. As $(N \wedge G)/K \cong [N, G]$, it follows that $\exp (N \wedge G)$ is a $\pi$-number bounded by a function depending only on $\exp G$. Since the group $L = \langle x \otimes x : x \in N \rangle$ is central in $N \otimes G$ and $G$ acts trivially on it, it follows from here that $\exp L$ divides $\exp N$ by a similar argument as above. This shows that we can bound $\exp (N \otimes G)$ with a function depending only on $\exp G$. In addition to that, $N \otimes G$ is locally finite and $\exp (N \otimes G)$ is a $\pi$-number.

It remains to consider the case when $G = MN$, where $M$ and $N$ are proper normal subgroups of $G$. Consider the group extensions

$$1 \to [M, N] \to M \to M/[M, N] \to 1$$

and

$$1 \to [M, N] \to N \to N/[M, N] \to 1$$

of locally finite groups. The actions between groups in the same columns are induced by conjugation in $G$, and it is clear that the homomorphisms preserve the actions. Note also that $M/[M, N]$ and $N/[M, N]$ act trivially on each other. Using [16, Lemma 3], we obtain an exact sequence

$$[[M, N] \otimes N] \otimes (M \otimes [M, N]) \to M \otimes N \to (M/[M, N])^{ab} \otimes [N/[M, N]]^{ab} \to 0,$$

where $\otimes$ denotes a semidirect product. From the above special case it now follows that $M \otimes N$ is locally finite, and that $\exp (M \otimes N)$ is a $\pi$-number that can be bounded by a function depending only on $e$ and $f$. This proves the result.

Theorem 1 can be extended in the following way. Suppose that $M$ and $N$ are two normal subgroups of a group $G$. Let $X_M$ and $X_N$ be generating sets for $M$ and $N$, respectively, and set $X = X_M \cup X_N$. Let $\tilde{X}_M$ and $\tilde{X}_N$ be generating sets of $M$ and $N$, respectively, that are closed under conjugation by $X$. Let $J$ be the normal closure in $M \ast N$ of $\langle x[m, n]x^{-1} \ast n, \ast m : x \in X, m \in \tilde{X}_M, n \in \tilde{X}_N \rangle$. Denote $\eta(M, N) = (M \ast N)/J$. This group was introduced by Ellis and Leonard [17], and played a crucial role in their computation of nonabelian tensor products of finite groups. We have canonical inclusions $\iota_M : M \hookrightarrow \eta(M, N)$ and $\iota_N : N \hookrightarrow \eta(M, N)$. By [17, Theorem 3] there is an isomorphism $\eta(M, N) \cong (M \otimes N) \times M$ which restricts to an isomorphism $[\iota_M(M), \iota_N(N)] \cong M \otimes N$. In particular, Theorem 1 implies the following.

**Corollary 5.** Let $M$ and $N$ be locally finite groups that are normally embedded in a group $G$. Then the group $\eta(M, N)$ is also locally finite. If furthermore $M$
and $N$ have finite exponents that are $\pi$-numbers, the exponent of $\eta(M,N)$ is also a $\pi$-number and can be bounded in terms of $\exp M$ and $\exp N$.

Let $G$ be a group given by a free presentation $G \cong F/R$. Define $\gamma_1(R,F) = R$ and $\gamma_{k+1}(R,F) = [\gamma_k(R,F),F]$ for $k \geq 1$. Beside that define $\gamma_k(F) = \gamma_k(F,F)$ for all $k \geq 1$. Given a positive integer $c$, the $c$-nilpotent multiplier $[2]$ of the group $G$ is the abelian group

$$M^{(c)}(G) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R,F).$$

The groups $M^{(c)}(G)$ were shown to be invariants of $G$ [2]. Note also that $M^{(1)}(G)$ is just the usual Schur multiplier $M(G)$. Theorem 1 yields the following result.

**Corollary 6.** Let $c$ be a positive integer and let the group $G$ be locally finite of exponent $n$. Then $M^{(c)}(G)$ is locally finite, and its exponent can be bounded in terms of $c$ and $n$ only. If $n$ is a $\pi$-number, then $\exp M^{(c)}(G)$ is also a $\pi$-number.

In order to prove this result, we need a generalization of $G \otimes G$ and $G \wedge G$. The group $G$ acts diagonally on $G \otimes G$, and the tensor square $G \otimes G$ acts on $G$ by conjugation in $G$ via the commutator map $\partial : G \otimes G \rightarrow G$. These actions are obviously compatible. We can thus construct the triple tensor product $(G \otimes G) \otimes G$. This construction can be iterated to form the $c$-fold tensor product $\otimes^c(G)$ defined inductively by $\otimes^1(G) = G$ and $\otimes^{k+1}(G) = \otimes^k(G) \otimes G$ for $k \geq 1$. Similarly, we can define the $c$-fold exterior product $\wedge^c(G)$.

**Proof of Corollary 6.** By a result of Burns and Ellis [9, Theorem 2.6] it follows that $M^{(c)}(G)$ is a homomorphic image of the group $\ker(\partial : \wedge^{c+1}(G) \rightarrow G)$, where $\partial$ is the commutator map. Clearly, $\wedge^{c+1}(G)$ is a homomorphic image of $\otimes^{c+1}(G)$ under canonical map. Thus the assertion follows directly from Theorem 1. \qed

### 3. Nonabelian tensor squares

Let $G$ be a finite group and $G^*$ any of its covering groups. In general, covering groups are not unique, but they all have isomorphic derived subgroups [5]. Let $\pi : G^* \rightarrow G$ be the canonical surjection. By [8, Proposition 7] there exists a homomorphism $\xi : G \otimes G \rightarrow G^*$ such that $\pi \xi$ is the commutator map. Thus the map $\xi$ induces a homomorphism $\eta : G \wedge G \rightarrow [G^*,G^*]$, where $G \wedge G = G \otimes G/(x \otimes x : x \in G)$. As $G$ is finite, it follows from [8, Corollary 2] that $\eta$ is an isomorphism. Arguments from [24, 26] can thus be adapted to show the following result.

**Proposition 7** (cf. [24, 26]). Let $G$ be a finite nilpotent group of class $c$ and exponent $e$, and let $f = \exp(G \wedge G)$.

(i) If $c \leq 3$, then $f$ divides $e$.
(ii) If $c \geq 2$, then $f$ divides $e^{\min\{2\log_2 c,\lfloor e/2 \rfloor\}}$.
(iii) If $e \in \{2,3\}$, then $f$ divides $e$.
(iv) If $e = 4$, then $f$ divides 8.

The bounds in (i), (iii) and (iv) are best possible.

**Proof.** Since the group $G$ is finite, hence it is the direct product of its Sylow subgroups, $G = S_1 \times \cdots \times S_k$. We can apply the distributive law [8, Proposition 11] to compute the exterior square $G \wedge G$. When $i \neq j$, the groups $S_i$ and $S_j$ act trivially upon each other by conjugation, therefore the cross term $S_i \wedge S_j$ is the ordinary abelian exterior product. It follows from here that it suffices to consider the case where $G$ is a finite $p$-group. With this reduction the result follows directly from [24] and [26]. \qed
To this we add the following observation.

**Proposition 8.** Let $G$ be a finitely generated group of exponent six. Then $\exp(G \land G)$ divides 36.

**Proof.** By Hall’s solution of the Burnside Problem for exponent six [29, Theorem 5.3.1], $G$ is finite. From [24] it follows that the exponent of $\mathcal{M}(G)$ divides 6. Let $G^*$ be a covering group of $G$. Then there exists $A \leq [G^*, G^*] \cap Z(G^*)$ isomorphic to $\mathcal{M}(G)$ such that $G^*/A \cong G$. It follows from here that the exponent of $G^*$, and hence also $\exp(G^*, G^*)$, divides 36.

We claim that the bound in Proposition 8 is best possible. More precisely, we show that $\exp(B(2, 6) \land B(2, 6)) = 36$. The exterior square of $B(2, 6)$ can be computed as follows. Let $G \cong F/R$ be a free presentation of a finite group $G$. Then we can construct the universal central extension $H = F/[R, F]$. The group $K = R/[R, F]$ is central in $H$. Let $G^*$ be a covering group of $G$. By Remarks 2.8 (iii) in [5], p. 142 we have that the central extensions $K \twoheadrightarrow H \twoheadrightarrow G$ and $\mathcal{M}(G) \twoheadrightarrow G^* \twoheadrightarrow G$ are isoclinic, hence we have that $[G^*, G^*] \cong [H, H]$. From [8, Proposition 7] it therefore follows that $G \land G \cong [H, H]$. When $G$ is polycyclic, the group $[H, H]$ can be effectively computed using standard methods for polycyclic groups, cf. [11]. For instance, if $G = B(2, 6) = F/F^6$, where $F$ is a free group of rank two, a consistent polycyclic presentation for $G$ has been obtained by Havas et al. [21]. Using the package Polycyclic [11] in GAP [18], we can compute a consistent polycyclic presentation for $H = F/[F^6, F]$. From here it is not difficult to compute a consistent polycyclic presentation for $[H, H] \cong G \land G$. Using GAP, we can easily find elements of order 4 and 9 within the given polycyclic generating sequence for $[G^*, G^*]$, whence $\exp(G \land G) = 36$ by Proposition 8.

Next we obtain a bound for the exponent of the nonabelian exterior square of finite metabelian groups:

**Proposition 9.** Let $G$ be a finite metabelian group. Then the exponent of $G \land G$ divides $(\exp G)^2 \exp[G, G]$.

**Proof.** Let $\exp G = p_1^{e_1} \cdots p_r^{e_r}$ be the prime power decomposition of $\exp G$. By [5, Proposition I.6.9], the $p_i$-part of $\mathcal{M}(G)$ is a direct summand of $\mathcal{M}(P_i)$, where $P_i$ is a Sylow $p_i$-subgroup of $G$, $i = 1, \ldots, r$. Using [24, Theorem 2.11], we get that $\exp \mathcal{M}(P_i)$ divides $p_i^{2e_i}$. From here it follows that $\exp \mathcal{M}(G)$ divides $p_1^{2e_1} \cdots p_r^{2e_r} = (\exp G)^2$. We have an exact sequence [8]

$$0 \to \mathcal{M}(G) \to G \land G \to [G, G] \to 0,$$

hence the result readily follows.

As observed in [24], computational results indicate that this bound is probably not best possible. Determination of precise bounds will be dealt with elsewhere. On the other hand, there exists a finite metabelian group $G$ with $\exp(G \land G) > \exp G$, see Example 12 below.

**Remark.** From the exact sequence $\mathcal{M}(G) \to G \land G \to [G, G]$ it clearly follows that $\exp(G \land G)$ is finite if and only if $G$ is (finite exponent)-by-abelian and $\exp \mathcal{M}(G) < \infty$. Thus even if $G \land G$ is a finite group, this does not necessarily imply that $G$ is locally finite of finite exponent. Consider for instance infinite metacyclic groups that have been classified by Beuerle and Kappe [4]. Let $G = \langle a, b \mid a^m = 1, [a, b] = a^{1-r} \rangle$, where $m > 0$, $r \geq 0$, $(r, m) = 1$, $r \not\equiv 1 \mod m$. It is shown in [4] that $G \land G \cong C_m$, a finite group. On the other hand, $G \land G$ is nonperiodic [4, Theorem 4.3].
Recall that for any group $G$, the exponent of $G \otimes G$ divides $(\exp G)(\exp(\langle G \rangle \wedge G))$. What we show in the rest of the section is that this bound can be improved at least when $G$ is nilpotent of class $\leq 3$. To facilitate the proof of Theorem 2, we briefly describe another presentation of nonabelian tensor squares. Let $G$ be a group and let $G^\#$ be its isomorphic copy via an automorphism $\varphi : G \to G$ sending $g \in G$ to $g^\# \in G^\#$. Set $\nu(G) = \eta(G, G^\#)$. This group was already studied by Rocco [28] who proved that there is a canonical isomorphism $G \otimes G \to [G, G^\#]$ induced by $x \otimes y \mapsto [x, y^\#]$. In addition to providing a psychological advantage of calculating with commutators instead of tensors, this presentation is particularly useful for computer implementations of computing the nonabelian tensor square [6, 14, 17].

Note also that, given a group $G$ with commutators instead of tensors, this presentation is particularly useful for it follows from here that if $G$ who proved that there is a canonical isomorphism $G \otimes G \to [G, G^\#]$ induced by $x \otimes y \mapsto [x, y^\#]$. In addition to providing a psychological advantage of calculating with commutators instead of tensors, this presentation is particularly useful for computer implementations of computing the nonabelian tensor square [6, 14, 17].

Note also that, given a group $G$, there is an exact sequence [17]

$$1 \longrightarrow [G, G^\#] \longrightarrow \nu(G) \longrightarrow G \times G \longrightarrow 1.$$ 

It follows from here that if $G$ is locally finite of finite exponent, then $\exp \nu(G)$ is finite and divides $\exp G \cdot \exp(\langle G \rangle \wedge G)$.

When $G$ is a nilpotent group, it follows from [28] that $\nu(G)$ is also nilpotent, and $\text{cl}(\nu(G)) \leq \text{cl}(G) + 1$. Beside that, $\text{cl}(G, G^\#) \leq \text{cl}(G, G) + 1$, cf. [1, Proposition 2.2]. When $G$ is nilpotent is nilpotent of class $\leq 3$, we have the following additional result.

**Lemma 10** (Blyth, and Morse [6]). Let the group $G$ be nilpotent of class $\leq 3$. Then we have the following.

(i) $[a, [b, c]] = [c, b, a]$ for all $a, b, c \in \nu(G)$.

(ii) Let $\omega \in \nu(G)$ be a commutator of weight at least three. Then $\varphi$ may be removed or introduced arbitrarily within $\omega$, as long as both the initial commutator and the resulting commutator contain at least one entry from $G$ and one from $G^\#$.

Finally, we mention some well-known identities which hold true in metabelian groups.

**Lemma 11.** Let $G$ be a metabelian group and let $a, b, x_1, \dotsc, x_r \in G$.

(i) $[a, b^n] = \prod_{k=0}^{n-1} \overbrace{[b, [b, \dotsc, [b, \overbrace{[x_1, [x_2, \dotsc, [x_r, \overbrace{[x_1, [x_2, \dotsc, [x_r, \dotsc, [x_r, \dotsc, [x_r, [a, b]] \dotsc]] \dotsc]] \dotsc]] \dotsc]}^{\text{k times}} \overbrace{[x_1, [x_2, \dotsc, [x_r, a, b]] \dotsc]}^{\text{(n-1) times}}$ for all $n \in \mathbb{Z}$.

(ii) $[x_1, x_2, \dotsc, x_n, [a, b]] = [x_{\sigma(1)}, x_{\sigma(2)}, \dotsc, x_{\sigma(n)}, [a, b]]$ for any permutation $\sigma$ of the set $\{1, 2, \dotsc, n\}$.

**Proof of Theorem 2.** As in the proof of Proposition 7 we may assume without loss of generality that $G$ is a finite $p$-group. Denote $n = \exp G$. Since $\nu(G)$ is nilpotent of class $\leq 4$, every 2-generator subgroup of $\nu(G)$ is metabelian. For $x, y \in G$ we have $[y^r, x^n] = 1$. Using Lemma 11 and inverting, we get

$$1 = [x, y^r][x, [x, y^r]][x, x]\quad(1)$$

Replacing $x$ by $[x, z]$, where $z$ is an arbitrary element of $G$, and using the class restriction, we obtain

$$[x, z, y^r]^n = 1.\quad(2)$$

If $p > 3$, then the equation (1) implies that $[x, y^r]^n = 1$. Let $\omega \in [G, G^\#]$ and $x, y \in G$. Since $[G, G^\#]$ is nilpotent of class $\leq 2$, we have that $(\omega [x, y^r])^n = \omega^n [x, y^r]^n [y^r, x, \omega^{-1}]^{n(n-1)/2} = \omega^n [y^r, x]^{n(n-1)/2}, \omega^{-1} = \omega^n$. This shows that the exponent of $[G, G^\#]$ divides $n$. Hence it remains to consider the case when $p \leq 3$. 

First assume that \( p = 2 \). Then (1) can be rewritten as

\[
1 = [x, y^n][x, [x, y^n]](\text{2}).
\]

Replacing \( x \) by \( xy \) in (3) and using (3) and the fact that \([y, y^n] \in Z(\nu(G))\), we obtain \([y, [x, y^n]](\text{2}) = 1\), and similarly also

\[
[x, [x, y^n]](\text{2}) = 1.
\]

Thus it follows from (3) that \([x, y^n]^n = 1\). Next replace \( x \) by \( xz \) in the equation (4). Keeping in mind that \( \nu(G) \) is nilpotent of class \( \leq 4 \) and using (4), we get, after expansion,

\[
([x, z, y^n][z, [x, y^n]][z, [x, z, y^n]][x, [z, [x, y^n]]]) (\text{2}) = 1.
\]

Commuting this equation with \( z \) from the left, we conclude that

\[
[z, [x, [z, y^n]]]^n = 1.
\]

Replacing \( z \) by \( zw \), where \( w \in G \), in the equation (6) and using the same equation, we obtain

\[
w, [x, [z, y^n]]]^n = 1.
\]

From here it is not difficult to conclude that all commutators of weight 4 with entries from \( G \) or \( G^{\varphi} \) have order dividing \( n(n - 1)/2 \). In particular, the equation (5) can be rewritten as

\[
([x, [x, y^n]][z, [x, y^n]]) (\text{2}) = 1.
\]

By the Hall-Witt identity (see, for instance, [6, Lemma 28]) we obtain after expansion that

\[
([x, z, y^n][z, [x, y^n]]) = [z, x, y^n][x, [z, y^n]]^2 c_4,
\]

where \( c_4 \) is a product of commutators of weight 4 with terms in \( \{x, z, y^n\} \). Using (7), we can thus rewrite the equation (8) as

\[
[x, z, y^n] (\text{2}) = 1.
\]

In view of Lemma 10 we therefore have that all commutators of weight 3 with entries from \( G \) or \( G^{\varphi} \) have order dividing \( n(n - 1)/2 \). Because of the class restriction it follows that \( \exp[\nu(G), \nu(G), \nu(G)] \) divides \( n(n - 1)/2 \). For \( \omega \in [G, G^{\varphi}] \) and \( x, y \in G \) we again have that \( \omega^n = \omega^n [y^{\varphi}, x, \omega^{-1}]^{n(n - 1)/2} = \omega^n \), whence the exponent of \( [G, G^{\varphi}] \) divides \( n \).

We are left with the case when \( p = 3 \). From (2) and Lemma 10 we get that \( \exp[\nu(G), \nu(G), \nu(G)] \) divides \( n \). Besides, (1) can now be rewritten as

\[
1 = [x, y^n][x, [x, y^n]](\text{2}).
\]

Note that since \([y, y^n] \in Z(\nu(G))\), we have that \( 1 = [y^n, y^n] = [y, y^n]^n \). Now replace \( x \) by \( xy \) in (10). Using the class restriction, (10) and Lemma 11, we get

\[
[y, [x, [x, y^n]]]^2 (\text{2}) = [y, [y, [x, y^n]]] (\text{2}) = 1,
\]

whence

\[
[y, [x, [x, y^n]]] (\text{2}) = [y, [y, [x, y^n]]] (\text{2}) = 1.
\]

Replacing \( x \) by \( xy \) in (11), we obtain that \([y, [x, [x, y^n]]] (\text{2}) = 1\), thus also \([x, y^n]^n = 1\) by (10). As in the previous paragraph we conclude that \( \exp[G, G^{\varphi}] \) divides \( n \). □

In a sense Theorem 2 cannot be extended to groups of class \( \geq 4 \). The following example suggested by the referee shows that there exists a metabelian group \( G \) of exponent 4 and class 4 with \( \exp(G \wedge G) = 8 \). Note that this example has minimal order.
Example 12. There is precisely one group $G$ of order 64, exponent 4, and class 4, that can be generated by an involution and an element of order 4. This can be easily verified by looking into the GAP library of small groups. The group $G$ is identified in this library as SmallGroup(64, 34). Using the GAP package Polycyclic, we can compute $G \wedge G$ and $G \otimes G$. In particular, $G \wedge G$ is a group of order 64 and exponent 8.

Our next example shows that for each prime $p > 3$ there exists a finite group of exponent $p$ such that the exponent of its nonabelian exterior square is greater than $p$.

Example 13. Let $p > 3$ be a prime and denote by $R(r, p)$ the largest finite $r$-generator group of exponent $p$ (note that $R(r, p)$ exists by the solution of the Restricted Burnside Problem). Let $F$ be a free group on $r$ generators. For any $c \geq 1$, the group $G = F/F^{p\gamma}_{c+1}(F)$ is a homomorphic image of $R(r, p)$. Let $H$ be the free central extension of $G$, i.e., $H = F/[F^{p}, F]^{\gamma}_{c+2}(F)$. Since $p > 3$, Razmyslov’s theorem [29, Theorem 4.4.8] implies that the nilpotency class of $R(r, p)$ cannot be bounded independently of $r$. Using the collection process (cf. [24, Lemma 2.7]), we can thus choose $r$ large enough so that $\omega^{p} \notin [F^{p}, F]^{\gamma}_{c+2}(F)$ for some $c \geq 1$ and $\omega \in [F, F]$. Then it follows that $\exp[H, H] > p$. As $[H, H] \cong G \wedge G$ [8, Corollary 2], we conclude that $\exp(G \wedge G) > p$.

For example, when $p = 5$, we can take $r = 2$ and $c = 10$. The group $H$ can be constructed with GAP using the Nilpotent Quotient algorithm. It can be verified from the polycyclic presentation of $H$ that the map $x \mapsto x^{25}$ is an endomorphism of $H$. As in the proof of Lemma 4 it follows from here that $\exp[H, H]$ divides 25. On the other hand, if $a$ and $b$ generate $F$ and $\bar{a}$ and $\bar{b}$ are their canonical images in $H$, then GAP calculation shows that $[\bar{a}, \bar{b}]^{5} \neq 1$. This shows that $\exp[H, H]$, and hence also $\exp(G \wedge G)$, is equal to 25.

The construction in Example 13 does not apply for groups of exponent 3. In our next example we construct a finite group $G$ of exponent 9 with $\exp(G \wedge G) = 27$.

The construction follows the lines of Example 13.

Example 14. Let $F$ be a free group on two generators and put $G = F/F^{9\gamma}_{11}(F)$. Then $G$ is a group of exponent 9 and class 10. Its free central extension is $H = F/[F^{9}, F]^{\gamma}_{12}(F)$. This group can again be constructed with the Nilpotent Quotient algorithm. As before, $G \wedge G \cong [H, H]$, and it can be verified from the polycyclic presentation of $H$ that the map $x \mapsto x^{27}$ is an endomorphism of $H$. As in Example 13 it follows from here that $\exp[H, H] = 27$.

Remark. Let $G$ be a finite group. If $G$ has exponent 2, then $G$ is clearly abelian, hence $\exp(G \otimes G)$ is 2. Similarly, if $G$ has exponent 3, then $\exp(G \otimes G)$ is 3 by Theorem 2, since the groups of exponent 3 are all nilpotent of class $\leq 3$ [29, Theorem 5.2.1]. In the case when $G$ has exponent 4, the situation is less clear. We have that $\exp(G \otimes G)$ divides 32 by Proposition 7. This bound is probably not best possible. Note that if $G$ is a 2-generator group of exponent 4, then $G \otimes G$ is a homomorphic image of the group $B(2, 4) \otimes B(2, 4)$. This group was computed in [17]. It turns out that $\exp(B(2, 4) \otimes B(2, 4)) = 8$. We conjecture that 8 is the exact bound in general, for arbitrary number of generators. To confirm this conjecture, it would suffice to compute $\exp(B(r, 4) \otimes B(r, 4))$ for all $r \geq 1$.

References


