ON PRO-\(p\) GROUPS WITH POTENT FILTRATIONS

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Abstract. In this note we prove that if \(G\) is PF-group of finite exponent, then the exponent of the second homology group \(H_2(G, M)\) divides the exponent of \(G\) for every profinite trivial \([\hat{\mathbb{Z}}G]\)-module \(M\). We introduce the notion of the exponential rank of a pro-\(p\) group, and find a bound for the exponential rank of a PF-group.

1. Introduction

In 1987, Lubotzky and Mann \[10, 11\] introduced the notion of powerful \(p\)-groups and powerful pro-\(p\) groups. These groups had been implicitly studied before by Lazard \[8\] and Arganbright \[2\]. Powerful groups have a particularly nice power-commutator structure, and have had an important role in the theory of finite \(p\)-groups and pro-\(p\) groups. In their paper \[10\], Lubotzky and Mann obtained some properties of the Schur multiplier \(H_2(G, \mathbb{Z})\) of a powerful \(p\)-group \(G\). In particular, they showed that if \(G\) is a powerful \(p\)-group, then the exponent of \(H_2(G, \mathbb{Z})\) divides the exponent of \(G\). The question whether \(\text{exp} H_2(G, \mathbb{Z})\) divides \(\text{exp} G\) for every finite group seems to have been a longstanding open problem, probably going all the way back to Schur. It is now known that the answer is negative in general, see, for example, \[13\]. On the other hand, the counterexamples seem to be quite rare. It is still not known whether or not there exists a finite group \(G\) of odd order such that \(\text{exp} H_2(G, \mathbb{Z})\) does not divide \(\text{exp} G\).

Recently, Fernández-Alcober, González-Sánchez, and Jaikin-Zapirain \[4\] defined a new family of pro-\(p\) groups, the so called PF-groups. These groups generalize the concepts of powerful pro-\(p\) groups and potent pro-\(p\) groups \[6\]. They have been used successfully in studying the power structure of pro-\(p\) groups \[4\]. Furthermore, González-Sánchez \[5\] proved that a torsion-free pro-\(p\) group is a PF-group if and only if it is \(p\)-saturable (in the sense of Lazard). The purpose of this paper is to study the power structure of central extensions of PF-groups. As a consequence we generalize the above mentioned result of Lubotzky and Mann by proving that if \(G\) is a PF-group of finite exponent, then \(\text{exp} H_2(G, M)\) divides \(\text{exp} G\) for every profinite trivial \([\hat{\mathbb{Z}}G]\)-module \(M\). This also generalizes a result of Ellis \[3\]. In the second part of the paper we follow the approach from \[13\] and define the exponential rank \(\text{exprank}(G)\) of a center-by-finite-exponent pro-\(p\) group \(G\). We first examine the relationship between the exponential rank of a pro-\(p\) group and exponential rank of its finite quotients. Then we prove that if \(G\) is a PF-group, then \(\text{exprank}(G) \leq 1\). We show by an example that this estimate is best possible. When \(G\) is potent, then this result can be further refined. We namely show that potent pro-\(p\) groups have zero exponential rank if \(p\) is odd. When \(p = 2\), the exponential rank is precisely 1 unless the group in question is abelian.
A word about the notations. If $G$ is a pro-$p$ group, then all the subgroups will be considered in a topological sense, i.e., as topological closures of corresponding abstract subgroups. For other unexplained notations we refer to the book of Ribes and Zaleskii [14], and [4].

2. Central extensions of PF-groups and homology

Let $G$ be a pro-$p$ group. Following [4], we say that a descending chain $(N_i)_{i \in \mathbb{N}}$ of closed subgroups of $G$ is a potent filtration of $G$ if its intersection $\cap_{i \in \mathbb{N}} N_i$ is trivial, and $[N_i, G] \leq N_{i+1}$ and $[N_{i+p}, G] \leq N_{i+1}^p$ for all $i \in \mathbb{N}$. A subgroup $N$ of $G$ is said to be PF-embedded in $G$ if there is a potent filtration of $G$ starting at $N$. We also say that $G$ is a PF-group if it is PF-embedded in itself. The notion of PF-groups is a generalization of that of potent pro-$p$ groups [6], and powerful pro-$p$ groups [10].

The main result of this section is the following.

Theorem 2.1. Let $G$ be a PF pro-$p$ group and let $H$ be a pro-$p$ group with $Z \leq Z(H)$ such that $H/Z \cong G$. Then $[H^p, H] = [H, H]^p$ for all nonnegative integers $i$.

Before proving this theorem, we mention the following two auxiliary results proved in [4].

Lemma 2.2 ([4]). Let $G$ be a pro-$p$ group and let $M$ and $N$ be closed normal subgroups of $G$. Then

$$[N^k, M] \equiv [N, M]^k \bmod \prod_{i=1}^{k} [M, p^i N]^{p^{k-i}}$$

for all nonnegative integers $k$.

Lemma 2.3 ([4]). Let $G$ be a pro-$p$ group, and $N$ a PF-embedded subgroup of $G$. Then we have the following:

(a) $N/K$ is PF-embedded in $G/K$ for every closed normal subgroup $K$ of $G$.

(b) Both $N^p$ and $[N, G]$ are PF-embedded in $G$.

(c) $[N^p, G^p] = [N^i, G]^p$ for all $i, j \geq 0$.

(d) $(N^p)^{p^j} = N^{p^{j+i}}$ for all $i, j \geq 0$.

Proof of Theorem 2.1. Let $G$, $H$, and $Z$ be as above. Let $U$ be the collection of all open normal subgroups of $H$. Let $U \subseteq U$. Then $ZU/U$ is a central subgroup of $H/U$, and $(H/U)/(ZU/U) \cong H/ZU$ is a PF pro-$p$ group. If we prove that the conclusion of the theorem holds true for all $H/U$, where $U \subseteq U$, then $[H^p, H] = [H, H]^p U$ for all $U \subseteq U$, and therefore $[H^p, H] = [H, H]^p$. Thus in order to prove that $[H^p, H] = [H, H]^p$, it suffices to show this for every finite quotient of $H$, therefore we may assume without loss of generality that $H$ is a finite $p$-group. Let $G = N_1 \geq N_2 \geq \cdots \geq N_k = 1$ be a potent filtration of $G$. Taking preimages in $H$, we obtain a descending chain $H = M_1 \geq M_2 \geq \cdots \geq M_k = Z$ of closed subgroups of $H$ such that $[M_i, H] \leq M_{i+1},$ and $[M_{i-p+1}, H] \leq M_{i+1}^p Z$ for all $i = 1, \ldots, k$. The last condition implies that

$$(2.3.1) \quad [M_{i+p}, H] \leq [M_{i+1}, H]^p$$

for all $i = 1, \ldots, k$. We claim that $([M_i, H])_{i \in \mathbb{N}}$ is a potent filtration for $H$. The only nontrivial thing to be verified is that $[M_{i+p}, H] \leq [M_{i+1}, H]^p$. Using Lemma 2.2, we get $[M_{i+1}, H] \leq [M_{i+1}, H]^p [M_{i+1+p}, H] \leq [M_{i+1}, H]^p [M_{i+2}, H]$. By induction, $[M_{i+1+p}, H] \leq [M_{i+1}, H]^p [M_{i+1+p}, H]$ for all $j \geq 1$. As $M_k = Z$, we conclude that $[M_{i+p}, H] \leq [M_{i+1}, H]^p$, hence also $[M_{i+p}, H] \leq [M_{i+1}, H]^p$, as required.
We now claim that $[M_i^{p^j}, H] = [M_i, H]^{p^j}$ for all positive integers $i$ and $j$. We prove this by induction on $j$. The above argument implies that $[M_i^{p^j}, H] \leq [M_i, H]^{p^j}$. On the other hand, Lemma 2.2 gives $[M_i, H]^{p^j} \leq [M_i^{p^j}, H][M_i, H] \leq [M_i, H][M_i^{p^{j+1}}/M_i^{p^j}, H] = [M_i, H]^{p^{j+1}}$, therefore $[M_i, H]^{p^j} = [M_i, H]^{p^{j+1}}$. Suppose now that $[M_i^{p^r}, H] = [M_i, H]^{p^r}$ for all positive integers $i$ and $r < j$, where $j > 1$. We have that

$$[M_i^{p^j}, H] = [M_i, H]^{p^j} \mod \prod_{\ell=1}^j [H, p^\ell M_i]^{p^{\ell-i}}$$

by Lemma 2.2. As $[M_i, H]$ are PF-embedded in $H$, induction argument gives that $[M_i, e_i^{p+1}H] \leq [M_i, H]^{p^j}$ for all $t \geq 0$. As $p^j \geq (p - 1) + 1$ for all $\ell \geq 1$, we therefore conclude that $[H, p^\ell M_i]^{p^{\ell-i}} \leq [M_i, H]^{p^{\ell-i}} \leq ([M_i, H]^{p^j})^{p^{\ell-i} = [M_i^{p^j}, H]}$. This shows that $[M_i^{p^j}, H] \leq [M_i, H]^{p^j}$. To prove the reverse inclusion, note first that $p^j \geq (p - 1) + 2$ for all $\ell \geq 2$, therefore $[H, p^\ell M_i]^{p^{\ell-i}} \leq [M_i, H]^{p^{\ell-i}} \leq ([M_i, H]^{p^j})^{p^{\ell-i}} = ([M_i^{p^j}, H]^{p^{\ell-i}} \leq [M_i^{p^j}, H])$ for all $\ell \geq 2$. It remains to consider $[H, p^\ell M_i]^{p^{\ell-i}}$. We clearly have that $[H, p^\ell M_i]^{p^{\ell-i}} \leq [M_i, H]^{p^{\ell-i}} \leq [M_i^{p^j}, H]^{p^{\ell-i}} = [M_i^{p^j}, H]$ by (2.3.1). Let us prove that the equation (2.3.1) still holds when $M_i$ are replaced by $M_i^{p^j}$ throughout.

We prove this by reverse induction on $i$. Using Lemma 2.2 and induction assumption, we get $[M_i^{p^j}, H] = [M_i, H]^{p^j} \leq [M_i^{p^j}, H]^{p^j} \leq [M_i^{p^{j+1}}, H][M_i, H]^{p^{j+1}} \leq ([M_i^{p^j}, H]^{p^j})^{p^{j+1}} = ([M_i^{p^{j+1}}, H])$, as required. Thus we can apply the induction assumption on $j$ to conclude that $[H, p^\ell M_i]^{p^{\ell-i}} \leq [M_i^{p^{j+1}}, H]^{p^{\ell-i}} = [M_i, H]^{p^{\ell-i}}$. By Lemma 2.3 we have that the equality $(N_i^{p^j})^{p^{\ell-i}} = N_i^{p^j}$ holds, hence $(M_i^{p^j})^{p^{\ell-i}} = Z_i^{p^j}$. Commuting with $H$, we get $[(M_i^{p^j})^{p^{\ell-i}}, H] = [M_i^{p^j}, H]$. This concludes the proof. 

The above result has the following consequence for the homology of PF-groups.

**Corollary 2.4.** Let $G$ be a PF pro-p group of finite exponent and let $M$ be a profinite trivial $[\mathbb{Z}G]$-module. Then $exp H_2(G, M)$ divides $exp G$.

**Proof.** First assume that $G$ is finite. Applying Theorem 2.1 to a covering group of $G$, we get that $exp H_2(G, Z)$ divides $exp G$. Let $M$ be a trivial $\mathbb{Z}G$-module. Then the Universal Coefficient Theorem implies that $H_2(G, M) \cong (H_2(G, Z) \otimes M) \oplus Tor_2^G(G^{ab}, M)$, hence $exp H_2(G, M)$ divides $exp G$. This proves the theorem in the finite case. As for the pro-p case, let $U$ be the collection of open normal subgroups of $G$, and $M$ a profinite trivial $[\mathbb{Z}G]$-module. Then we have [14, Corollary 6.5.8] that

$$H_2(G, M) = \lim_{\mathcal{U}} H_2(G/U, M_U),$$

hence the result follows from the above conclusion. 

Corollary 2.4 also holds for potent pro-p groups, i.e, pro-p groups satisfying $\gamma_{p-1}(G) \leq G^p$ if $p$ is odd, or $\gamma_2(G) \leq G^4$ when $p = 2$ [6]. For, it is straightforward to see that every potent pro-p group is a PF-group. Another related class of groups was considered by Ellis [3]. He introduced the class $C_p$ consisting of finite p-groups $G$ satisfying $[G^{p^{-1}}, G, G] \leq G^p$ for all $1 \leq i \leq e$, where $exp G = p^e$. Ellis proved that if $G$ is a finite p-group belonging to $C_p$, then $exp H_2(G, Z)$ divides $exp G$. Extending this notion, we define $C_{p}^*$ to be the class of all pro-p groups $G$ satisfying $[G^{p^{-1}}, G, G] \leq G^p$ for all $i \in N$. It is now clear that if $p > 3$, then every $C_{p}^*$-group
is potent. Thus Corollary 2.4 also applies to pro-\( p \) groups belonging to \( \hat{\mathcal{E}}_p \), where \( p > 3 \).

3. Exponential rank

Let \( n \) be an integer. A group \( G \) is said to be \( n \)-abelian if it satisfies the law \((xy)^n = x^ny^n\). The study of \( n \)-abelian groups was initiated by Levi in [9]. Alperin [1] showed that if \( G \) is \( n \)-abelian for some \( n \neq 0, 1 \), then both \( \exp G/Z(G) \) and \( \exp \gamma_2(G) \) divide \( n(n-1) \). Kappe [7] considered the sets \( \mathcal{E}(G) = \{ n \in \mathbb{Z} \mid G \text{ is } n\text{-abelian} \} \). She found arithmetic characterizations of these sets. In the case of finite \( p \)-groups these were further refined in [13].

Let \( G \) be a pro-\( p \) group and suppose that \( \exp G/Z(G) = p^e \). Then \( G/Z(G) \) is locally finite by a result of Zelmanov [15]. Using a result of Mann [12], we conclude that \( \exp G \) is \((p, e)\)-bounded (Mann’s result holds true for abstract groups, but can be extended to the topological setting, since taking powers is continuous). It follows that there exists \( n = n(p, e) > 1 \) such that \( G \) is \( n \)-abelian. Adapting the argument from [13], we have that there exists a nonnegative integer \( r \) such that \( \mathcal{E}(G) = p^{e+r} \mathbb{Z} \cup (p^{e+r} \mathbb{Z} + 1) \). As in [13] we say that \( r \) is the exponential rank of \( G \), and we write \( r = \exp \text{rank}(G) \). Our first result shows that there is a relationship between \( \exp \text{rank}(G) \) and the exponential rank of finite quotients of \( G \).

**Proposition 3.1.** Let \( G \) be a pro-\( p \) group with \( \exp G/Z(G) = p^e \). Then

\[
s = \sup(\exp \text{rank}(G/U) \mid U \text{ an open normal subgroup of } G)
\]

is finite, and \( \exp \text{rank}(G) \leq s \).

**Proof.** Let \( r = \exp \text{rank}(G) \) and let \( Q \) be any finite quotient of \( G \). Let \( \exp Q/Z(Q) = p^f \) and \( \exp \text{rank} Q = t \). Then \( f \leq e \). As \( G \) is \( p^{e+r} \)-abelian, so is \( Q \). This implies that \( t \leq r + e - f \), therefore \( s < \infty \). To prove the second part, note that, by definition, \( Q \) is \( p^{f+1} \)-abelian, hence it is also \( p^{e+s} \)-abelian. Since this is true for any finite quotient of \( G \), it follows that \( G \) is \( p^{e+s} \)-abelian. From here we conclude that \( r \leq s \), as required.

The next lemma is an elementary consequence of Hall’s collection process.

**Lemma 3.2.** Let \( G \) be a group and let \( x, y \in G \). Then

\[
(xy)^{p^k} \equiv x^{p^k} y^{p^k} \mod \gamma_2((x, y))^{p^k} \prod_{i=1}^{k} \gamma_{p^i}((x, y))^{p^{k-i}}
\]

for all nonnegative integers \( k \).

**Theorem 3.3.** Let \( G \) be a center-by-finite-exponent PF pro-\( p \) group. Then its exponential rank is at most 1.

**Proof.** By Proposition 3.1 we may assume that \( G \) is a finite PF \( p \)-group. Let \( \exp G/Z(G) = p^e \). Lemma 2.3 implies that \( \exp \gamma_2(G) = p^e \). Let \( G = N_1 \geq \cdots \geq N_k = 1 \) be a potent filtration for \( G \). By induction on \( i \) we can prove that \( \gamma_{i(p-1)+1}(G) \leq N_{p^i+1} \) for all \( i \geq 1 \). Now, Lemma 3.2 gives

\[
(xy)^{p^{i+1}} \equiv x^{p^{i+1}} y^{p^{i+1}} \mod \gamma_2((x, y))^{p^{i+1}} \prod_{i=1}^{i+1} \gamma_{p^i}((x, y))^{p^{i+1-i}}
\]

for all \( x, y \in G \). Clearly, \( \gamma_2((x, y))^{p^{i+1}} = 1 \). Furthermore, we have that \( p^i > (i-1)(p-1) + 1 \) for all \( i \geq 1 \), hence \( \gamma_{p^i}(G)^{p^{i+1-i}} \leq \prod_{i=1}^{i+1} \gamma_{p^i}((x, y))^{p^{i+1-i}} \leq \prod_{i=1}^{i+1} \gamma_{p^i}((x, y))^{p^{i+1-i}} \leq 1 \).
\[(N_i^{p-1})^{p^{i+1}-1}, G) = [N_i^p, G] = [N_i, G]^{p^i} = 1. \) Thus the equation (3.3.1) can be rewritten as \((xy)^{p^{i+1}} = x^{p^{i+1}} y^{p^{i+1}}, \) hence \(\text{exp}(\text{rank}(G)) \leq 1. \)

The following example, taken from [4], shows that for each prime \(p\) there exists a finite \(p\)-group \(G\) with \(\text{exp}(\text{rank}(G)) = 1.\)

**Example 3.4.** Let \(p\) be a prime and \(n\) a positive integer. Let \(H = (x_1) \times \cdots \times (x_p),\) where \(|x_1| = \cdots = |x_{p-1}| = p^n\) and \(|x_p| = p^{n+1}.\) Form \(G = H \rtimes \langle \alpha \rangle,\) where \(\alpha\) is an automorphism of \(H\) of order \(p^n\) acting on \(H\) in the following way: \(x_i^\alpha = x_i x_{i+1}\) for \(1 \leq i \leq p-2, x_{p-1}^\alpha = x_{p-1} x_p\), and \(x_p^\alpha = x_p.\) Then it can be verified [4] that \(G\) is a PF-group. As \([x_p^{p^n}, \alpha] = 1,\) we conclude that \(\text{exp}G/Z(G) = p^n.\) Short calculation shows that \((\alpha x_1)^{p^n} \neq 1,\) whereas \(\alpha^{p^n} x_1^{p^n} = 1.\) Thus \(\text{exp}(\text{rank}(G)) = 1.\)

**Theorem 3.5.** Let \(G\) be a center-by-finite-exponent potent pro-\(p\) group. If \(p\) is odd, then \(\text{exp}(\text{rank}(G)) = 0.\) If \(p = 2\) and \(G\) is nonabelian, then \(\text{exp}(\text{rank}(G)) = 1.\)

**Proof.** If \(p = 2,\) then \(G\) is powerful and the conclusion follows from [13]. Thus we assume from here on that \(p\) is odd. Let \(\text{exp}G/Z(G) = p^e.\) Then \(\text{exp} \gamma_2(G) = p^e.\)

We have that

\[(xy)^{p^e} \equiv x^{p^e} y^{p^e} \mod \gamma_2((x, y))^{p^e} \prod_{i=1}^e \gamma_i((x, y))^{p^{e-i}}.\]

We prove by induction on \(i\) that \(\gamma_i^p(G)^{p^{e-i}} = 1\) for all \(i \geq 1.\) The case \(i = 1\) follows from \(\gamma_1^p(G)^{p^{e-1}} = 1\). For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) For the induction step observe that \(\gamma_i^p(G)^{p^{e-i}} \leq [G^p, G]^{p^{e-i}} = \gamma_2(G)^{p^e} = 1.\) This shows that \(G\) is \(p^e\)-abelian.

**References**


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