The non-abelian tensor product of polycyclic groups is polycyclic

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1. Introduction

Let $M$ and $N$ be groups acting on each other on the left. The action of $M$ on $N$ is written $^m n$, and the action of $N$ on $M$ is written $^n m$, where $m \in M$, $n \in N$. The groups $M$ and $N$ are assumed to act upon themselves by conjugation $y x = y x y^{-1}$. These actions are said to be compatible if

\[
^m m' = m(n(m^{-1} m')),
\]
\[
^n n' = n(m(n^{-1} n')),
\]

for $m, m' \in M$ and $n, n' \in N$. The non-abelian tensor product $M \otimes N$ is the group generated by the symbols $m \otimes n$ with defining relations

\[
mm' \otimes n = (^m m' \otimes ^m n)(m \otimes n),
\]
\[
m \otimes nn' = (m \otimes n)(^n m \otimes ^n n'),
\]

where $m, m' \in M$ and $n, n' \in N$. When $M = N$ and all actions are conjugations, the group $M \otimes M$ is called the non-abelian tensor square of $M$. Note also that whenever the groups $M$ and $N$ act trivially on each other, then their tensor product $M \otimes N$ is isomorphic to the usual tensor product $M_{\text{ab}} \otimes N_{\text{ab}}$ of the abelianisations. The concept of the non-abelian tensor product of groups was introduced by Brown and Loday in [3], following the ideas of Dennis [5]. This construction has its origins in algebraic K-theory as well as in homotopy theory, and it has become interesting from a purely group-theoretical point of view since the paper of Brown, Johnson and Robertson [4].

Non-abelian tensor products of groups subject to various finiteness conditions have been studied by several authors. Ellis [6] proved that if $M$ and $N$ are finite, then $M \otimes N$ is also finite. Nakaoka [9] showed that if the group $[M, N] = \langle m^n m^{-1} : m \in M, n \in N \rangle$ is solvable, then so is $M \otimes N$. In [2], Blyth, Morse and Redden proved that tensor squares of polycyclic groups are also polycyclic. The purpose of this note is to extend this result to arbitrary non-abelian tensor products of groups. Our main result goes as follows.

**Theorem.** Let $M$ and $N$ be polycyclic groups acting on each other in a compatible way. Then the group $M \otimes N$ is also polycyclic.

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This theorem is proved in Section 2. In Section 3 we obtain a generating set for $M \otimes N$ in the case when $M$ and $N$ are normal polycyclic subgroups of a common overgroup, in terms of polycyclic generating sequences of $M$ and $N$.

2. Proof of the main theorem

Let $G$ and $H$ be groups. A crossed module is a group homomorphism $\mu : G \to H$ together with an action of $H$ on $G$ satisfying $\mu(hg) = h\mu(g)h^{-1}$ and $\mu(g)h' = gg'h^{-1}$ for all $g, g' \in G$ and $h \in H$. An immediate consequence of the above definition is that if $\mu : G \to H$ is a crossed module, then $\text{im}\mu$ is a normal subgroup of $H$ and $\ker\mu$ is central in $G$.

Now we proceed to the proof of Theorem. Let $M$ and $N$ be polycyclic groups acting compatibly on each other. Let $G$ be the Peiffer product $[10]$ of $M$ and $N$. To be more precise, let $G = M \ast N/\langle IJ \rangle$, where $I$ and $J$ are the normal closures in $M \ast N$ of $\langle mm\cdots m^{-1} : m \in M, n \in N \rangle$ and $\langle mm\cdots m^{-1} : m \in M, n \in N \rangle$, respectively. Note that $G$ is an image of the semidirect product $M \times N$, hence $G$ is polycyclic. Let $\mu : M \to G$ and $\nu : N \to G$ be the natural maps and denote $\bar{M} = \mu(M)$ and $\bar{N} = \nu(N)$. Then $\bar{M}$ and $\bar{N}$ are normal subgroups of $G$ and $G = \bar{M}\bar{N}$. As $\mu : M \to G$ and $\nu : N \to G$ are crossed modules, it follows that $\ker\mu$ is central in $M$ and $\ker\nu$ is central in $N$. A similar argument as in the proof of [4, Proposition 9] shows that we have an exact sequence

$$(M \otimes \ker\nu) \times (\ker\mu \otimes N) \xrightarrow{\iota} M \otimes N \longrightarrow \bar{M} \otimes \bar{N} \longrightarrow 1,$$

where $\iota$ is induced by $(m \otimes n', m' \otimes n) \mapsto (m \otimes n')(m' \otimes n)$. Furthermore, it can be seen that $\text{im}\iota$ is a central subgroup of $M \otimes N$, and that $\ker\mu$ and $\ker\nu$ act trivially on $N$ and $M$, respectively. We thus have [6] that $M \otimes \ker\nu \cong I(M) \otimes_{ZM} \ker\nu$ and $\ker\mu \otimes N \cong \ker\mu \otimes_{ZN} I(N)$, where $I(M)$ and $I(N)$ are the augmentation ideals in $ZM$ and $ZN$, respectively. Consider the homomorphism $\kappa : I(M) \otimes_{ZM} \ker\nu \longrightarrow \ker\nu$ induced by $(m-1) \otimes a \mapsto maa^{-1}$, where $m \in M$, $a \in \ker\nu$. Note that $\ker\kappa \cong H_1(M, \ker\nu)$ [8], hence $\ker\kappa$ is finitely generated by a result of Baumslag, Cannonito and Miller [1]. We thus have that the group $I(M) \otimes_{ZM} \ker\nu$ is finitely generated, and the same conclusion holds true for $\ker\mu \otimes_{ZN} I(N)$. It follows from here that the group $(M \otimes \ker\nu) \times (\ker\mu \otimes N)$ is finitely generated, whence $\text{im}\iota$ is also finitely generated. As $\bar{M} \otimes \bar{N} \cong (M \otimes N)/\text{im}\iota$, it suffices to show that $\bar{M} \otimes \bar{N}$ is polycyclic. Thus from now on we may assume that $M$ and $N$ are normal subgroups of $G$ and $G = \bar{M}\bar{N}$. Define

$$M \wedge N = (M \otimes N)/D,$$

where $D = \langle x \otimes x : x \in M \wedge N \rangle$, and let $K$ be the kernel of the commutator map $M \wedge N \to [M, N]$. In order to finish the proof of Theorem it suffices to show that $K$ and $D$ are finitely generated. By [3, Theorem 4.5] we have an exact sequence

$$\longrightarrow H_3(G/M) \oplus H_3(G/N) \longrightarrow K \longrightarrow H_2(G) \longrightarrow .$$

Since $G$, $G/M$ and $G/N$ are polycyclic, $H_2(G)$, $H_3(G/M)$ and $H_3(G/N)$ are finitely generated by [1]. From here we conclude that $K$ is polycyclic. As for the group $D$, consider the map $\phi : (M \wedge N) \times (M \wedge N) \to M \otimes N$ defined by $\phi(g, h) = (g \otimes h)(h \otimes g)$. It is straightforward to verify that $\phi$ is a bilinear map.
Thus, if \((M \cap N)/[M, N] = \langle x_1[M, N], \ldots, x_r[M, N] \rangle\), then \(D\) is generated by the set \(\{(x_i \otimes x_j)(x_j \otimes x_i), x_i \otimes x_i : i, j = 1, \ldots, r\}\). This concludes the proof.

3. Generating sets

Let \(M\) and \(N\) be polycyclic groups acting compatibly on each other. For computational reasons it would be convenient to obtain a generating set for \(M \otimes N\) in terms of polycyclic generating sequences of \(M\) and \(N\). For the non-abelian tensor squares of polycyclic groups this has been done in [2]. Here we use a similar approach, following Ellis and Leonard [7]. Let \(J\) denote the normal subgroup of \(M \ast N\) normally generated by the elements \(x[m, n]x^{-1}[\bar{m}, \bar{n}]\) for \(m \in M\), \(n \in N\), \(x \in M \cup N\), where \(\bar{m} = xmx^{-1}\) and \(\bar{n} = xnx^{-1}\). Then there is an isomorphism [7]

\[
((M \otimes N) \rtimes N) \rtimes M \cong (M \ast N)/J.
\]

This isomorphism restricts to an isomorphism \(M \otimes N \cong [\bar{M}, \bar{N}]\), where \(\bar{M}\) and \(\bar{N}\) are the normal closures in \((M \ast N)/J\) of \(M\) and \(N\). Thus the algorithm is the following. First note that \(\bar{M}\) and \(\bar{N}\) are polycyclic, since they can be embedded into the group \((M \otimes N) \rtimes M\) which is polycyclic by our main theorem. Thus one can obtain polycyclic generating sequences \(\bar{m}_1, \ldots, \bar{m}_k\) and \(\bar{n}_1, \ldots, \bar{n}_l\) of \(\bar{M}\) and \(\bar{N}\), respectively. By [2, Lemma 22], the group \([\bar{M}, \bar{N}]\), which is isomorphic to \(M \otimes N\), can be generated by the set

\[
\{[\bar{m}_i^{\epsilon_i}, \bar{n}_j^{\delta_j}] : 1 \leq i \leq k, 1 \leq j \leq l\}
\]

where

\[
\epsilon_i = \begin{cases} 1 : |\bar{m}_i| < \infty \\ \pm 1 : |\bar{m}_i| = \infty \end{cases}
\quad \text{and} \quad
\delta_j = \begin{cases} 1 : |\bar{n}_j| < \infty \\ \pm 1 : |\bar{n}_j| = \infty \end{cases}.
\]

REFERENCES


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