SA Note on Self-Maps Inducing Identity Automorphisms of Homology Groups

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Abstract

A normal series for the group $\text{Aut}_\ast(X)$ of self-homotopy equivalences which induce identity automorphisms of homology groups is derived when $X$ is a countable, simply-connected and finite-dimensional CW-complex.

0 Introduction

Let $X$ be a pointed CW-complex and let $\text{Aut}(X)$ denote the set of homotopy classes of self-maps of $X$ that are homotopy equivalences. This set is a group, called group of self-homotopy equivalences, with respect to the operation induced by the composition of maps. The excellent survey paper [1] gives an idea of the extensive literature on these groups. Still, their structure is very often unknown, and one of the main difficulties originates from the fact that a cellular (or even homology) decomposition of $X$ does not lead to a corresponding decomposition of $\text{Aut}(X)$. In fact, there are elementary examples when a self-equivalence of a space cannot be represented by a cellular map whose restrictions on skeletons are also self-equivalences (cf. Remark 1.1 of [6]).

The main purpose of this note is to show that the situation is more favorable when dealing with $\text{Aut}_\ast(X)$, the subgroup of $\text{Aut}(X)$ consisting of classes that induce identity automorphisms of homology groups (or, in other words, with the kernel of the obvious representation $\text{Aut}(X) \to \text{Aut}H_\ast(X)$). The crucial step is Theorem 1.4 that for a large class of spaces $X$ any element of $\text{Aut}_\ast(X)$ can be represented by a cellular map which yields a self-equivalence inducing identity on homology when restricted to any skeleton of $X$. With this fact at hand it is easy to derive a finite normal series for $\text{Aut}_\ast(X)$. The existence of such a normal series paves the way for a construction of a spectral sequence converging to $\text{Aut}_\ast(X)$ which we are going to consider in a forthcoming paper.

We wish to remark that much of this work was motivated by the proof of the main theorem in [4] and that we used similar arguments in proving our results.

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1991 Mathematics Subject Classification. Primary: 55P10
1 A normal series for \(\text{Aut}_*(X)\)

Let \(\varphi: V \to A\) be a map from a wedge of \(n\)-dimensional \((n \geq 2)\) spheres \(V\) to an \(n\)-dimensional, 1-connected CW-complex \(A\), and let \(C_\varphi\) be the mapping cone of \(\varphi\). By Theorem 7.3’ of [3] for every self-map \(f\) of \(C_\varphi\) there are self-maps \(f_A\) and \(f_V\) of \(A\) and \(V\) respectively, such that the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & A \\
\downarrow{f_V} & & \downarrow{f_A} \\
V & \xrightarrow{\varphi} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{f} & & \downarrow{f} \\
X & \xrightarrow{i} & X \\
\end{array}
\]

Note that maps \(f_A\) in \(f_V\) are by no means unique. For an \(f \in \text{Aut}_*(C_\varphi)\) a chosen \(f_A\) will not be in general an element of \(\text{Aut}_*(A)\). However, under suitable assumptions that choice can be modified in order to obtain maps \(f'_A\) and \(f'_V\) that fit the above diagram, and such that \(f'_A \in \text{Aut}_*(A)\). This can be achieved by means of an action of \([A, V]\) on \([A, A]\), which we describe in the following paragraph.

Let \(F\) be the homotopy fibre of the projection \(C_\varphi \to C_\varphi/A \simeq \Sigma V\). The homotopy fibre of \(F \hookrightarrow C_\varphi\) is homotopy equivalent to \(\Omega \Sigma V\). By the universal property of Puppe fibration sequences there are maps \(j, k\) such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & A \\
\downarrow{j} & & \downarrow{k} \\
\Omega \Sigma V & \xrightarrow{u} & F \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{i} & & \downarrow{i} \\
C_\varphi & \xrightarrow{i} & C_\varphi \\
\end{array}
\]

\[
\begin{array}{ccc}
C_\varphi & \xrightarrow{i} & C_\varphi/A \\
\downarrow{C_\varphi} & & \downarrow{C_\varphi} \\
\Sigma V & \xrightarrow{i} & \Sigma V \\
\end{array}
\]

commutes up to homotopy.

Let \(P_n\) denote the functor that assigns the \(n\)-th Postnikov section \(P_nX\) to a space \(X\).

**Lemma 1.1** The maps \(P_nj: P_nV \to P_n\Omega \Sigma V\) and \(P_nk: P_nA \to P_nF\) are homotopy equivalences.

**Proof:** Consider the following diagram with exact rows

\[
\begin{array}{cccccccc}
\pi_q(C_\varphi) & \to & \pi_q(C_\varphi, A) & \to & \pi_{q-1}(A) & \to & \pi_{q-1}(C_\varphi) & \to & \pi_{q-1}(C_\varphi, A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_q(C_\varphi) & \to & \pi_q(C_\varphi/A) & \to & \pi_{q-1}(F) & \to & \pi_{q-1}(C_\varphi) & \to & \pi_{q-1}(C_\varphi/A) \\
\end{array}
\]

Since the space \(C_\varphi\) is obtained by attaching \((n+1)\)-cells to \(A\), the homomorphism \(\pi_q(A) \to \pi_q(C_\varphi)\) is an isomorphism for \(q < n\) and an epimorphism for \(q = n\). It follows that \(\pi_q(C_\varphi, A) = 0\) when \(q \leq n\) so, by Blakers-Massey theorem, \(\pi_q(C_\varphi, A) \to \pi_q(C_\varphi/A)\) is an isomorphism for \(q \leq n + 1\). Applying the 5-lemma to the above diagram we get that \(k_2: \pi_q A \to \pi_q F\) is an isomorphism for \(q \leq n\), hence \(P_nk\) is a homotopy equivalence. That \(P_nj\) is a homotopy equivalence is proved similarly (alternatively, one can use the Freudenthal theorem). \(\square\)
As \( \dim(A) \leq n \) for any space \( X \), the function \([A, X] \rightarrow [A, P_n X]\) is a bijection. It follows that \( j \) and \( k \) induce bijections

\[
j^* = j^\circ - : [A, V] \rightarrow [A, \Omega \Sigma V] \quad \text{and} \quad k^* = k^\circ - : [A, A] \rightarrow [A, F].
\]

The lifting of paths induces an action of \( \Omega \Sigma V \) on \( F \) which we denote by \( \mu : \Omega \Sigma V \times F \rightarrow F \). Through the above bijections this yields an action of \([A, V]\) on \([A, A]\): given \( \alpha \in [A, V] \) and \( g \in [A, A] \) define \( \alpha \cdot g \) to be \((k^*)^{-1}\) of the composition

\[
A \overset{(\alpha,g)}{\rightarrow} V \times A \overset{j \times k}{\rightarrow} \Omega \Sigma V \times F \overset{\mu}{\rightarrow} F.
\]

Note that \( i^*(\alpha \cdot g) \simeq i \cdot g : A \rightarrow C_\varphi \) for every \( g : A \rightarrow A \).

**Lemma 1.2** The action of \( \alpha \cdot g \) on the homology of \( A \) is given by \((\alpha \cdot g)_* = \varphi_* \alpha_* + g_*\). In other words, \((\alpha \cdot g)_q = g_q\) when \( q \neq n \), and \((\alpha \cdot g)_n = \varphi_n \alpha_n + g_n\).

**Proof:** It is sufficient to consider the case \( q \leq n \). Recall that \( P_n Y \) can be obtained by attaching to \( Y \) cells of dimension greater than \( n + 1 \), therefore \( H_q(Y) \rightarrow H_q(P_n Y) \) is an isomorphism for \( q \leq n \). In particular, \( H_q(\Omega \Sigma V) = 0 \) for \( q < n \), and by the Eilenberg-Zilber theorem we get that for \( q \leq n \) there is a direct sum representation \( H_q(\Omega \Sigma V \times F) = H_q(\Omega \Sigma V) \oplus H_q(F) \) induced by the inclusions of \( \Omega \Sigma V \) and \( F \) in the product as in the following commutative diagram \((q \leq n)\):

\[
\begin{array}{ccc}
H_q(\Omega \Sigma V) & \overset{u_*}{\longrightarrow} & H_q(F) \\
\downarrow & & \downarrow \\
H_q(\Omega \Sigma V) \oplus H_q(F) & \overset{\mu_*}{\longrightarrow} & H_q(F)
\end{array}
\]

We conclude that in dimensions under consideration the effect of \( \mu_* \) on an element \((x, y) \in H_q(\Omega \Sigma V \times F)\) is given by \( \mu_* (x, y) = u_*(x) + y \). The action of

\[
\alpha \cdot g = (k^*)^{-1}[\mu^* (j \times k)^*(\alpha, g)]
\]

on \( x \in H_q(A) \) is now easily computed:

\[
\mu_*(j \times k)_*(\alpha, g)_*(x) = u_* j_* \alpha_* (x) + k_* g_* (x) = k_\varphi \alpha_* (x) + k_* g_* (x).
\]

As \( k_* \) is an isomorphism when \( q \leq n \), the assertion of the lemma is proved. \( \square \)

In order to prove our main result we need another peace of structure. Let us say that the attaching map \( \varphi : V \rightarrow A \) is canonical if the wedge of spheres \( V \) can be decomposed into two sub-wedges \( V = V_r \vee V_g \) such that \((\varphi|_{V_r})_* : H_n(V_r) \rightarrow H_n(A)\) is injective and that \( \varphi(V_g) \) is contained in the \((n-1)\)-skeleton of \( A \) (and hence \((\varphi|_{V_g})_* = 0\)). When this is the case the exact sequence

\[
0 \longrightarrow H_{n+1}(C_\varphi) \longrightarrow H_n(V) \overset{\varphi_*}{\longrightarrow} H_n(A) \longrightarrow H_n(C_\varphi) \longrightarrow 0,
\]
which determines the homology of $C_\varphi$ in dimensions $n$ and $n + 1$, can be decomposed into two shorter exact sequences

$$0 \rightarrow H_{n+1}(C_\varphi) \xrightarrow{\sim} H_n(V_g) \rightarrow 0$$

and

$$0 \rightarrow H_n(V_r) \xrightarrow{(\varphi|V_r)_*} H_n(A) \rightarrow H_n(C_\varphi) \rightarrow 0$$

A CW-complex has a canonical decomposition if the attaching maps in all dimensions are canonical. By Theorem 2.3 of [6] any simply-connected countable CW-complex $X$ has the cellular homotopy type of a CW-complex with a canonical decomposition $Y$, i.e. there are cellular maps $f : X \to Y$ and $g : Y \to X$, such that $f \cdot g$ and $g \cdot f$ are homotopic to $1_Y$ and $1_X$ by cellular homotopies.

**Lemma 1.3** Assume that the attaching map $\varphi : V \to A$ is canonical. Then for every map $f : X \to X$ satisfying $f_{sn} = 1_{H_n(X)}$ there exists a map $\tilde{f} : A \to A$ such that $i\tilde{f} \simeq fi$, $\tilde{f}_{sn} = 1_{H_n(A)}$ and $\tilde{f}_{sq} = f_{sq}$ for $q < n$.

**Proof:** As mentioned before, there are maps $f_A$ and $f_V$ with $i \cdot f_A = f \cdot i$ and $\varphi \cdot f_V \simeq f_A \cdot \varphi$. Since $\varphi$ is canonical, we obtain the following commutative diagram with exact rows (where $\iota : V_r \hookrightarrow V$ and $\pi : V \to V_r$ are the natural inclusion and projection respectively):

$$
\begin{array}{cccc}
0 & \rightarrow & H_n(V_r) & \xrightarrow{\varphi_*} \rightarrow H_n(A) & \xrightarrow{i_s} \rightarrow H_n(C_\varphi) & \rightarrow 0 \\
\downarrow{(i\pi f)_*} & & \downarrow{(f_A)_*} & & \downarrow{f_s} & \\
0 & \rightarrow & H_n(V_r) & \xrightarrow{\varphi_*} \rightarrow H_n(A) & \xrightarrow{i_s} \rightarrow H_n(C_\varphi) & \rightarrow 0
\end{array}
$$

The equality

$$i_s(1 - (f_A)_*) = i_s - i_s(f_A)_* = 0$$

implies that $s := \varphi^{-1}_*(1 - (f_A)_*)$ is a well-defined homomorphism from $H_n(A)$ to $H_n(V_r)$. Using Hopf’s theorem and the fact that $H_n(A)$ is a free group we deduce the existence of a map $\alpha : A \to V$ such that $\alpha_{sn} = s$. Finally, $\tilde{f} := \alpha \cdot f_A$ is a map satisfying the conditions of the lemma. \(\square\)

Using this lemma as inductive step we can prove the following theorem.

**Theorem 1.4** Assume that $X$ is countable, simply-connected and finite-dimensional. Then every element of $\text{Aut}_*(X)$ can be represented by a cellular map whose restriction to every skeleton $X^{(q)}$ belongs to $\text{Aut}_*(X^{(q)})$.

**Proof:** Let $h : X \to Y$ be a cellular homotopy equivalence where $Y$ is an $n$-dimensional CW-complex with a canonical decomposition, and let $\bar{h}$ be a cellular homotopy inverse of $h$. For every map $f : X \to X$ representing a class in $\text{Aut}_*(X)$ the composition $g := h \cdot f \cdot \bar{h}$ represents a class in $\text{Aut}_*(Y)$.

We will use induction to prove that there is a cellular map $\tilde{g}$ homotopic to $g$, such that $g|_{Y^{(q)}} \in \text{Aut}_*(Y^{(q)})$ for every $q$. Assume inductively that there is a cellular map $g : Y \to Y$, such that $g|_{Y^{(n-q)}} \in \text{Aut}_*(Y^{(n-q)})$ for $q = 0, 1, \ldots, k - 1$. As the inclusion $i_{n-k} : X^{(n-k)} \hookrightarrow X^{(n-k+1)}$ is a
cofibration, the map $\bar{g}_{n-k} \in \text{Aut}_s(Y^{(n-k)})$ satisfying $i_{n-k} \circ \bar{g}_{n-k} \simeq g \circ i_{n-k}$, which exists by the previous lemma, can be extended to a map $\bar{g}_{n-k+1} \in \text{Aut}_s(Y^{(n-k+1)})$ homotopic to $g|_{Y^{(n-k+1)}}$. Iterations of that construction eventually yield a cellular map $\bar{g} \in \text{Aut}_s(Y)$ with the property $g|_{Y^{(n-q)}} \in \text{Aut}_s(Y^{(n-q)})$ for $q = 0, 1, \ldots, k$. Since the assertion for $g$ is obviously true when $k = 0$, the claim is proved.

The map $f := \bar{h} \circ \bar{g} \circ h$ is homotopic to $f$, and due to the cellularity of $h$ and $\bar{h}$, $f|_{X^{(q)}} \in \text{Aut}_s(X^{(q)})$. □

The proof of our main result is now at hand. Let us denote by $G_q = G_q(X)$ the subgroup of $\text{Aut}_s(X)$ whose elements are represented by maps $f$ with the property $f|_{X^{(q)}} = 1_{X^{(q)}}$.

**Theorem 1.5** Let $X$ be a countable, simply-connected $n$-dimensional CW-complex. Then $G_q$ is a normal subgroup of $\text{Aut}_s(X)$ for every $q$. Moreover,

$$1 \trianglelefteq G_{n-1} \trianglelefteq \ldots \trianglelefteq G_2 \trianglelefteq \text{Aut}_s(X).$$

is a finite normal series for $\text{Aut}_s(X)$.

**Proof:** Let $[g] \in \text{Aut}_s X$, $\bar{g}$ a homotopy inverse of $g$, and $[f] \in G_q$. We must prove that $g \circ f \circ \bar{g}$ is homotopic to a map whose restriction to $X^{(q)}$ equals $1_{X^{(q)}}$. Because of the previous theorem, we can assume that $g$ is cellular and that $g|_{X^{(q)}} \in \text{Aut}_s(X^{(q)})$. We can also assume without loss of generality that $\bar{g}|_{X^{(q)}}$ is a homotopy inverse of $g|_{X^{(q)}}$. The commutativity of the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\bar{g}} & X \\
\| & f & \downarrow g \\
X^{(q)} & \xrightarrow{g|_{X^{(q)}}} & X^{(q)}
\end{array}$$

implies that $(g \circ f \circ \bar{g})|_{X^{(q)}}$ is homotopic to $1_{X^{(q)}}$. As the inclusion of $X^{(q)}$ in $X$ is a cofibration, $g \circ f \circ \bar{g}$ is homotopic to a map that restricts to $1_{X^{(q)}}$, hence it belongs to a class in $G_q$, which proves the normality. □

**2 Applications**

As already mentioned, the existence of a normal series is a necessary condition if one wants to construct a spectral sequence for a noncommutative group. In the Shih’s spectral sequence (see [7]) such a normal series for $\text{Aut}(X)$ is obtained for free, due to the naturality of the Postnikov’s decomposition. The apparent impossibility to construct a normal series for $\text{Aut}(X)$ corresponding to a cellular decomposition is a real obstacle for the study of this group by means of spectral sequences.

However, spectral sequences for $\text{Aut}_s(X)$ will be treated elsewhere, but there are also some immediate implications, which we consider in this section. In the presence of a normal series it is natural to ask what are its subquotients. Unfortunately, the description of those turns out to be quite complicated but nonetheless, some estimates are possible.
When $X$ is a subspace of $Y$ let us denote by $\text{aut}_sX(Y)$ the space consisting of self-maps of $Y$ which induce identity automorphisms of homology groups and which restrict to the identity on $X$, and let $\text{Aut}_sX(Y) := \pi_0(\text{aut}_sX(Y))$.

**Lemma 2.1** Let $X$ be as in theorem 1.5. For a fixed $q$, if the group $\text{Aut}_{sX(q)}(X^{(q+1)})$ is trivial, then so is the quotient $G_q/G_{q+1}$ (i.e. $G_{q+1} = G_q$).

**Proof:** Every element of $G_q$ can be represented by a cellular map $f$ such that $f|_{X^{(p)}} \in \text{Aut}_s(X^{(p)})$ for every $p$ and $f|_{X^{(q)}} = 1_{X^{(q)}}$. The restriction $f|_{X^{(q+1)}}$ represents an element of $\text{Aut}_{sX(q)}(X^{(q+1)})$, which is by the assumptions trivial, so there is a homotopy, which is fixed on $X^{(q)}$, between $f|_{X^{(q+1)}}$ and the identity. As $X^{(q+1)} \to X$ is a cofibration, this homotopy can be extended over all of $X$, therefore $f$ represents an element in $G_{q+1}$. □

The space $X^{(q+1)}$ is the mapping cone of the attaching map $\varphi_q : V_q \to X^{(q)}$, where $V_q$ is a wedge of $q$-dimensional spheres. The application of the functor $[-, X^{(q+1)}]$ to the Puppe cofibration sequence

$$V_q \xrightarrow{\varphi_q} X^{(q)} \xrightarrow{p} X^{(q+1)} \xrightarrow{\Sigma V_q}$$

yields an exact sequence of pointed sets

$$[\Sigma V_q, X^{(q+1)}] \xrightarrow{p^*} [X^{(q+1)}, X^{(q+1)}] \xrightarrow{[\,, X^{(q)}]} [X^{(q)}, X^{(q+1)}],$$

hence a self-map of $X^{(q+1)}$ restricting to the identity on $X^{(q)}$ corresponds to an element coming from $[\Sigma V_q, X^{(q+1)}]$. It is well-known (cf. §1 of [5]) that $p^*(f)$ of an $f : \Sigma V_q \to X^{(q+1)}$ is homotopic to the composition

$$X^{(q+1)} \xrightarrow{\nu} X^{(q+1)} \vee \Sigma V_q \xrightarrow{1 \vee f} X^{(q+1)} \vee X^{(q+1)} \xrightarrow{F} X^{(q+1)},$$

where $\nu$ is the standard coaction and $F$ is the folding map. An elementary computation shows that on homology $(p^*(f))_* = 1 + f_*$. We conclude that the elements of $\text{Aut}_{sX(q)}(X^{(q+1)})$ are in bijection with classes in $[\Sigma V_q, X^{(q+1)}]$ which induce trivial homomorphisms in homology. Since $[\Sigma V_q, X^{(q+1)}] \cong \text{Hom}(\pi_{q+1}(\Sigma V_q), \pi_{q+1}(X^{(q+1)}))$, the effect of an $f : \Sigma V_q \to X^{(q+1)}$ in homology can be determined from the following commutative diagram (both $h$ are Hurewicz maps):

$$\pi_{q+1}(\Sigma V_q) \xrightarrow{f} \pi_{q+1}(X^{(q+1)})$$

$$\xrightarrow{h} \xrightarrow{h} H_{q+1}(\Sigma V_q) \xrightarrow{f} H_{q+1}(X^{(q+1)})$$

The following theorem, which is dual to the main result of [8], now readily follows:

**Theorem 2.2** Let $X$ be a simply-connected, countable, finite-dimensional CW-complex. If the Hurewicz homomorphism $h : \pi_q(X^{(q)}) \to H_q(X^{(q)})$ is injective for all $q$ such that $X$ has $q$-dimensional cells, then $\text{Aut}_s(X) = \{1\}$. 


Let us conclude with some applications of the last theorem:

1. The homotopy groups of the complex projective spaces are given by \( \pi_2(\mathbb{C}P^n) \cong \mathbb{Z} \) and \( \pi_i(\mathbb{C}P^n) \cong \pi_i(S^{2n+1}) \) when \( i > 2 \). The dimension of \( \mathbb{C}P^n \) is \( 2n \) and the first non-trivial Hurewicz homomorphism is bijective so by the previous theorem \( \text{Aut}_*(\mathbb{C}P^n) = \{1\} \).

2. A Moore space \( M(G, n) \) has a cellular decomposition with cells only in dimension \( n \) if \( G \) is a free group, and in dimensions \( n \) and \( n + 1 \) otherwise. In those dimensions the homotopy groups are \( \pi_n(M(G, n)) \cong G \) and \( \pi_{n+1}(M(G, n)) \cong G \otimes \mathbb{Z}_2 \) (we are assuming \( n > 2 \)). Consequently, when \( G \) is a free group or when \( G \otimes \mathbb{Z}_2 = 0 \) the group \( \text{Aut}_*(M(G, n)) \) is trivial.

3. Let us consider spaces \( S^n \vee S^{n+4} \) when \( n \geq 6 \). The usual cellular decomposition has cells in dimensions \( n \) and \( n + 4 \), and by the Hilton-Milnor theorem

\[
\pi_n(S^n \vee S^{n+4}) \cong \pi_{n+4}(S^n \vee S^{n+4}) \cong \mathbb{Z},
\]

induced by the inclusions of \( \pi_n(S^n) \) and \( \pi_{n+4}(S^{n+4}) \) respectively. For that reason the Hurewicz homomorphisms in dimensions \( n \) and \( n + 4 \) are bijective, hence \( \text{Aut}_*(S^n \vee S^{n+4}) \) is trivial when \( n \geq 6 \). By the same argument \( \text{Aut}_*(S^n \vee S^{n+5}) \) is trivial when \( n \geq 7 \). Note that these cases are exceptional as it can be easily seen that \( \text{Aut}_* \) of a wedge of spheres is generally non-trivial.

4. By the results of [2] the group \( \text{Aut}_*(X) \) is nilpotent, so it can be localized with respect to a set of primes. Maruyama showed in [4] that \( \text{Aut}_* \) commutes with the localization when \( X \) is a simply-connected finite CW-complex, i.e. the obvious map

\[
\text{Aut}_*(X) \to \text{Aut}_*(X_P)
\]

is a \( P \)-localization for any set of primes \( P \). As the localization of a finite simply-connected complex is countable and finite-dimensional, the previous theorem can be applied on primary components.

References


