Table of Contents

Chapter 1

FROM GRAPHS TO GEOMETRIES

1.1 Graphs

§ 4. Simple graphs. There are several cryptomorphic definitions of graphs available in the literature. The choice of definition and notation is usually governed by the classes of graphs under investigation.

For *simple graphs* the following definition suffices. A *graph* X is a pair $(X, \sim)$, where $X = V(X)$ is the vertex set and $\sim$ is an irreflexive, symmetric relation on $V(X)$, called *adjacency*. We let $E(X)$ denote the edge set, that is the set of unordered pairs of adjacent vertices of X. If X is a graph we let $V(X) = V$ and $E(X) = E$ denote the respective sets of vertices and edges. The edge, corresponding to the adjacent vertices u and v is denoted by $u \sim v = \{u, v\} = uv = vu$. u and v are called *endpoints* (or *endvertices*) of the edge uv. The number of vertices adjacent to a given vertex v is called the *valence* of v and denoted by $\nu(v)$. We say that an edge is *incident* to its endpoints. Since every edge is incident to two endpoints, summing over the vertex valencies yields twice the number of edges. This simple but useful observation is called the Handshaking Lemma.

Theorem 1.1 (Handshaking Lemma).

$$\sum_{v \in V} \nu(v) = 2|E|$$

Let X and Y be simple graphs. A bijection $\phi : V(X) \mapsto V(Y)$ is an *isomorphism* between graphs if two vertices $x, x' \in V(X)$ are adjacent in X if and only if their images under ϕ, say $y = \phi(x), y' = \phi(x')$ are adjacent in Y. Graphs X and Y are isomorphic if and only if there exists and isomorphism between them.

A property of a graph is called a *graph invariant*, if it is preserved under isomorphisms. Here are some graph invariants: number of vertices, number of edges, maximal valence, or minimal valence.

preserving adjacency, i.e. $u \sim v$ if and only if $i(u) \sim' i(v)$.

We usually represent a graph by a figure using a "dot" for a vertex and a (curved) line segment connecting the two dots corresponding to the endpoints u and v for the edge uv. There are many graph drawing programs that produce such figures from the combinatorial information. A graph is a purely abstract concept and its representation as a figure leaves a lot of freedom. A nice figure should not only be aesthetically pleasing, but convey moreover additional information about the properties of the graph. There is some interaction taking place here. Knowing about special properties of a graph influences the rendering of it and conversely, a nice figure might make some special properties of a graph transparent. We use the computer system VEGA[26] to produce many of our figures.

1.1.1 Examples of Graphs

For the following list of examples the reader is encouraged to take pencil and paper and produce several drawings for each abstract definition given. These examples are basic and will be used many times. Names such as path, cycle, complete graph, etc. will be used for all graphs isomorphic to the ones defined below.

§ 5. Paths. P_n, the path on n vertices.

$$V = \{v_1, v_2, \ldots, v_n\} \qquad E = \{v_i v_{i+1} \mid i = 1, \ldots, n-1\}.$$

The vertices v_1 and v_n in the above example are called *endvertices* of P_n, its other vertices are called *internal*.

§ 6. Cycles. C_n, the cycle of length n.

$$V = \{v_1, v_2, \ldots, v_n\}$$

$$E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$$

.

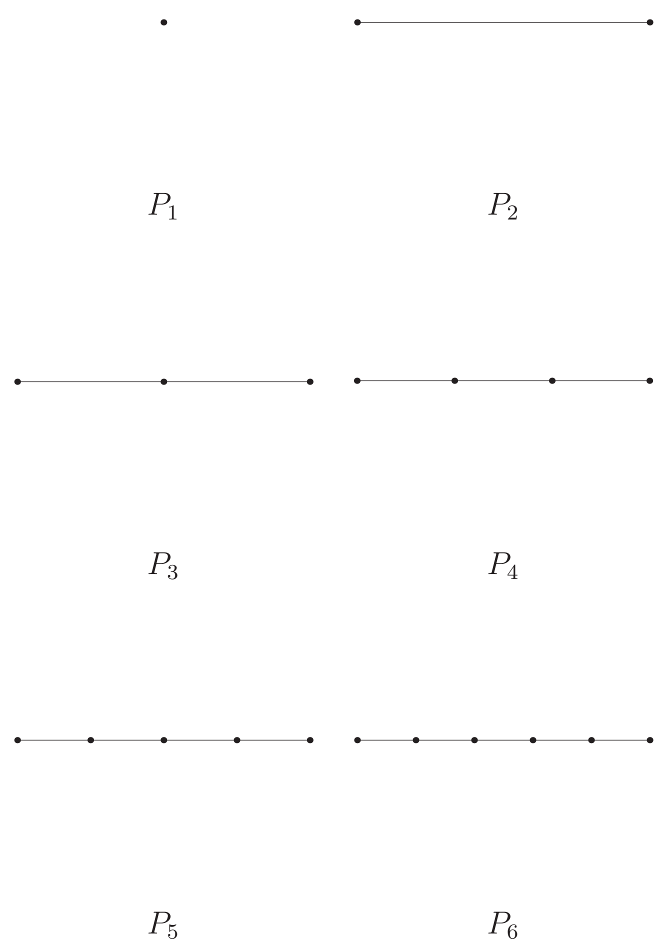


Figure 1.1: Paths $P_n, n = 1, 2, \cdots, 6$.

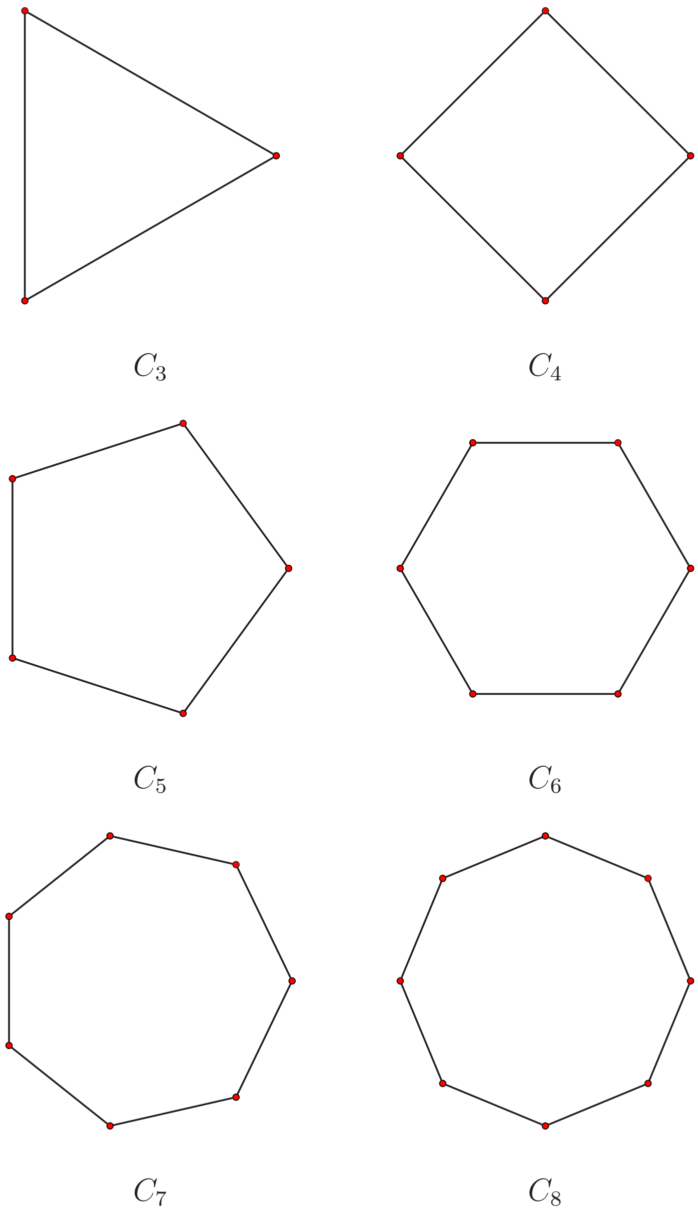


Figure 1.2: Small cycles $C_n, n = 3, 4, \ldots, 8$.

§ 7. Complete Graphs. K_n, the complete graph on n vertices.

$$V = \{v_1, v_2, \ldots, v_n\}$$

$$E = \{v_i v_j \mid i < j; i, j = 1, \ldots, n\}$$

.

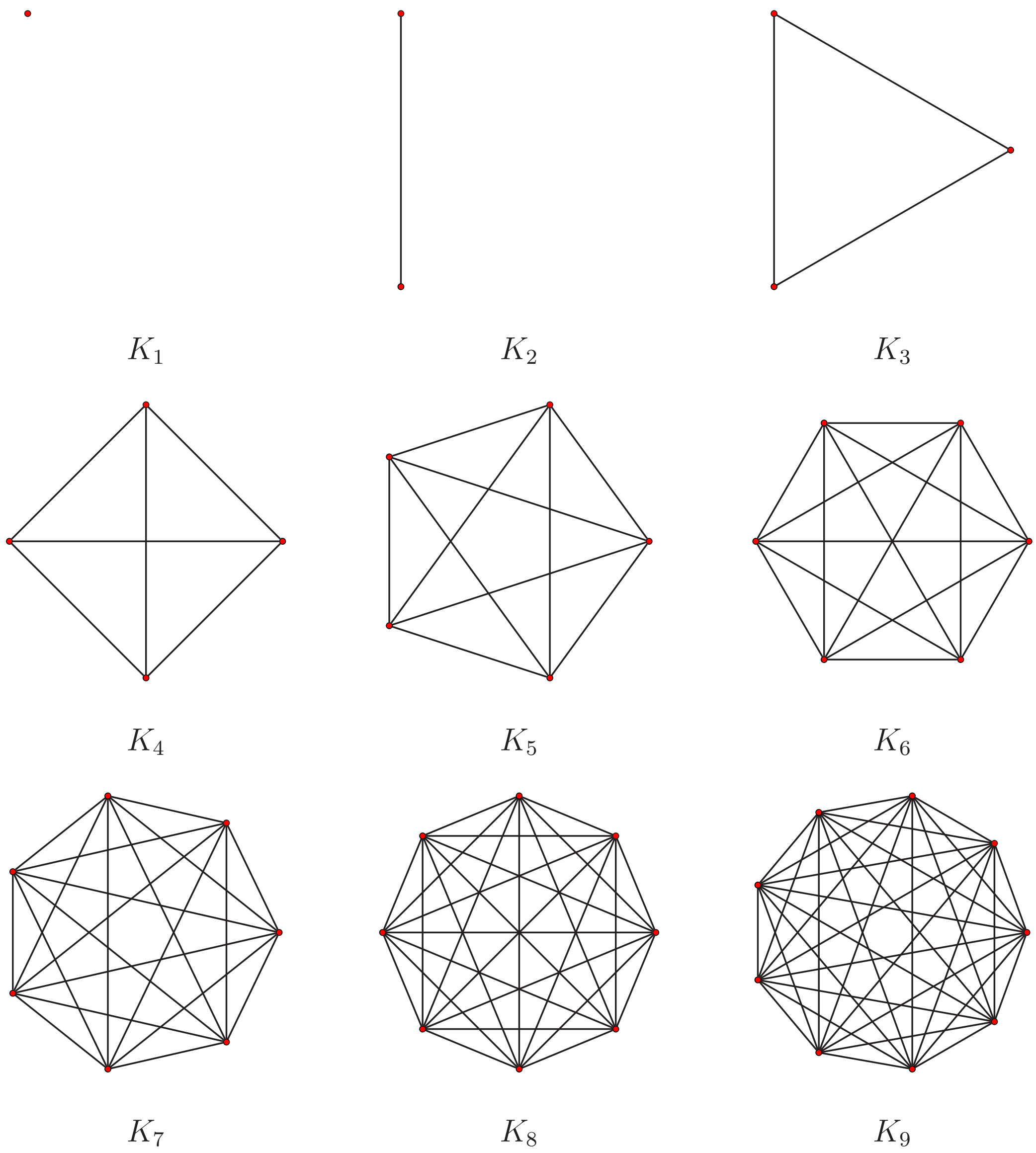


Figure 1.3: Complete graphs $K_n, n = 1, 2, \ldots, 9$.

§ 8. Complete Bipartite Graphs. $K_{m,n}$, the complete bipartite graph.

$$V = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots b_n\}$$

.

$$E = \{a_i b_j \mid i = 1, \ldots, m; j = 1, \ldots, n\}$$

.

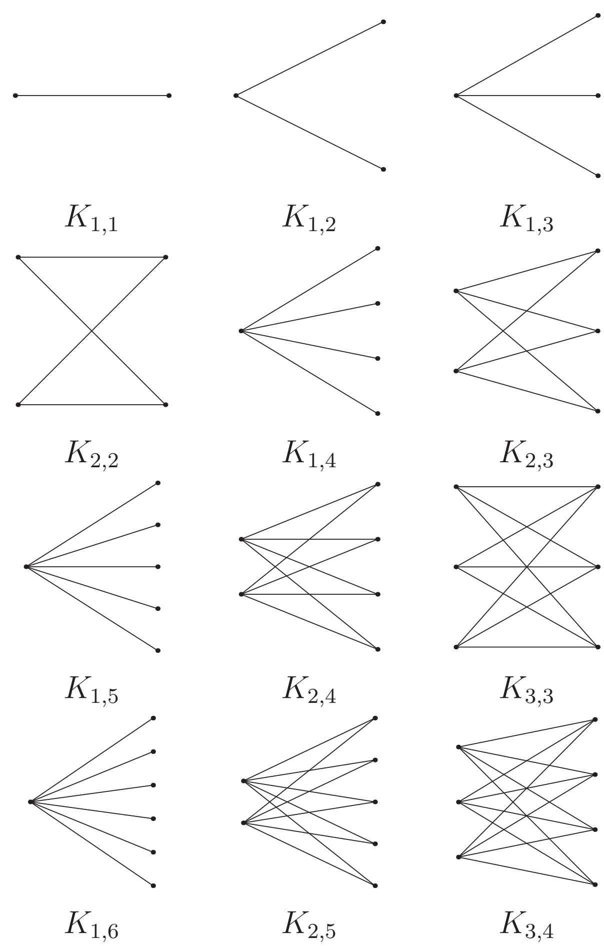


Figure 1.4: Small complete bipartite graphs $K_{m,n}$.

Alternatively, we can define a complete bipartite graph as a graph whose vertex set is partitioned into two sets (the a's and b's in the example above), and two vertices are adjacent if they are in different sets of the bipartition.

§ 9. Complete Multipartite Graphs $K_{n_1,n_2,\ldots,n_p}$, the complete multipartite graph.

$$V = \{a_{ij} \mid 1 \leq i \leq p; 1 \leq j \leq n_i\}$$

$$E = \{a_{ij}a_{kl} \mid i \neq k\}$$

.

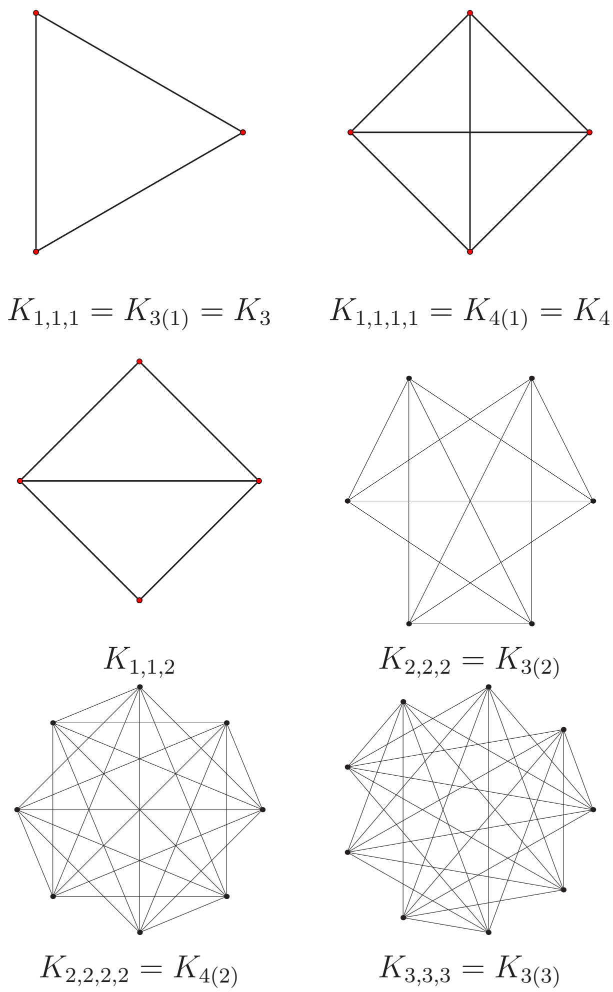


Figure 1.5: Small complete multipartite graphs.

A complete multipartite graph has a vertex set partitioned into p sets and two vertices are adjacent if they belong to different sets of the partition.

For the graph $K_{n_1,n_2,\ldots,n_p}$ with $n_1 = n_2 = \cdots = n_p = n$ the more economical notation $K_{p(n)}$ is frequently used.

§ 10. Generalized Petersen Graphs. For a positive integer $n \geq 3$ and $1 \leq r < n/2$, *the generalized Petersen graph* $G(n,r)$ has vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edges of the form $u_i v_i, u_i u_{i+1}, v_i v_{i+r}, i \in \{0, 1, \ldots, n-1\}$ with arithmetic modulo n.

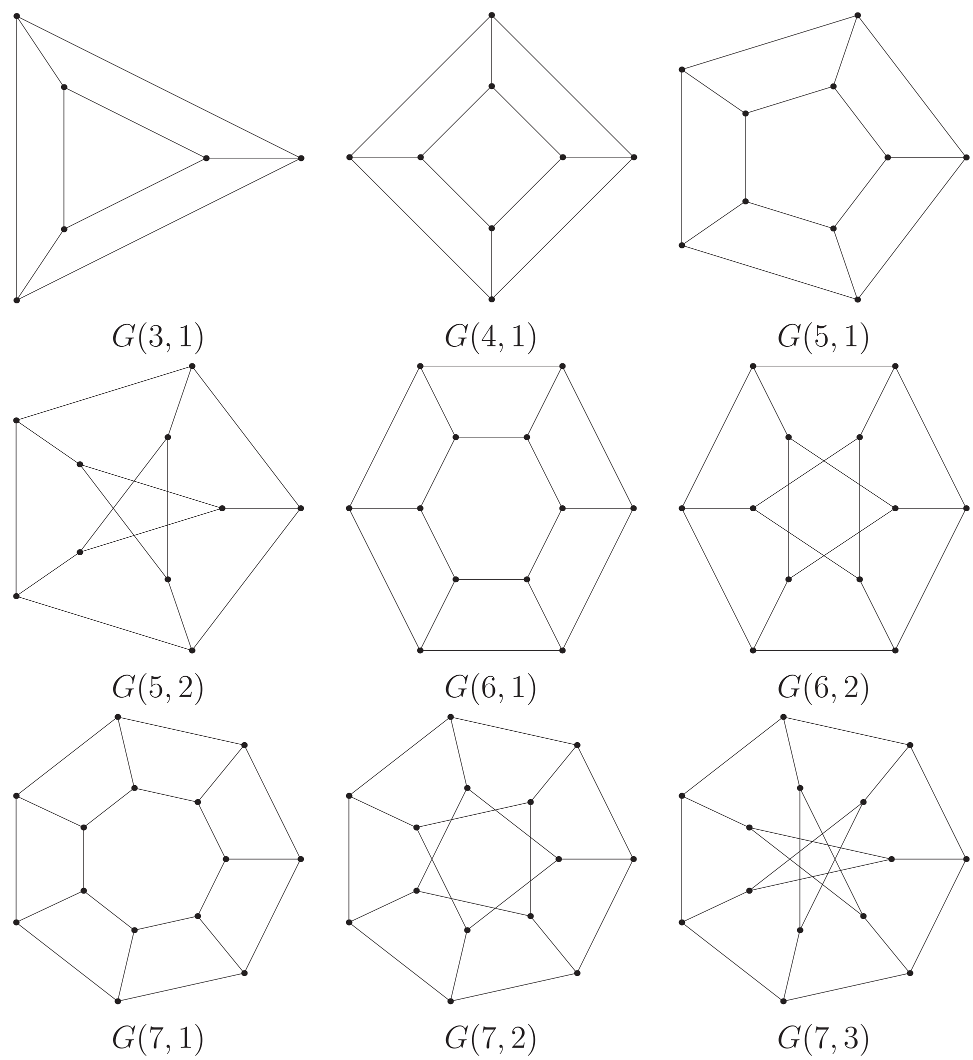


Figure 1.6: Small Generalized Petersen graphs $G(n,r)$.

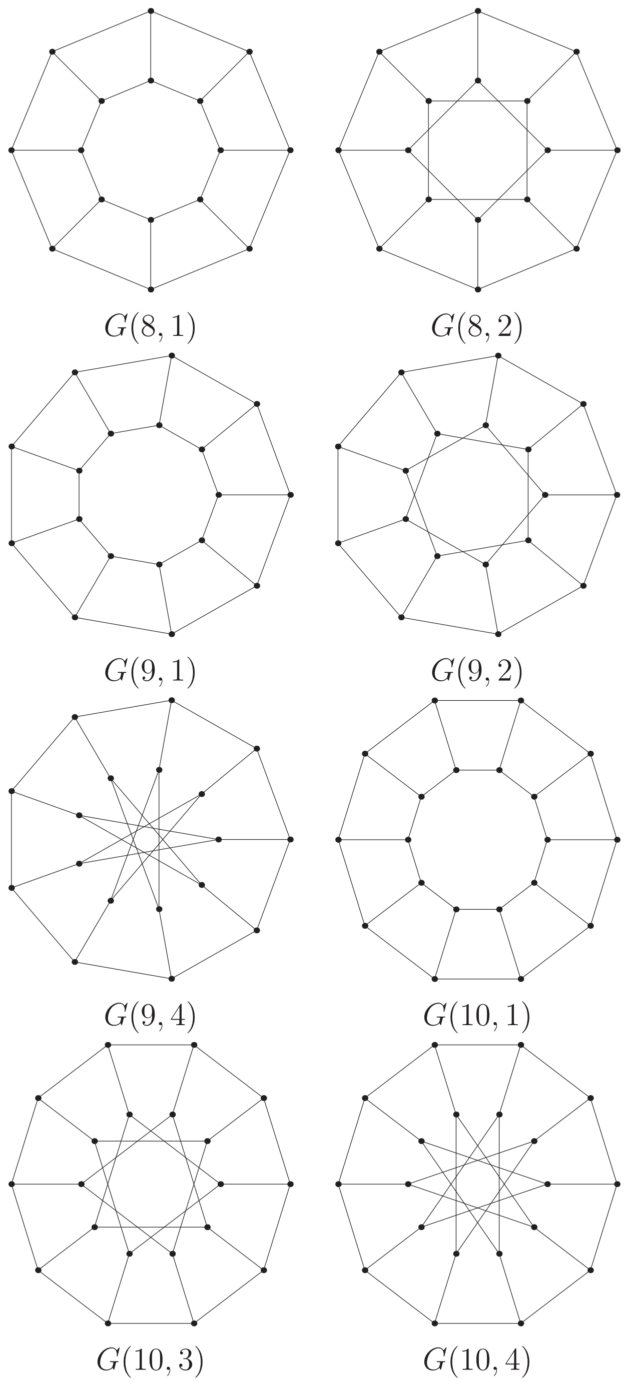


Figure 1.7: More generalized Petersen graphs $G(n,r)$.

§ 11. Dürer Graph. If we cut away (truncate) the opposite corners of a cube $Q_3 = \Pi_4 = G(4,1)$ we obtain the so-called *Dürer graph* $G(6,2)$.

§ 12. Subgraphs, Induced Subgraphs, Spanning Subgraphs. A *subgraph* Y of a graph X is a graph $(Y, \sim')$ such that $V(Y) \subseteq V(X)$ and the relation $\sim'$ is contained in the relation $\sim$ restricted to Y. This implies that $E(Y) \subseteq E(X)$. If $\sim'$ equals the restriction of $\sim$ to S, $S = (S, \sim')$ is called *induced* subgraph of X. A subgraph Y with $V(Y) = V(X)$ is called a *spanning subgraph.* Obviously, a graph with v vertices and e edges contains 2^v induced subgraphs and 2^e spanning subgraphs. These numbers indicate that it is usually difficult to find subgraphs of a particular kind in a large graph. A spanning cycle is called a *hamilton cycle* A graph is called *hamiltonian* if it contains a hamilton cycle as subgraph.

§ 13. Connected Graphs. A graph X is called *connected* if it contains a path between any pair of its vertices.

A graph is called *n-connected* if it contains n internally disjoint paths between any pair of its vertices. The *connectivity* of a graph X is the largest k for which X is k-connected. Connectivity is a graph invariant. The connectivity of C_n, for example is 2, P_n has connectivity 1, while K_n has connectivity $n-1$. Note that in an n–connected graph every vertex must have valence at least n.

§ 14. Trees. A graph that contains a *unique* path between any pair of its vertices is called a *tree*. Equivalently, a tree is a connected graph that contains no cycles. Every connected graph contains a *spanning* tree. It is easy to see that a spanning tree for a graph on n vertices must contain exactly $n-1$ edges and that a graph on n vertices with at least n edges must contain at least one cycle. For our purposes we define a binary tree in a slightly more general way. A *binary tree* is a tree with exactly one vertex of valence k, which is called the *root*, and all other vertices of valence 3 (internal vertices) or of valence 1 (leaf vertices). Figure 1.8 shows a binary tree with a root of valence 3.

A graph that has no cycles is called a *forest.* Observe that a forest is the disjoint union of trees. A connected forest is a tree.

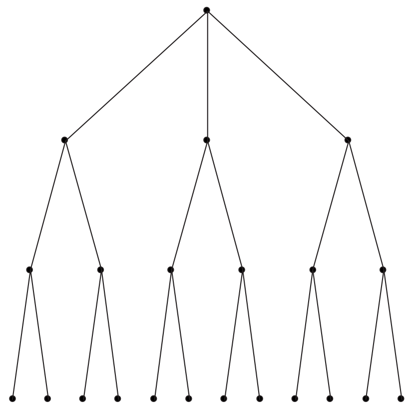


Figure 1.8: A binary tree with a root of valence 3.

§ 15. Girth. The length of the shortest cycle in a graph is called the *girth* of the graph. The girth of a simple graph is at least 3. The girth of a tree or forest is infinite. Note that girth is a graph invariant.

§ 16. Cages. The smallest trivalent graph of girth g is called a g-cage. K_4 is the unique 3-cage and $K_{3,3}$ is the only 4-cage. The Petersen graph $G(5,2)$ is the only 5-cage. We now establish a lower bound on the number of vertices a g-cage must have.

Theorem 1.2. *A $2k$ cage must have at least $3(2^{k-1}+2^{k-3})-2$ vertices. A $2k+1$ cage must have at least 3.2^k-2 vertices.*

Proof. To establish a lower bound on the number of vertices of a g-cage, we start with a binary tree whose root vertex is of valence 3. On the first level, we have 3 vertices, on the second level, we have 6 vertices, etc, see Figure 1.8. For the case $g = 2k+1$, edges between vertices on the same level are only allowed from level k on, so we have $1+3+3.2+3.4+\dots 3.2^{k-1}$ vertices, yielding the desired bound. For even girth we could possibly pair off two vertices on level $k-1$ and connect them to a vertex on level k. □

It turns out that the 6-cage is the Heawood graph; see Figure 4.22. The 7-cage has 24 vertices and is depicted on Figure 2.12. The 8-cage is known as the Cremona-Richmond graph. However, graph theorists prefer to call it the Tutte 8-cage. We will learn more about the Heawood graph and the Tutte

8-cage later and the relationship of the former to projective planes and the latter to the *hexagrammum mysticum* of Pascal.

It is interesting that the 9-cage was not found until quite recently (1995). The search for the 9-cage involved a lot of computer checking and the result came as a surprise. There are 18 non-isomorphic 9-cages. All smaller cages have regular structure and are unique. However, the 9-cages do not show any apparent structure; they are computed in [15].

Balaban found one of the three 10-cages which is shown in Figure 1.9. It is perhaps of interest to note that the 10-cages were known before all the 9-cages were computed. The reason is simply in the fact that the gap between the easily proven lower bound and the actual size of the cage is larger for the 9-cage than for the 10-cage. By Theorem **??** there is no trivalent graph of girth 9 on fewer than 46 vertices and that there is no such graph of girth 10 on fewer than 62 vertices. Since the 9-cage has 58 vertices and the 10-cage has 70 vertices the respective gaps are 12 for the 9-cage and only 8 for the 10-cage. For a survey on cages, see [84].

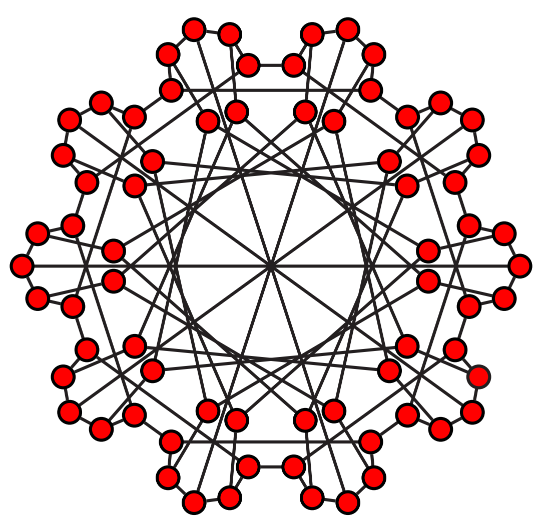


Figure 1.9: Balaban's 10-cage.

§ 17. Planar Graphs. A graph that can be drawn in the plane so that edges intersect only at vertices, is called planar. A drawing without edge crossings is called a *plane embedding* of the graph. Clearly any tree can be drawn without edge crossings. Let G be a connected planar graph and

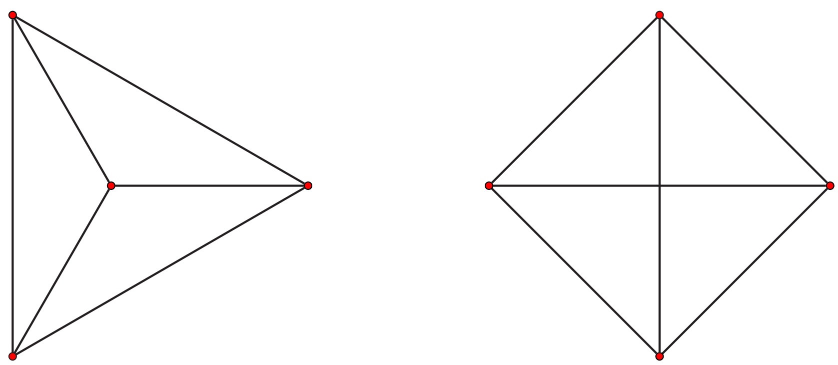


Figure 1.10: Planar and non-planar drawing of K_4.

consider a plane embedding of it. Such a drawing subdivides the plane into regions, which we call the *faces* of G. For example in the plane embedding of K_4 in Figure 1.10, we count four faces, namely three triangles and the infinite outer face.

Given a connected planar graph G on n vertices, together with a plane embedding, choose a spanning tree T in G. T has $n-1$ edges. The plane drawing of T, induced by the plane drawing of G, has only one face. Inserting an additional edge of $E(G) - E(T)$ will divide this region into two. Inductively, we get one more face by inserting an additional edge. We started out with n vertices, $n-1$ edges and 1 face. We add $e-(n-1)$ edges to get f faces, so $e-(n-1) = f-1$, or $n-e+f=2$. The alternating sum $n-e+f$ is called the *Euler characteristic*. It is a property of the surface in which the graph is embedded and we say that the Euler characteristic of the plane equals 2.

§ 18. Semiedges, Pregraphs and General Graphs. Note that in the above examples we are tacitly assuming that n is large enough. C_n fits our definition of graph only if $n \geq 3$. Using this definition we cannot describe the so-called *multi-graphs,* the graphs having parallel edges and loops. In order to accomplish a mathematical description of multi-graphs, we consider two disjoint sets $V(X)$ and $E(X)$, the vertex set and the edge set of the graph X, as well as a function that assigns each edge e at most two vertices, which are, as before, called the *endvertices* of e. Let $S(X) \subseteq V(X) \times E(X)$ denote the collection of vertex - edge incident pairs. $s \in S(X)$ if and only if $s = (v, e)$ and e is an edge whose end-vertex is v. $S(X)$ is called the set of *semi-edges*, of X.

The most general definition that we will use defines a graph X as a quadruple (V, S, i, r) such that V and S are sets, i is a map $i : S \to V$ that assigns each semi-edge $s \in S$ its end-vertex $i(s) \in V$, and $r : S \to S$ is an involution $r^2 = 1$ mapping each semi-edge to its opposite semi-edge. In this model, the set of edges E is given as the set of orbits of r. If r is allowed to have fixed points the corresponding orbits have a single element and the corresponding edge is called a half-edge. Structures with half-edges are sometimes called *pre-graphs*.

Let us summarize: for any graph X we will use the sets $V(X), E(X), S(X)$, the relation $\sim_X$, the mapping i_X and the involution r_X. Note that our definition of graph isomorphism was only given for simple graphs. In Exercise **??** we discuss this notion for general graphs.

1.1.2 Skeleta.

There are several natural bridges between graphs and all kinds of geometric objects. It is time we explore some of them.

Platonic and Archimedean solids are well studied polyhedra. For some history, definitions and theory see [**?**].

Each polyhedron gives rise to a graph composed of its vertices and adjacency defined by its edges. Such a graph is called a *skeleton* or *1-skeleton*.

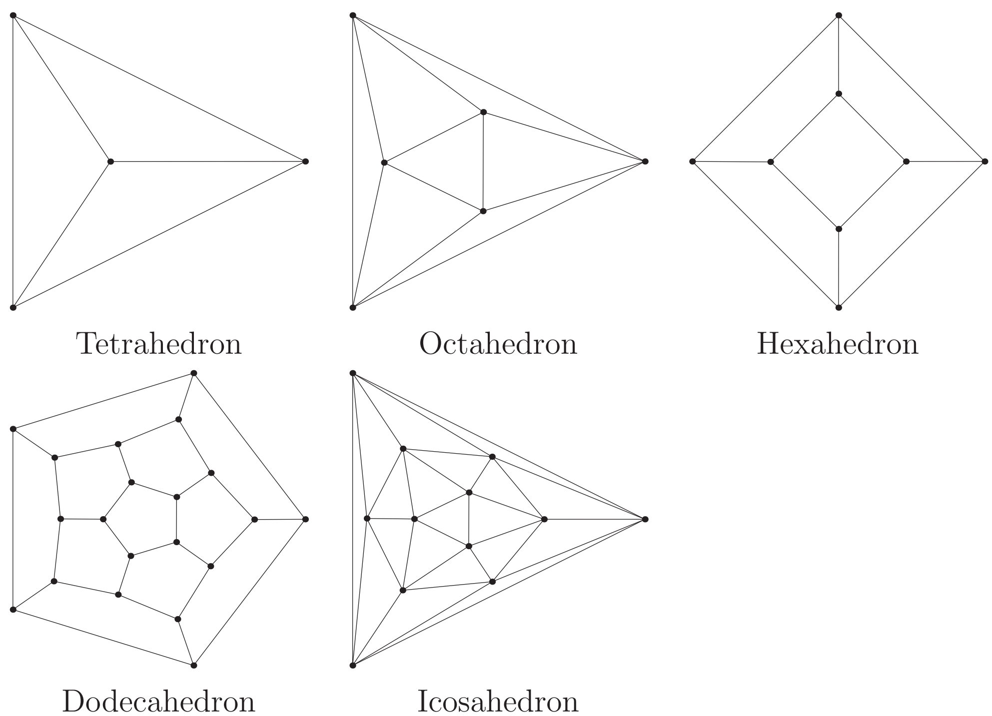


Figure 1.11: Five platonic graphs: tetrahedron, octahedron, hexahedron (cube), dodecahedron and icosahedron.

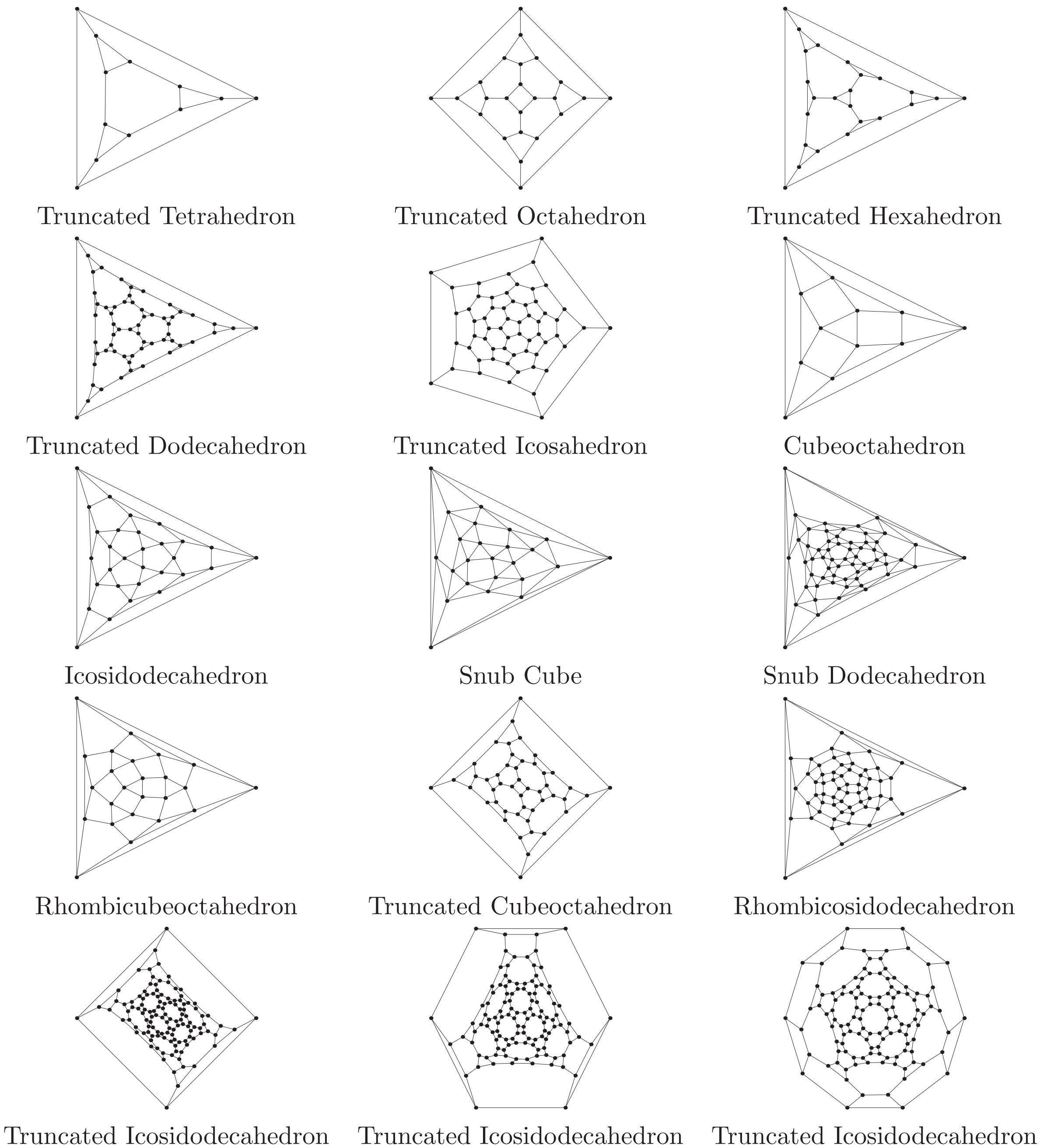


Figure 1.12: Thirteen Archimedean graphs. (The last one is shown in three different forms.)

§ 19. Platonic and Archimedean Graphs. The five platonic solids, namely tetrahedron, octahedron, hexahedron (cube), dodecahedron and icosahedron give rise to the *platonic graphs*, see Figure 1.11. In a similar way the thirteen archimedean solids give rise to the *archimedean graphs*, see Figure 1.12.

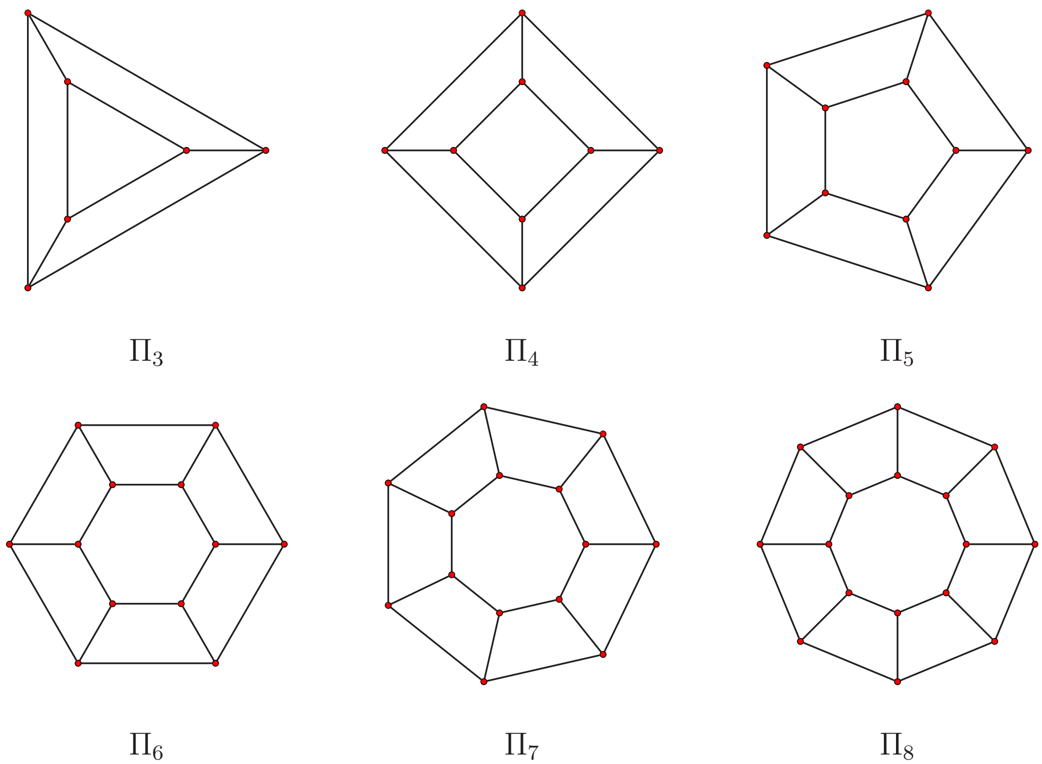


Figure 1.13: Prisms $\Pi_n, n = 3, 4, ..., 8$.

§ 20. Prisms. A prism is a polyhedron with two parallel opposite faces, called bases, that are congruent polygons. All the other faces, called lateral faces, are parallelograms formed by the straight lines through corresponding vertices of the bases. Π_n, the $n-$sided prism graph, is the skeleton of a prism whose base is an n-gon.

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

$$E = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1, v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1, u_1v_1, u_2v_2, \ldots, u_nv_n\}$$

.

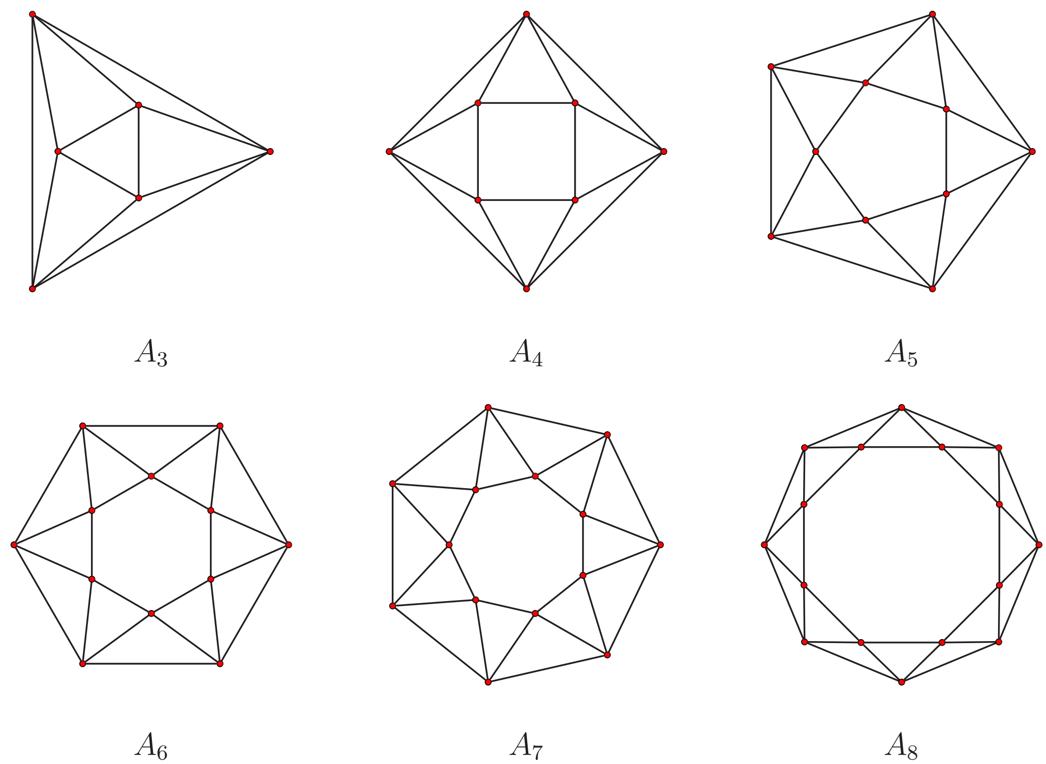


Figure 1.14: Antiprisms $A_n, n = 3, 4, \ldots, 8$.

§ 21. Antiprisms. An antiprism is a polyhedron with two n-gons as bases and $2n$ triangles as side faces. Its 1-skeleton is A_n, the $n-$sided antiprism graph.

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

$$E = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1,$$

$$v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1,$$

$$u_1v_2, u_2v_3, \ldots, u_{n-1}v_n, u_nv_1,$$

$$u_1v_1, u_2v_2, \ldots, u_nv_n\}$$

.

§ 22. Regular Graphs. Cycles, complete graphs, prisms and antiprisms are all examples of *regular* graphs. A graph is *k-regular* if all of its vertices have valence k. Cycles are 2-regular, and conversely, 2-regular graphs are collections of cycles. Note that 1-regular graphs are collections of non-incident edges. 3-regular graphs exist in much greater variety, however. Prisms and antiprisms are simple examples, so are the generalized Petersen graphs.

§ 23. Cubic Graphs and LCF Notation. 3–regular graphs are also called *cubic*. If a cubic graph is Hamiltonian, we can draw it as a $|V|$-gon with inserted diagonals which leads to a convenient notation, due to Frucht et al. [30]. Given the Hamilton cycle, all we have to do is to list the lengths of chords measured in jumps when we traverse the vertices along the Hamilton cycle. Such a list is called the LCF notation.

For instance, K_4 can be described by $[2, 2, 2, 2]$. $K_{3,3}$ is $[3, 3, 3, 3, 3, 3]$, the cube Q_3 is $[3, -3, 3, -3, 3, -3, 3, -3]$. Note that we may use negative integers. In the last example we could have used 5 instead of -3. We can also used exponent notation in order to shorten repeated subsequences. Here is an equivalent shorthand notation for the above examples: $LCF(K_4) = [2^4], LCF(K_{3,3}) = [3^6], LCF(Q_3) = [(3, -3)^4]$.

Example 1.3. *The graph G whose LCF notation is*

$$LCF(G) = [-5, 2, 4, -2, -5, 4, -4, 5, 2, -4, -2, 5]$$

is depicted in Figure 1.15.

§ 24. Wheels or Pyramid Graphs. The 1-skeleton of an n-sided pyramid is called *pyramid graph*. Pyramid graphs are also known as *wheels*, an indication that these graphs can be obtained without help from geometry. In the next section we will describe operations on graphs that can efficiently be used to obtain for example prism or pyramid graphs from vertices, edges or cycles.

1.1.3 Operations on graphs.

There are several operations on graphs available that can be used to generate new, large graphs from old, simple ones.

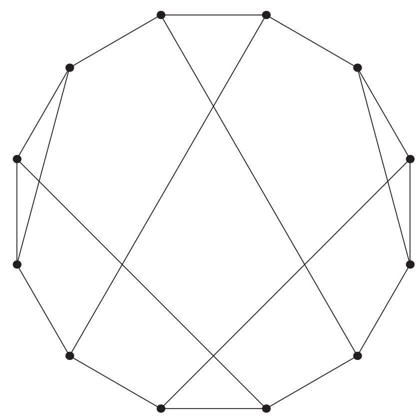


Figure 1.15: LCF code: $[-5, 2, 4, -2, -5, 4, -4, 5, 2, -4, -2, 5]$.

§ 25. Cartesian Product. The hypercube Q_n can be expressed in terms of K_2 if we introduce the *cartesian product* (denoted by $\square$ of graphs. In the same way, the prism $\Pi_n = C_n \square K_2 = G(n, 1)$.

Let X and Y be any two simple graphs. The *cartesian product* $X \square Y$ has the vertex set

$$V(X \square Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \square Y)$ are adjacent if and only if either $x = y$ and $x' \sim y'$ or $x \sim y$ and $x' = y'$.

The prism Π_n, for example can be expressed as the cartesian product of a cycle of length n and the complete graph on 2 vertices, $\Pi_n = C_n \square K_2$.

Since $\square$ is associative (see Exercise 1.9), we can consider the cartesian product of several factors. Taking n factors equal to K_2, we obtain the *hypercube* Q_n. $Q_n = K_2 \square K_2 \square \ldots \square K_2$. Small hypercubes are depicted in Figure 1.18.

§ 26. Tensor Product. Let X and Y be any two simple graphs. The *tensor product* $X \times Y$ has the vertex set

$$V(X \times Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \times Y)$ are adjacent if and only $x \sim y$ and $x' \sim y'$.

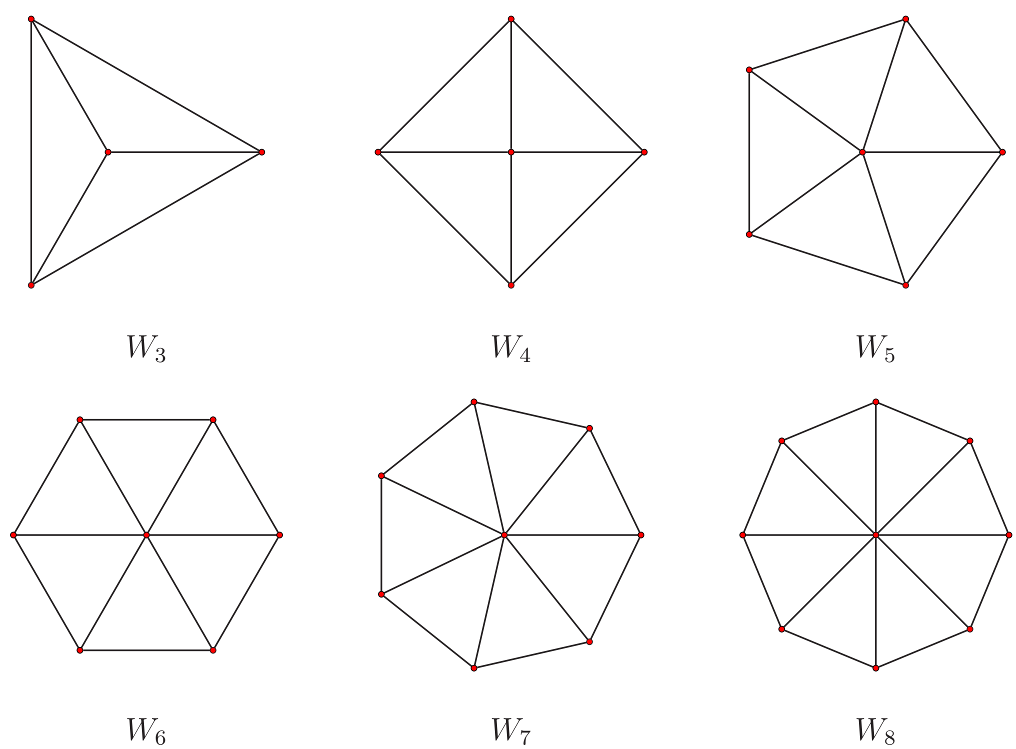


Figure 1.16: Small wheels or pyramid graphs $W_n, n = 3, 4, \ldots, 8$.

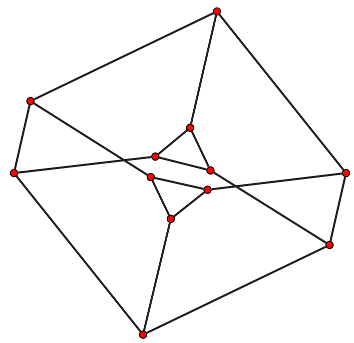


Figure 1.17: The Dürer graph $G(6,2)$ obtained as a skeleton of the cube whose two antipodal vertices have been truncated.

§ 27. Strong Product. Let X and Y be any two simple graphs. The *strong product* $X \boxtimes Y$ has the vertex set

$$V(X \boxtimes Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \boxtimes Y)$ are adjacent if and only if either $x \sim y$ and $x' \sim y'$ or $x = y$ and $x' \sim y'$ or $x \sim y$ and $x' = y'$.

For product graphs see the monograph [49]

§ 28. Line Graph. Among the operations on graphs we need the transformation into the *line graph.* For any simple graph X let $L(X)$ denote the graph whose vertex set $V(L(X))$ is $E(X)$ and two vertices e and e' from $V(L(X))$ are adjacent if and only if e and f are incident (as edges of X) with a common vertex of X. The line graph of K_3 for example is the octahedron graph.

§ 29. Subdivision graph. The *subdivision graph* $S(X)$, has: $V(S(X)) = V(X) \cup E(X)$ and two vertices x and f of $S(X)$ are adjacent if and only if $x \in V(X)$ and $e \in E(X)$ and x is incident to e in X.

§ 30. Graph Complement. The *graph complement* $Y = X^c$ has $V(Y) = V(X)$ and $x \sim y$ in Y if an only if x is not adjacent to y in X.

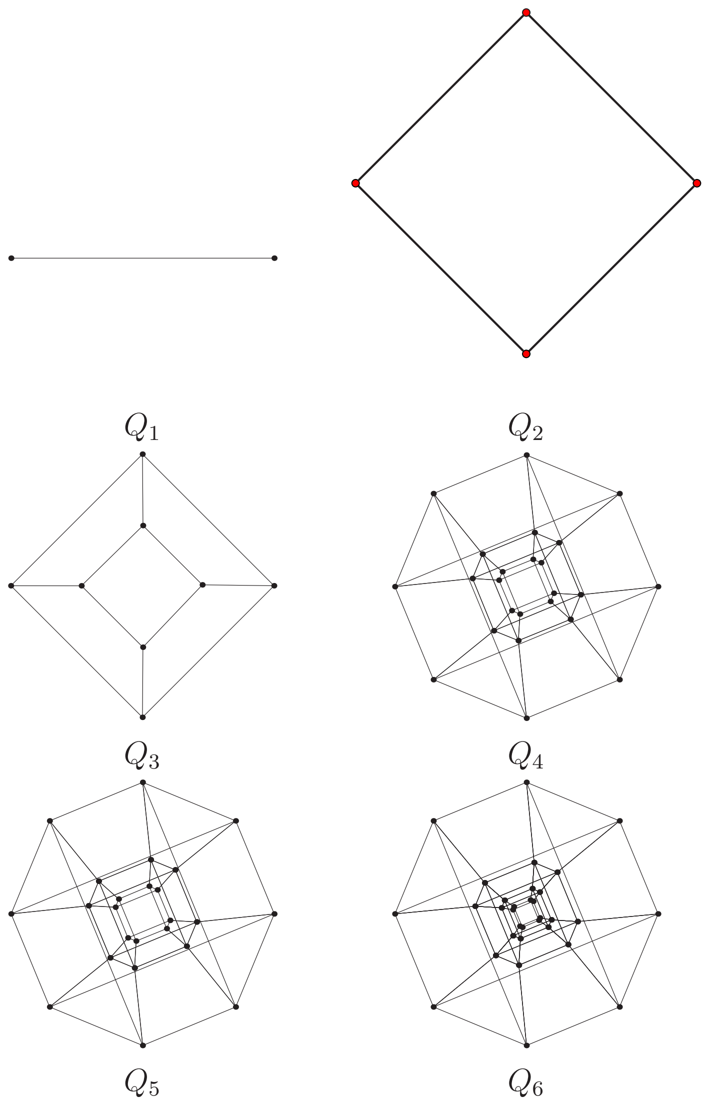


Figure 1.18: Small hypercubes $Q_n; n = 1, 2, 3, 4, 5, 6$.

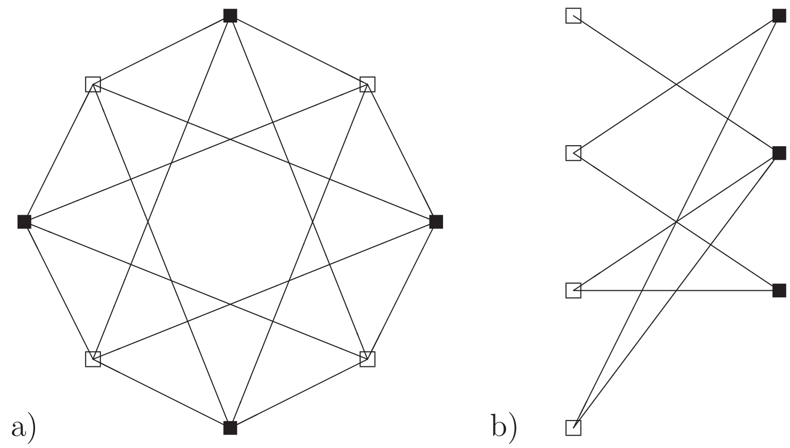


Figure 1.19: Bipartite graphs with black and white bipartitions.

§ 31. **Graph Union.** We define the *graph union* $X \cup Y$ as the disjoint union of two graphs. In particular, if two graphs are isomorphic, we write $2X$ for $X \cup X$, $3X$ for $X \cup X \cup X$, etc.

§ 32. **Graph Join, Cone, Suspension.** The *graph join* $X * Y$ of graphs X and Y can be defined in terms of graph union and graph complement.

$$X * Y = (X^c \cup Y^c)^c.$$

For instance, for a given graph X we denote by $C(X)$ the *cone* over X. It is obtained from X by adding a new vertex called the *apex* of the cone and attaching it to each vertex of X. This operation generalizes to a k-fold cone $C^{(k)}(X)$ in which k new vertices are introduced. A 2-fold cone is known as *suspension*. Finally, $K_{m,n} = K_m^c * K_n^c$. (See Exercise 1.17.)

§ 33. **Graph Square.** For a given graph X its *square* $X^{(2)}$ is a graph on the same vertex set with two vertices adjacent if and only if they are at distance at most 2 in X.

§ 34. **Bipartite Graphs.** A graph $G = (V, E)$ is bipartite if V can be partitioned into two (nonempty) sets V_1 and V_2 such that each edge has one of its endvertices in V_1 the other in V_2. Note that if G is connected and bipartite, the bipartition of the vertex set is uniquely determined, namely two vertices are in the same set of the bipartition if and only if their distance in G is even. For disconnected graphs, bipartiteness clearly implies bipartiteness of its connected components. If G has $c+1$ nonempty bipartite connected components, there are 2^c bipartitions of the vertex set. There are several standard ways to indicate in a drawing that a graph is bipartite, for example to arrange the vertex sets V_1 and V_2 on two different lines, see Figure § 33. b. One also indicates the bipartition by coloring the vertices, say, black and white. The it is easy to check if indeed every edge has one black and one white endpoint, see for example Figure § 33. a.

We first want to explore the structure of regular bipartite graphs. If every vertex of $G = (V_1 \cup V_2, E)$ has valence k, then $k|V_1| = k|V_2|$, and unless $k = 0$, we have $|V_1| = |V_2|$, i. e. V must be partitioned into two sets of equal cardinality, in particular, $|V|$ must be even. For $k = 1$ we get a set of mutually non-incident edges. If a graph G contains a spanning subset of non-incident edges, we say that G contains a *1-factor*.

We now can prove the following theorem.

Theorem 1.4. *Every k-regular bipartite graph G can be written as the edge disjoint union of k 1-factors.*

Proof. We use induction on k. For $k = 1$, there is nothing to show. We assume $k > 1$ and want to show that a k-regular bipartite graph G contains a 1-factor F. We then use the induction hypothesis on $K - F$ to obtain the desired decomposition of the edge set.

To construct a 1-factor, select mutually non-incident edges until every edge not yet selected is incident with at least one of the edges selected so far. Let us call this maximal set of mutually non-incident edges M. If M is not spanning, let v be a vertex not covered by M and consider the set A of all paths starting at v, then using an edge of M, an edge not in M, then an edge in M, etc,. We can find a set of mutually non-incident edges which is of larger cardinality than M if there is at least one path in A that ends at another uncovered vertex u, by removing from M all edges of M on this $u-v$-path and adding its edges not in M. If A does not contain such a path, then the subgraph of G induced by A has only one vertex, namely u, which is not covered by M and all of its vertices in the same bipartition class as u are of valence k, which is impossible if none of its vertices is of valence more than k. □

This theorem enables us to encode regular bipartite graphs on $2n$ vertices by k permutations of n.Given n black and n white vertices and a k-regular graph on these $2n$ vertices with a bipartition indicated by the vertex color, we may list, for each 1-factor, the black endpoints of the edges starting at the white vertices. Conversely, k permutations on n symbols give rise to a k regular bipartite (simple) graph, provided that distinct permutations move a symbol to distinct symbols.

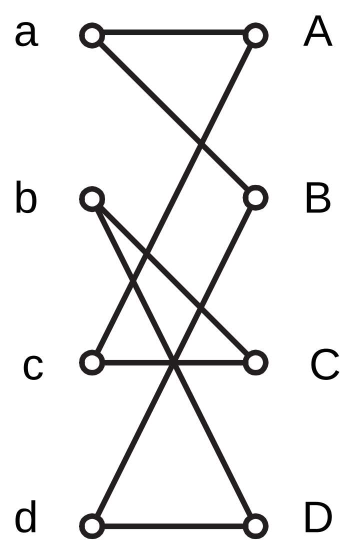


Figure 1.20: The graph from Example 1.5.

Example 1.5.

$$\begin{array}{cccc} 1 & 4 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{array}$$

encodes the graph on the vertex set $\{a, b, c, d, A, B, C, D\}$ *and edge set* $\{(a, A), (b, D), (c, C), (d, B), (a, B), (b, C), (c, A), (d, D)\}$. *See Figure 1.20.*

If we want to relax the condition of regularity to allow unequal bipartition, we can at least require that all vertices of the same color have the same valence. In this case we call the bipartite graph *semiregular*. We write $G = (V_1 \cup V_2; k_1, k_2)$ to indicate bipartition and vertex valences. Prescribing the size of the bipartition imposes restrictions on the values of k_1 and k_2. Certainly k_1 is bounded by $|V_2|$ for a simple graph G, moreover for $G = (V_1 \cup V_2; k_1, k_2)$ we must have $|V_1|k_1 = |V_2|k_2$.

Given $|V_1|$ and $|V_2|$, we might ask for all possible values of k_1 and k_2 so that a semiregular bipartite $G = (V_1 \cup V_2; k_1, k_2)$ exists. $|V_1| = 5$ and $|V_2| = 3$, for example, allow the only possible solution $k_1 = 3$ and $k_2 = 5$, yielding the complete bipartite graph $K_{3,5}$ as the unique structure satisfying the requirements.

Given k_1 and k_2 we may ask for the smallest vertex set, on which there is a semi-regular bipartite graph with the prescribed regularity, again we get as a unique answer a complete bipartite graph.

It is not difficult to show, see Exercise 1.25, that the obvious necessary conditions on the parameters, mamely $|V_1|k_1 = |V_2|k_2$, $k_1 \leq |V_2|$ and $k_2 \leq |V_1|$ are also sufficient for the existence of a simple semi-regular bipartite graph $G = (V_1 \cup V_2; k_1, k_2)$. This situation changes drastically, if we add the extra requirement for G to have girth larger than 4.

To construct a graph $G = (V_1 \cup V_2; k_1, k_2)$ of girth larger than 4 we need to insure that all k_1 neighbors of a vertex in $|V_1|$ have disjoint sets of $k_2 - 1$ neighbors in $|V_1|$ and we get the necessary condition $|V_1| \geq 1 + k_1(k_2 - 1)$. By symmetry, we have the corresponding requirement on the size of $|V_2|$, namely $|V_2| \geq 1 + k_2(k_1 - 1)$. Unfortunately these obvious necessary conditions are not sufficient to ensure the existence of G. According to Gropp [41], there does not exist any 5 regular bipartite graph on 44 vertices of girth larger than 6. The smallest parameter set satisfying the necessary conditions, but for which the existence of a bipartite semi-regular graph is not known is: $|V_1| = 30$, $|V_2| = 20$, $k_1 = 4$, $k_2 = 6$. [41] gives several more examples.

1.2 From Geometry to Graphs and Back

There are numerous paths leading from geometry to graphs and back. We have already met the skeleta of polyhedra as a rich source of interesting graphs. Here we mention some more of such interesting connections. But first let us recall the concept of metric space. This structure lies somewhere between geometry and topology. It captures those properties of usual Euclidean space that measure distance between any two points in space.

§ 35. Metric Space and Distance Function. A set M together with a function $d : M \times M \to \mathbb{R}$ is called a *metric space* if the following is true:

1. For any two points $x, y \in M$ there is $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. For any two points $x, y \in M$ there is $d(x, y) = d(y, x)$
3. For any three points $x, y, z \in M$ there is $d(x, y) \leq d(x, z) + d(z, y)$.

Function d is called the *distance function* of M.

Example 1.6. *Euclidean plane* $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$ *is a metric space for* $d((x, y),(x', y')) = \sqrt{(x - x')^2 + (y - y')^2}$

The same set can be endowed with different distance functions. Given a metric space, we can define a *closed ball* $B(x, r)$ with center $x \in M$ and radius $r > 0$ as follows:

$$B(x, r) = \{y \in M|d(x, y) \leq r\}.$$

§ 36. Distances in Graphs. In a connected graph G we define the *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ to be the length of the shortest path between u and v. Clearly d_G defines a metric space on the vertex set $V(G)$. Usually we describe the metric space by the *distance matrix* $D(G)$ with entry $D_{i,j} = d_G(v_i, v_j)$ for a given ordering $v_1, v_2, ..., v_n$ of the vertices of G. For an arbitrary vertex $v \in V(G)$ we define the distance sequence $d_{G,v} = (1, d_1, d_2, \cdots)$ where d_k denotes the number of vertices at distance k from v. Usually we only consider $d_k > 0$.

Example 1.7. *Since all vertices in a prism graph are indistinguishable there is only one distance sequence. For instance, for* Π_3 *we have* $d_(\Pi_3, v) = (1, 3, 2)$. *Similarly we get:* $d_(\Pi_4, v) = (1, 3, 3, 1), d_(\Pi_5, v) = (1, 3, 4, 2)$.

§ 37. Intersection Graphs. Given a family of sets $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ we may define its *intersection graph*. The vertex set is $\mathcal{B}$ and two vertices are adjacent if and only if the corresponding sets have non-empty intersetion. Note: There is a variation to this construction. Namely, we may construct a general graph by putting $|B_i \cap B_j|$ edges between B_i and B_j.

Example 1.8. *Consider the following 7 sets in the plane: 6 segments: the three sides of a regular triangle, the three heights, and the inscribed circle (see Figure xx?). It is not hard to see that the corresponding intersection graph is* K_7.

Intersection graphs are universal, in the sense that any graph can be represented as an intersection graph. However, if we restrict our attention to some special classes of sets we get sometimes interesting mathematical questions.

§ 38. Intersection Graphs of a Family of Balls. Given a set of n points $V = \{v_1, v_2, \ldots, v_n\}$ in some metric space and a positive number $r > 0$, we may draw n closed balls $B_i := B(v_i, r), i = 1, 2, \ldots, n$, each ball B_i centered at v_i and having radius r. Define a graph $G(V, r)$ as follows: The vertices are the n selected points. Two vertices v_i and v_j are adjacent iff the corresponding balls intersect, i.e. if $B_i \cap B_j \neq \emptyset$. The radius r will be called a *unit* and the graph the *unit sphere graph*.

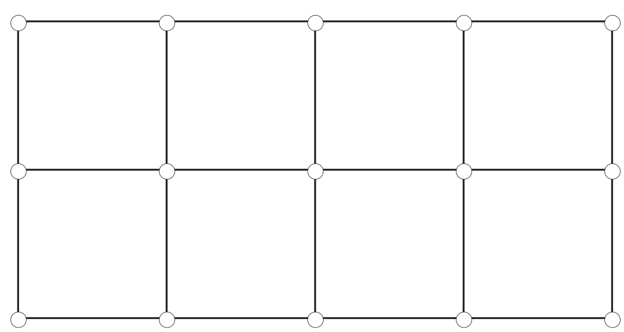


Figure 1.21: The grid graph $Gr(3, 5)$. This is the same as the Cartesian product $P_3 \Box P_5$ of paths P_3 and P_5.

Here are some specific examples:

Example 1.9. *Let us select the following points in the Euclidean plane:* (x, y), $x \in \{1, 2, \ldots, a\}$, $y \in \{1, 2, \ldots, b\}$. *Hence* $n = ab$. *Let* $r = 0.5$. *The unit sphere graph is the well-known* $a \times b$ *grid graph* $Gr(a, b)$. *The grid graph is also known in graph theory as the Cartesian product of the paths* P_a *and* P_b. *Figure § 38. shows the case for* $a = 3$ *and* $b = 5$.

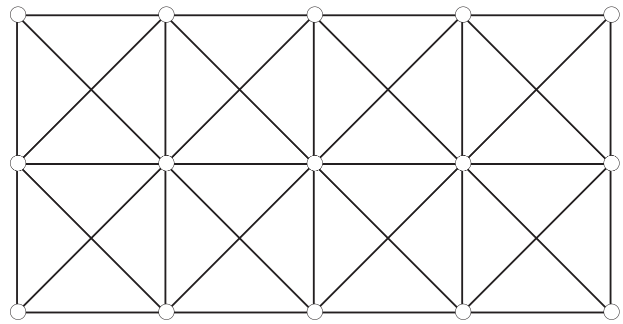


Figure 1.22: The strong product of paths P_3 and P_5.

§ 39. Unit Sphere Graphs. *Unit sphere graphs* are a special case of intersection graphs. Given a family of sets $\{S_1, S_2, \ldots, S_n\}$ we define a graph on n vertices as follows: the vertex set is $\{S_1, S_2, \ldots, S_n\}$. Two vertices S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \emptyset$. By selecting various geometric objects we get interesting families of graphs. For instance, the so-called *interval graphs* are intersection graphs of finite families of line segments in the $\mathbb{R}^1$ line.

It would be interesting and useful to characterize the unit sphere graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance, all platonic graphs arise as unit sphere graphs in $\mathbb{R}^3$. One has to take the vertices of the corresponding platonic solid and radius r the half of the edge length.

Example 1.10. *In order to obtain cube Q_3 one can take*

$$V = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

and $r = 1/2$.

This example shows that the cube graph can be described by a careful choice of 8 points in some metric space. There is another approach to this construction. It involves convex sets.

§ 40. Convex sets. A set of points $K \subseteq \mathbb{R}^3$ is *convex,* if for any two points $x, y \in K$ any point z on the line segment from x to y belongs to K. For any set $S \subseteq \mathbb{R}^3$ we can find the smallest convex set $S \subseteq \operatorname{conv}(S) \subseteq \mathbb{R}^3$, called the *convex closure* of S.

Example 1.11. *The convex closure of the set*

$$V = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

is the cube.

This gives us another general mechanism for constructing graphs from simple geometric objects.:

$$\text{finite set} S \to \text{conv}\,(S) \to \text{skeleton}$$

Starting with a finite set of points in $\mathbb{R}^3$, its convex closure is a convex polyhedron whose 1-skeleton is a graph.

It is time we change the direction and construct geometries from graphs. The geometries we have in mind are mostly finite.

The intersection graph of the family of curves depicting the Fano configuration is K_7 and does not capture the whole combinatorial structure of the configuration.

There is another graph that does that much more accurately. Take in addition to the sets B_i also all the sets $C_{i,j} = B_i \cap B_j$ that are not empty. Define a graph that has all these vertices and there is an edge between $C_{i,j}$ and B_i (and B_j). In the case of Fano plane we get a cubic graph on 14 vertices that is bipartite. It captures all the combinatorial structure of the Fano plane.

§ 41. Representations and Drawings of Graphs. Let G be a graph and let S be a set. A pair of mappings $\rho_V : V(G) \to S, \rho_E : E(G) \to \mathcal{P}(S)$ is called a *graph representation* or an S-representation of graph G if $\rho_V(v) \in \rho_E(e)$ if v is incident with e. If there is no fear for confusion we omit the subscripts of ρ since the argument determines which mapping is considered. We only consider representations for which no pair of vertices is mapped to the same element of S.

Sometimes we only specify ρ_V and have no need for ρ_E. In such a case we may tacitly assume that for each $e = uv \in E(G)$ we have $\rho_E(uv) := \{\rho_V(u), \rho_V(v)\}$.

If S is a vector space, the representation is called a *vector representation*, if S is metric space, the representation is called a *metric representation.* In a metric representation we may define the *length* of each edge $e = uv$ relative

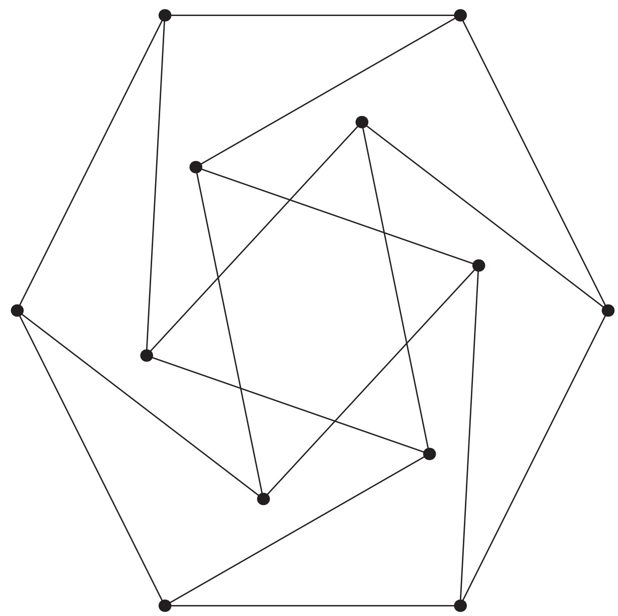


Figure 1.23: Some generalized Petersen graphs are unit distance graphs. In particular, this is true for $G(6,2)$.

to representation ρ as $||e||_\rho = d(\rho(u), \rho_(v)) > 0$. In a simple graph G the length of each edge is strictly positive.

We will consider mostly two cases when S is taken to be $\mathbb{R}^2$ or $\mathbb{R}^3$. In both cases the representation is both metric and vector representation. The first one is called *planar* the second one *spacial representation.* In both cases we define $\rho_E(uv) := conv(\rho(u), \rho(v))$. Each edge is therefore represented as the line-segment connecting the two represented vertices. Such a representation is called *graph drawing.*

Each figure depicting graph in this book has now a formal description as a graph drawing defined above. We have to define when two drawings are equal (or equivalent). Obviously we may consider two drawings that differ for an isometry equivalent. But we may also neglect the difference in scale. This means, for instance, we can always take the barycenter in the origin and take the shortest edge length to be 1. If we are given an ordering of vertices we may thus define the unique "standard" equivalent drawing. For instance, we may define the *energy* of a drawing to be the sum of the lengths of all line-segments representing the edges.

We may define the *dilation coefficient* as the quotient between the longest and shortest edge of the drawing. Graph drawings with dilation coefficient 1 are known as *unit distance graphs.*

§ 42. Generalized Petersen Graphs as Unit Distance Graphs. Some generalized Petersen graphs $G(n, k)$ can be drawn in the plane as unite distance graphs. We embed the outer rim as a regular polygon with a side of length 1. We also embed the inner rim as a collection of star polygons of the side of length 1. If $k = 1$, the inner polygon is congruent the outer polygon. After shifting it in a general direction for a unit, we get the appropriate coordinates for the representation. If $k \neq 1$ the radius of the inner circle is different from the radius of the outer circle. This means we can rotate the inner circle for an angle in such a way that the distance between the two adjacent vertices along a spoke becomes 1. The vertices of the outer rim are given the coordinates $\rho(v_i) = (R\cos(i\pi/n), R\sin(i\pi/n))$ and the vertices in the inner rim are given the coordinates $\rho(u_i) = (r\cos(\phi+i\pi/n), r\sin(\phi+i\pi/n)$, where $R = 1/(2\sin(\pi/n)), r = 1/(2\sin(r\pi/n))$ and $\phi = \arccos((R^2+r^2-1)/(2rR))$. This method works if $R-r < 1$. In particular, the case of Dürer graph $G(6, 2)$ is shown in Figure § 40..

Proposition 1.12. *Every prism graph Π_n graph can be drawn in the plane as a unit distance graph.*

§ 43. Graphs from Polyhedra. The following example shows how we can associate a graph to any convex polyhedron. We use the cube as an example.

Example 1.13. *The cube has 8 vertices:*

$$1, 2, 3, 4, 5, 6, 7, 8,$$

12 edges:

$$a, b, c, d, e, f, g, h, i, j, k, l,$$

and 6 faces:

$$A, B, C, D, E, F.$$

Define a graph on $8+12+6 = 26$ vertices with the property that two elements x and y are adjacent in the graph if and only if they are incident on the cube.

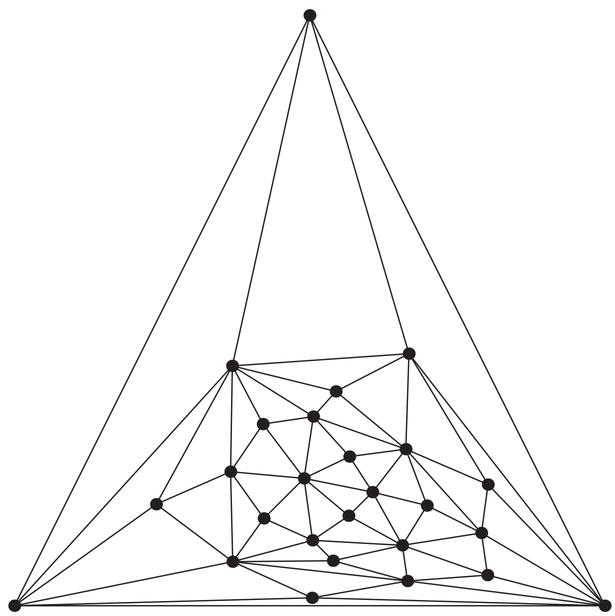


Figure 1.24: The graph defined in Example 1.13.

Observe that the graph we obtained in the Example **??** has an interesting property. The set of vertices can be partitioned in three subsets in such a way that edges connect vertices from different sets. This idea is very powerful and well-known in graph theory. It is called the vertex coloring.

§ 44. Vertex Colorings. An mapping c from $V(X)$ to any finite set of colors C is called a *vertex coloring* if not two adjacent vertices are assigned the same color of C. The smallest number of colors needed for a (proper) vertex coloring of a graph G is called a *chromatic number* of a graph and is denoted by $\chi(G)$.

Example 1.14. *It is not hard to see that the chromatic number of a cycle C_n is 2 if n is even and 3 if n is odd.*

Example 1.15. *Clearly, the chromatic number of tetrahedron is 4. Since octahedron graph is $K_{2,2,2}$, its chromatic number is 3. Cube is bipartite, therefore it has chromatic number equal to 2. Since dodecahedron contains an odd cycle, its chromatic number is at least 3. It is not hard to find a* *proper 3-coloring of* $G(10,2)$. *We leave the determination of the chromatic number of icosahedron to the Exercises.*

Study of colorings of graphs constitute an important branch of graph theory. The problem of determining the exact upper bound on the chromatic number of planar graphs was an outstanding open problem in graph theory and it's solution is known under the name *Four Color Theorem.* Clearly colorings of graphs played an important role in the development of topological graph theory. It may come as a surprise the fact that the graph with a given vertex coloring is a faithful image of a geometry, an *incidence geometry.*

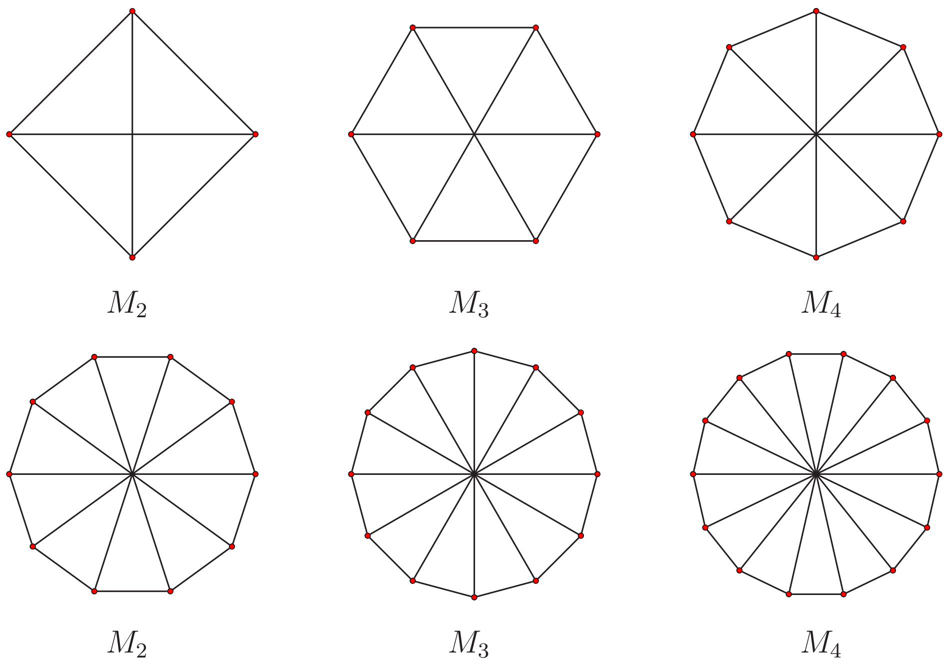


Figure 1.25: Small Möbius ladders M_n.

1.3 Exercises

Exercise 1.1. *Show that K_4, $K_{2,2,2}$, Q_3, $G(10, 2)$ are four out of the five platonic graphs. Design a graph-theoretical method for constructing the missing icosahedron graph.*

Exercise 1.2. *Show that the wheel graph W_n is isomorphic to the cone over C_n.*

Exercise 1.3. *Show that $K_{2,2,2}$ is isomorphic to the suspension over C_4.*

Exercise 1.4. *Show that K_{n+1} is isomorphic to the cone over K_n.*

Exercise 1.5. *Show that Π_n is isomorphic to the product $K_2 \square C_n$.*

Exercise 1.6. Möbius ladder M_n *is obtained from the cycle C_{2n} by adding n main diagonals.*

$$V = \{v_1, v_2, \ldots, v_{2n}\}$$

$$E = \{v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}, v_{2n}v_1, v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}\}$$

.

Exercise 1.7. *Prove that the Möbius ladder M_n can be obtained from the prism graph Π_n by deleting and reattaching only two edges.*

Exercise 1.8. *Prove that the Möbius ladder M_n is bipartite if and only if n is odd.*

Exercise 1.9. *Show that $\Box$ is associative.*

Exercise 1.10. *We defined Q_n, the hypercube in dimension n, as cartesian product of n factors equal to K_2. Show that Q_n may also be defined as follows: The vertex set of Q_n consists of n-tuples of 0's and 1's. Two vertices are adjacent if they differ in exactly one coordinate.*

Exercise 1.11. *Show that $G_1 \Box G_2$ is connected if and only if both G_1 and G_2 are connected.*

Exercise 1.12. *Show that if G_1 and G_2 are both connected, then $G_1 \Box G_2$ is 2–connected.*

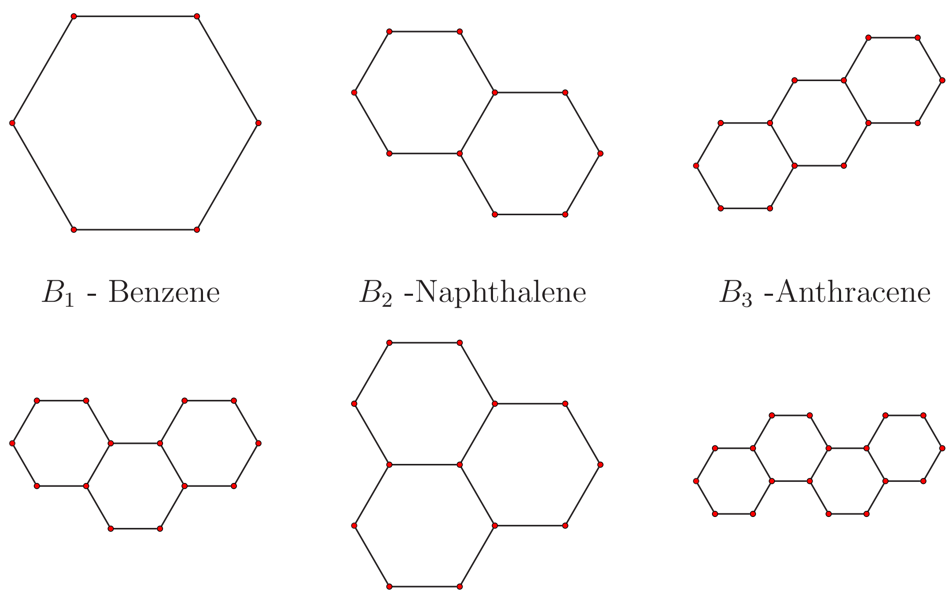
Exercise 1.13. *Given the valence of a vertex $v_1 \in V(G_1)$ and the valence of $v_2 \in V(G_2)$, what can you say about the valence of $(v_1, v_2) \in V(G_1 \Box G_2)$?*

Exercise 1.14. *The definition of a cycle C_n in paragraph § 6. applies to cycles with $n \geq 3$. Define C_1 (the loop) and C_2 a general graphs (see paragraph yy).*

Exercise 1.15. *Any regular $n-$gon $P(n), n \geq 3$ defines a graph $X(n)$ whose vertex set consists of vertices of $P(n)$ and two vertices are adjacent if and only if they belong to the same edge of $P(n)$. Prove that $X(n)$ is isomorphic to the cycle C_n.*

Exercise 1.16. *Prove that the strong product of any two paths is a unit sphere graph.*

Exercise 1.17. *Show that $K_{m,n} = K_n^c * K_m^c$.*



B_1 - Benzene B_2 -Naphthalene B_3 -Anthracene

B_4 -Phenanthrene B_5 - Phenalene B_6 -Chrysene

Figure 1.26: Small benzenoid graphs.

Exercise 1.18. *A graph is called a* benzenoid graph[1] *if it can be obtained by selecting a connected subset of hexagons in an infinite planar hexagonal lattice (representing graphite). Show that all benzenoid graphs can be described as unit sphere graphs in the plane.*

Exercise 1.19. *Let $K(X)$ denote the number of 1-factors in a graph G. Show that benezene has two 1-factors: $K(B_1) = 2$.*

Exercise 1.20. *Prove that a bipartite graph with bipartition sets of unequal size has no 1-factor. Use this result to show that $K(B_5) = 0$.*[2]

Exercise 1.21. *Determine $K(B_n)$, for the bezenoid graphs in Figure xx.*

Exercise 1.22. *Show that M_n admits descrption via LCF notation. Show that it is isomorphic to the graph $[(n)^{2n}]$.*

Exercise 1.23. *Prove that there is - up to an isomorphism - only one cubic graph on 4 vertices with 3 loops.*

Exercise 1.24. *Prove the following result. A graph is bipartite if and only if it contains no cycles of odd length.*

Exercise 1.25. *Given the parameters $|V_1|, |V_2|, k_1, k_2$, satisfying $|V_1|k_1 = |V_2|k_2$, $k_1 \leq |V_2|$ and $k_2 \leq |V_1|$ construct a semi-regular bipartite graph $G = (V_1 \cup V_2; k_1, k_2)$. Hint: Let the i'th vertex of V_1 be adjacent to vertices $\{i, i+1, \ldots, i+k_1-1 (mod |V_2|)\}$.*

Exercise 1.26. *Formulate and prove a structure theorem analogous to Theorem 1.4 for semi-regular bipartite graphs.*

Exercise 1.27. *Here is a table for the Fano configuration:*

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

Draw the corresponding regular bipartite graph and rewrite the table to reflect the partition into 1-factors.

[1] In theoretical chemistry a benzenoid graph is sometimes defined in various slightly different ways. Benzenoid graphs represent molecules of polyhexes, i.e. polycyclic aromatic hydrocarbons. Vertices correspond to carbon atoms while hydrogen atoms are not shown.

[2] In chemistry, 1-factor is called a *Kekule structure*. It is known that polyhex hydrocarbons without Keluke structures are extremely unstable. For instance, this is the case with phenalene.

Exercise 1.28. *What is the girth of the graph in Exercise 1.27?*

Exercise 1.29. *Determine the size of the smallest cubic bipartite graph of girth larger than* 4 *and construct an example.*

Exercise 1.30. *A* fullerene *is a trivalent convex polyhedron whose faces are only pentagons and hexagons. We also call its skeleton by the same name. Prove that the smallest fullerene has 20 vertices.*

Exercise 1.31. *Prove that any fullerene has exactly 12 pentagons.*

Exercise 1.32. *Prove that there are no fullerens on 22 vertices.*

Exercise 1.33. *Find all fullerenes among the generalized Petersen graphs.*

Exercise 1.34. *Find all fullerenes among the Platonic and Archimedean graphs. Prove that any fullerene has exactly 12 pentagons.*

Exercise 1.35. *Draw all nonisomorphic trees on n vertices for $n = 1, 2, 3, 4$.*

Exercise 1.36. *Show that a graph is a forest if and only if each of its connected component is a tree.*

Exercise 1.37. *Prove that the generalized Petersen graphs $G(7, 2)$ and $G(7, 3)$ are isomorphic.*

Exercise 1.38. *Prove that the generalized Petersen graphs $G(8, 2)$ and $G(8, 3)$ are not isomorphic.*

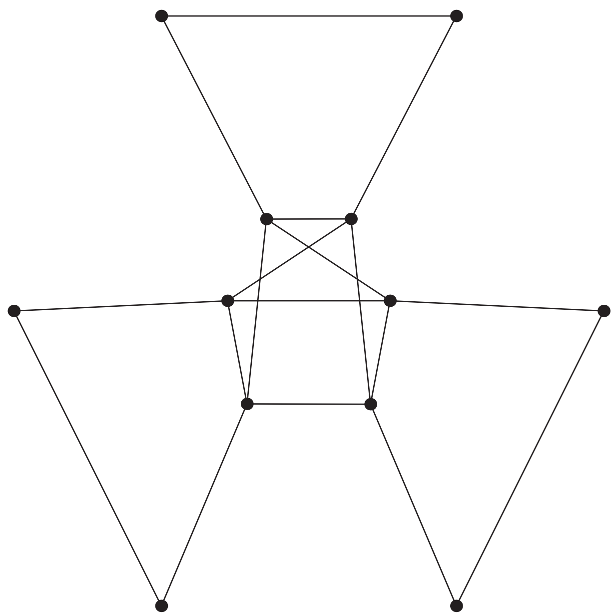
Exercise 1.39. *Prove that the girth of the Petersen graph $G(5, 2)$ is* 5.

Exercise 1.40. *Determine all generalized Petersen graphs $G(n, r)$ of girth* 5.

Show that the graph in Figure 1.41 is indeed the complement of the square of the Dürer graph.

Exercise 1.42. *Prove that every graph is an intersection graph of some family of sets. In particular, determine a family of sets whose intersection graph is isomorphic to $G(5, 2)$.*

Exercise 1.43. *Determine the coordinates for the vertices of a 4-valent polyhedron whose skeleton is isomorphic to the cuboctahedron graph.*



Exercise 1.41.

Figure 1.27: $G(6,2)^{2^c}$.

Exercise 1.44. *Determine the chromatic number of all Archimedean graphs.*

Exercise 1.45. *Define the notion of an isomorphism for general graphs and pregraphs.*

Exercise 1.46. *Find the optimal dilation coefficient of any planar representation of the Moser graph in Figure 1.3.*

Exercise 1.47. *Show that there exists a spatial drawing of Q_3 with dilation coefficient 1.*

Exercise 1.48. *(*) Find a planar drawing of Q_3 with the lowest dilation coefficient. Is it unique?*

Exercise 1.49. *(*) Find a planar drawing of a generalized Petersen graph $G(n,r)$ with dilation coefficient 1.*

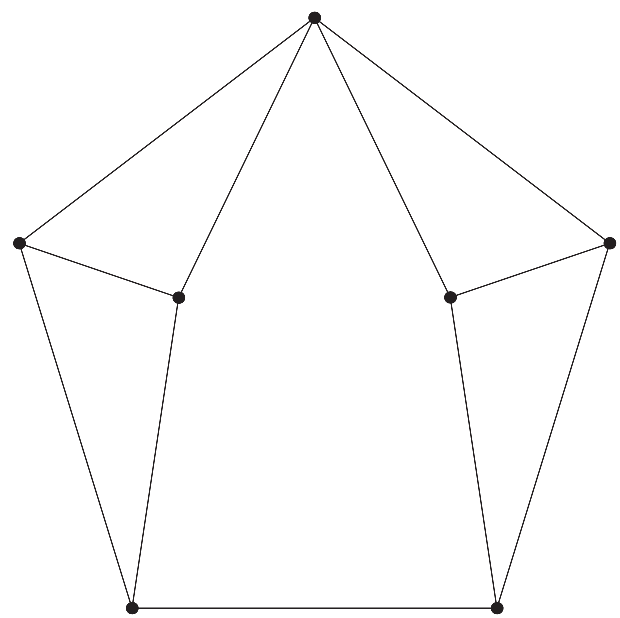


Figure 1.28: The Moser graph.

General Index

References

[1] B. Alspach, C. C. Chen, and K. McAvane. On a class of hamilton laceable 3-regular graphs. *Discrete Math.*, 151:19–38, 1996.

[2] B. Alspach and C. Q. Zhang. Hamilton cycles in cubic cayley graphs on dihedral groups. *Ars. Combin.*, 28:101–108, 1989.

[3] E. Merlin (and E. Steinitz). Configurations. *Encyclopédie des Sciences Mathematiques, Édition française, Tom III. Vol. 2, 1913, 144-160.*

[4] M. Aschbacher. Finite Group Theory. *Cambridge studies in advanced mathematics 10 Cambridge University Press, 1993.*

[5] H. F. Baker. Principles of Geometry; Volume II Plane Geometry - Conics, Circles, Non-Euclidean Geometry. *Cambrige, 1922.*

[6] L. M. Batten. Combinatorics of Finite Geometries, Second ed. *Cambridge University Press, 1997.*

[7] A. Betten, G. Brinkmann, and T. Pisanski. Counting symmetric configurations v_3. *Discrete Appl. Math., 99 (2000) 331–338.*

[8] Anton Betten and Dieter Betten:. Regular linear spaces. *Beiträge zur Algebra und Geometrie 38 (1997), No. 1, 111–124.*

[9] N. Biggs. Algebraic Graph Theory, second edition. *Cambridge Univ. Press, Cambridge, 1993.*

[10] M. Boben, T. Pisanski, and Coordinatizations of the configurations arising from the 10-Cages. `http://www.ijp.si/balaban/balweb.nb`.

[11] M. Boben, T. Pisanski, Polycyclic configurations, submitted.

[12] J. Bokowski and B. Strumfels. Computational Synthetic Geometry. *LNM 1355, Springer-Verlag, 1989.*

[13] I. Z. Bouwer. An edge but not vertex transitive cubic graph. *Bull. Can. Math. Soc. 11 (1968), 533–535.*

[14] I. Z. Bouwer. Vertex and edge-transitive but not 1-transitive graphs. *Canad. Math. Bull., 13 (1970), 231–237.*

[15] G. Brinkmann, B. D. McKay, , and C. Saager. The smallest cubic graph of girth 9. *Combinatorics, Probability and Computing, 5 (1995) 1–13.*

[16] F. Buekenhout. Foundations of incidence geometry. *in Handbook of Incidence Geometry, (F. Buekenhout, ed.) Elsevier, Amsterdam, (1995) 63–105.*

[17] P.J. Cameron and J.H. van Lint:. Designs, Graphs, Codes and their Links,. *Cambridge Univ. Press, Cambridge, 1991.*

[18] H. S. M. Coxeter. Projective Geometry. *Blaisdell Publishing Company, New York, 1964.*

[19] H. S. M. Coxeter, R. Frucht, , and D. L. Powers. Zero-Symmetric Graphs. *Academic Press, New York, 1981.*

[20] H. S. M. Coxeter and W. O. J. Moser. Generators and Relators for Discrete Groups, fourth edition. *Springer-Verlag, 1980.*

[21] H.S.M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc. 56, (1950) 413–455.*

[22] P. Dembowski. Finite Geometries. *Ergebnisse der Mathematik und ihre Greenzgebiete, Bd. 44 Springer-Verlag, New York, 1968.*

[23] A. Deza, M. Deza, and V. Grishukin. Fullerenes and coornidation polyhedra versus half-cube embeddings. *Discrete Math., 192 (1998) 41–80.*

[24] S. F. Du, D. Marušič, , and A. O. Waller. On 2-arc-transitive covers of complete graphs. *J. Combin. Theory B, to appear.*

[25] F. Buekenhout (ed.). Handbook of Incidence Geometry. *Elsevier, Amsterdam, 1995.*

[26] T. Pisanski (ed.):. Vega Version 0.2; Quick Reference Manual and Vega Graph Gallery,. *IMFM, Ljubljana, 1995.*`http://vega.ijp.si/`.

[27] I. Z. Bouwer et al., editor. *The Foster Census.* The Charles Babbage Research Centre, Winnipeg, Canada, 1988.

[28] J. Folkman. Regular line-symmetric graphs. *J. Combin. Theory, 3 (1967), 215–232.*

[29] P. Fowler and T. Pisanski. Leapfrog transformation and polyhedra of clar type. *J. Chem. Soc. Faraday Trans., 90 (1994) 2865–2871.*

[30] R. Frucht. A canonical representation of trivalent hamiltonian graphs. *J. Graph Theory, 1 (1977) 45–60.*

[31] R. Frucht, J. E. Graver, , and M. E. Watkins. The groups of the generalized petersen graphs. *Proc. Cambridge Philos. Soc., 70 (1971) 211–218.*

[32] David G. Glynn. On the anti-Pappian 10_3 and its construction. *Geom. Dedicata*, 77(1):71–75, 1999.

[33] Chris Godsil and Gordon Royle. Algebraic Graph Theory. *Springer, New York, 2001.*

[34] H. Gropp. Blocking set free configurations and their relations to digraphs and hypergraphs. *Discr. Math. 165/166 (1997), 359-370.*

[35] H. Gropp. Configurations. *In C. J. Colburn and J. H. Dinitz, editors The CRC Handbook of Combinatorial Designs CRC Press LLC, Boca Raton (1996), 253–255.*

[36] H. Gropp. Configurations and graphs. *Discr. Math. 111 (1993), 269–276.*

[37] H. Gropp. Configurations and graphs—ii. *Discr. Math. 164 (1997), 155–163.*

[38] H. Gropp. Configurations and their realizations. *Discrete Math. 174:137–151, 1997.*

[39] H. Gropp. The construction of all configurations $(12_4, 16_3)$. *In J. Nešetřil and M. Fiedler, editors, Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity, Elsevier, (1992) 85–91.*

[40] H. Gropp. The drawing of configurations. *Lecture Notes in Computer Science 1027 , F.J. Brandenburg (ed). Graph Drawing GD'95 Proceedings, Springer Verlag (1995), 267–276.*

[41] H. Gropp. On the existence and nonexistence of configurations n_k. *Graphs, designs and combinatorial geometries (Catania, 1989) J. Combin. Inform.System Sci.15 (1990), no.1-4, 34–48.*

[42] H. Gropp. On the history of configurations. *In A. Deza, J. Echeverria and A. Ibarra, editors Internat. Symposium on Structures in Math. Theories, Bilbao, (1990) 263–268.*

[43] J. L. Gross and T. W. Tucker:. Topological Graph Thoery,. *Wiley Intersicence, 1987.*

[44] B. Grünbaum. Astral (n_k) configrations. *Geombinatorics 3 (1993) 32–37.*

[45] B. Grünbaum. Configurations. *Special Topics in Geometry, Math 553B University of Washington, Spring 1999.*

[46] B. Grünbaum. Connected (n_4) configrations exist for almost all", journal = "n. geombinatorics 9 (2000) 24–29.

[47] M. Hladnik. Schur norms of bicirculant matrices. *Linear Alg. Appl. 286 (1999) 261–272.*

[48] M. Hladnik and T. Pisanski. Schur norms of haar graphs. *work in progress.*

[49] Wilfried Imrich and Sandi Klavžar. Product Graphs. *2000.*

[50] D. J. Klein. Graph geometry, graph metrics, and wiener. *MATCH, 35 (1997) 7–27.*

[51] F. Levi. Geometrische konfigurationen. *Leipzig, 1929.*

[52] M. Lovrečič-Saražin. A note on the generalized petersen graphs that are also cayley graphs. *J. Combin. Theory (B), 69 (1997) 226–229.*

[53] R. E. Maeder. Uniform polyhedra. *The Mathematica Journal, 3(4) (1993) 48–57.*

[54] D. Marušič. Half-transitive group actions on finite graphs of valency 4. *J. Combin. Theory, B 73 (1998) 41–76.*

[55] D. Marušič and M.-Y. Xu. A 1/2-transitive graphs of valency 4 with a nonsolvable group of automorphisms. *J. Graph Theory 25 (1997), 133–138.*

[56] M. Meringer. Fast generation of regular graphs and construction of cages. *J. Graph Theory 30 (1999), 137–146.*

[57] K. Metsch:. Linear Spaces with Few Lines,. *LNM 1490, Springer Verlag, 1991.*

[58] B. Mohar and P. Rosenstiehl. Tessellation and visibility representations of maps on the torus. *Discrete Comput. Geom., 19(2):249–263, 1998.*

[59] R. Nedela and M. Škoviera. Which generalized petersen graphs are cayley graphs. *J. Graph Theory, 19 (1995) 1–11.*

[60] T. Pisanski and A. Malnič. The diagonal construction and graph embeddings. *In Proceedings of the Fourth Yugoslav Seminar on Graph Theory pages 271–290. Novi Sad, 1983.*

[61] T. Pisanski and M. Randić. Bridges between geometry and graph theory. *to appear.*

[62] T. Pisanski, J. Shawe-Taylor, and J. Vrabec. Edge-coloring of graph bundles. *J. Combin. Theory B 35(198?)12–19.*

[63] T. Pisanski, A. Žitnik, A. Graovac, , and A. Baumgartner. Rotagraphs and their generalizations. *J. Chem. Inf. Comp. Sci. 34 (1994) 1090-1093.*

[64] B. Polster:. A geometrical picture book. *Springer, 1998.*

[65] D. L. Powers. Exceptional trivalent cayley graphs for dihedral groups. *J. Graph Theory, 6 (1982) 43–55.*

[66] Th. Reye.

[67] G. Ringel. Map Color Theorem. *Springer-Verlag, Berlin, 1974.*

[68] Doris Schattschneider. Escher's combinatorial patterns. *Electronic Journal of Combinatorics, 4 (1997), R#17.*

[69] A. Schroth. How to draw a hexagon. *Discrete Math. 199 (1999) 161 - 171.*

[70] E. Steinitz. Konfigurationen der projektiven geometrie. *Encyklop. Math. Wiss. 3 (Geometrie) (1910) 481–516.*

[71] E. Steinitz. *Über die Construction der Configurationen* n_3. PhD thesis, Kgl. Universität Breslau, 1894.

[72] B. Sturmfels and N. White. All 11_3 and 12_3 configurations are rational. *Aequationes Mathematicae, 39 (1990) 254-260.*

[73] D. Surowski. The Möbius-Kantor Regular Map of Genus Two and Regular Ramified Coverings. *presented at SIGMAC 98, Flagstaff, AZ, July 20-24, 1998* http://odin.math.nau.edu:80/s̃ew/sigmac.html.

[74] J. Tits. Buildings of spherical type and finite bn-pairs. *Lecture Notes in Math.382, Berlin, 1974. Springer–Verlag.*

[75] T. W. Tucker. There is only one group of genus two. *J. Combin. Theory B, 36 (1984) 269–275.*

[76] W. T. Tutte. Connectivity in graphs. *University of Toronto Press, Toronto, 1966.*

[77] W. T. Tutte. A family of cubical graphs. *Proc. Camb. Phil. Soc., 43 (1948), 459–474.*

[78] R. Daublebsky von Sterneck. Die configurationen 11_3. *Monatsh. Math. Physik, 5 (1894) 325–330.*

[79] R. Daublebsky von Sterneck. Die configurationen 12_3. *Monatsh. Math. Physik, 6 (1895) 223–255.*

[80] M. Škoviera and J. Širáň. Regular maps from cayley graphs, part 1: Balanced cayley maps. *Discrete Math., 109 (1992) 265–276.*

[81] D. Wells. The Penguin Dictionary of Curious and Interesting Geometry. *Penguin Books, London, 1991.*

[82] A. White. Graphs, Groups, and Surfaces. *North-Holland Pub. Co., Amsterdam, 1973*.

[83] E. Steinitz (with H. Rademacher). Vorlesungen über die Theorie der Polyeder. *Springer-Verlag, Berlin, 1934*.

[84] P. K. Wong. Cages—a survey. *J. Graph Theory, 6:1–22, 1982*.