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Chapter 2

FROM GROUPS TO SURFACES

2.1 Groups

§ 44. Introduction. An isomorphism of a graph into itself is called a *graph automorphism*. Every graph has an automorphism, e.g. the identity mapping. Automorphisms may be composed and the number of automorphisms a graph possesses is a graph invariant. Consider for example a path P_k of length $k > 1$. In addition to the identity, P_k has an automorphism interchanging its endpoints. The graph in Figure 2.1, which can be described by the notation developed as the cone $C(K_2 \bigcup 2K_1)$, has four automorphisms in all, one that interchanges the two vertices of valence 1, one that interchanges the two vertices of valence 2, one that switches both pairs, and one that fixes everything.

For simple graphs every automorphism is determined by its action on the vertex set and can therefore be considered a permutation of the vertices. In general of course not all permutations of $V(X)$ correspond to automorphisms of X, but for complete graphs or their complements this is the case. For the cycle C_n, cyclic permutations of the vertex set all correspond to automorphisms, while transposition of adjacent vertices does not, if $n > 3$. For the cone over C_n, we get the same set of automorphisms as for C_n, because the new vertex has to be fixed if $n > 3$.

The set of automorphisms $\operatorname{Aut} X$ of a graph X together with composition of mappings, form an algebraic structure, called a *group*.Group theory might

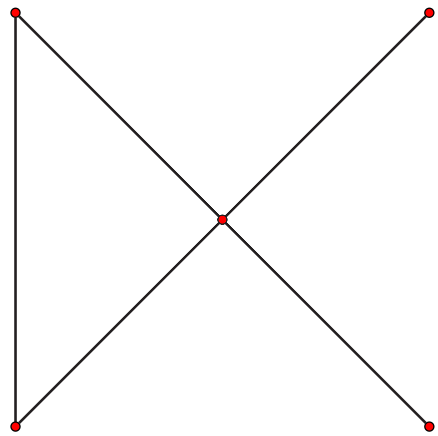


Figure 2.1: $C(K_2 \bigcup 2K_1)$

be a familiar topic for many readers, but we want to define concepts and summarize results used in later chapters. We use Hölder's [**?**] definition for a group, which was also used in Levi's book on configurations [**?**].

§ 45. Group. A *group* G of *order* n consists of n elements, $g_1, g_2, \ldots, g_n$ together with a law of composition which assigns to each ordered pair (g_i, g_j) a third element g_k, which we usually call the product and write $g_i \centerdot g_j = g_k$, provided the law of composition satisfies the following two axioms:

Axiom 1 Replacing any two of the symbols in the equation $a \centerdot b = c$ by group elements, uniquely determines a third group element.

Axiom 2 The associative law holds, $(g_i \centerdot g_j) \centerdot g_k = g_i \centerdot (g_j \centerdot g_k)$

Example 2.1. *The real numbers,* $\mathbb{R}$ *form a group with respect to addition. If we want to consider multiplication as group operation on* $\mathbb{R}$*, we have to exclude* 0*, because the equation* $a0 = 0$ *does not uniquely define a real number, in violation of axiom 1.*

Example 2.2. *The integers,* $\mathbb{Z}$ *form a group with respect to addition.*

Example 2.3. *The integers mod n,* $\mathbb{Z}_n = \{0, 1, 2, \cdots n-1\}$ *form a group with respect to addition mod* n.

From the group axioms it is easy to deduce, see Exercise 2.3, the existence of a *unit element*, e, for which $g \centerdot e = e \centerdot g = g$ for all $g \in G$.

We say that two group elements g_i and g_j *commute* if $g_i \centerdot g_j = g_j \centerdot g_i$. Thus every group element commutes with the unit element e.

For k-fold composition of a group element g with itself we use the notation g^k. We define $g^0 = e$. In a group G of finite order n, there must be a natural number k for which $g^k = e$. The smallest positive such k is called the order of the element g. The order of an element divides the order of the group (see Exercise 2.2). Consequently, in a group whose order is a prime number p, each non–unit element has order p.

Negative exponents are defined by $g^{-1} \centerdot g = e$. For finite groups we have $g^{-1} = g^{k-1}$, where k is the order of g. g^{-1} is called the *inverse* of g. Every group element commutes with its inverse, see Exercise 2.4.

Example 2.4. *Consider as an example the rotations of the plane about a fixed origin through multiples of* $2\pi/n$ *as group elements with composition of rotations. This group can also be described by the elements* $0, \ldots, n-1$ *with addition modulo* n *as composition and denoted by* $\mathbb{Z}_n$.

Example 2.5. *The set of numbers* $\{1, -1, i, -i\}$ *where* $i = \sqrt{-1}$ *with complex multiplication forms a group.*

Example 2.6. *Important examples for us are the groups of symmetries of geometric objects.* $\mathbb{Z}_4$ *can, for instance, be interpreted as the group of orientation preserving symmetries of the Keops Pyramid.*

§ 46. Subgroups. A *subgroup* H of a group G consists of a subset of the elements of G together with the same law of composition as for G, and so that H is indeed a group. To check if a finite subset H of G forms a subgroup, it is enough to check if H is closed under composition, i.e. if the product of any two elements of H is again an element of H. This is not true if H is infinite, see Exercise 2.5.

As subgroups of $\mathbb{Z}_n$ we can easily identify all $\mathbb{Z}_k$ where k divides n. There are no other subgroups. In particular $\mathbb{Z}_p$, p-prime, has only trivial subgroups, namely $\mathbb{Z}_1$ and $\mathbb{Z}_p$. In fact, any group of prime order has only trivial subgroups.

§ 47. Cosets. Let G be a group of order n and $H = \{h_1, \ldots, h_k\}$ a subgroup of G. For an element $g \in G$ we call the set $Hg = \{h_1 \centerdot g, h_2 \centerdot g, \ldots, h_k \centerdot g\}$ a *right coset of H generated by g.* Cosets are either identical or disjoint, since if Hg_1 and Hg_2 have an element $h_1 \centerdot g_1 = h_2 \centerdot g_2$ in common, then $g_2 \centerdot g_1^{-1}$ as well as its inverse $g_1 \centerdot g_2^{-1}$ are elements of H, so $Hg_1 = Hg_2$. Moreover, all cosets have the same number, k, of elements, therefore k divides n. The quotient $i = n/k$ is called the *index* of H in G. We formulate this as a theorem.

Theorem 2.7. *The order k of a subgroup H of a group G divides the order n of G and $n = ik$, where i is the index of H in G.*

Right cosets of H therefore partition the group G into equivalence classes of the same size. In a similar way one can define *left cosets*.

§ 48. Abelian Groups. A group in which every pair of elements commute is called *commutative* or *Abelian group*. In an Abelian group the collection of left cosets coincides with the collection of right cosets.

§ 49. Conjugation. Given a subgroup H of a group G, then the set $g^{-1}Hg$ consisting of elements of H multiplied by a fixed element of G from the right and by its inverse from the left, also forms a subgroup of G, which is easily seen by the fact that the set is closed under composition. $g^{-1}Hg$ is called a *conjugate* of H. H and all its conjugates have the same order.

§ 50. Normal subgroups. It may happen that a subgroup H of a group G is equal to all its conjugates. In this case H is called a *normal* subgroup of G. The unit element as well as the group itself trivially satisfy the conditions of a normal subgroup, subgroups other than those are called *nontrivial*. If Hg_1 and Hg_2 are cosets of a normal subgroup H of G, then we have $Hg_1Hg_2 = (g_1Hg_1^{-1})g_1Hg_2 = g_1Hg_2 = Hg_1g_2$. This means that the cosets themselves form a group, which is called the *quotient* G/H. Observe that the order of G/H equals the index i of H in G.

§ 51. Group Homomorphisms and Isomorphisms. Given two groups $(G_1, \odot)$ and $(G_2, \centerdot)$, a *homomorphism* $\phi : G_1 \to G_2$ is a map with the property that $\phi(g_i) \centerdot \phi(g_j) = \phi(g_i) \odot \phi(g_j)$ for all pairs (g_i, g_j) of elements of G_1. Every homomorphism maps the identity of G_1 to the identity of G_2 because

Table 2.1: The Klein 4-group

$\odot$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\phi(g) = \phi(1 \odot g) = \phi(1) \cdot \phi(g)$. Moreover, $\phi(g^{-1}) \cdot \phi(g) = \phi(1) = 1$ implies that $\phi(g^{-1}) = (\phi(g))^{-1}$. All the elements of G_1 which are mapped to the unit element of G_2 form a normal subgroup of G_1, see Exercise 2.10, which is called the *kernel* of ϕ. A homomorphism which is both injective and surjective is called *isomorphism*. As far as the abstract groups are concerned the group is interesting up to isomorphism. This means we do not distinguish between isomorphic groups.

Example 2.8. *The multiplicative group on* $\{1, -1, i, -i\}$ *is isomorphic to additive group* $\mathbb{Z}_4$ *on the elements* $\{0, 1, 2, 3\}$ *with* $\phi(1) = 0, \phi(-1) = 1, \phi(i) = 2, \phi(-i) = 3$ *as an isomorphism.*

For a prime p, all groups of order p are isomorphic, so up to isomorphism, there is only one group of order p. See Exercise 2.14.

§ 52. Group Table. It is customary and sometimes convenient to use a *group table* to represent a group of order n. The rows and columns of an $n \times n$ grid are labelled by group elements and in each column (row) are the results of right (left) multiplication of the group elements by the row (column) label. As an example we define the *Klein 4-group*, which is e.g. the automorphism group of the graph in Figure 2.1, by its group table, Table 2.1.

While the associative law has to be checked separately, the existence of a unit element is easy to read off the table. It is the element in the row and column that equals the row and column labels. Inverses are easy to read off the table as well. Moreover, every group element occurs in each row and column of the table exactly once, so all rows (columns) are actually permutations of the group elements. We see that each group of order n is isomorphic to a subgroup of the permutation group of n symbols: Each group element x is represented as a row X in its group table. This row is understood as a permutation of the leading row. Group multiplication xy

Table 2.2: The Quaternion Group

$\odot$	1	$\bar{1}$		i	$\bar{i}$		j	$\bar{j}$		k	$\bar{k}$
1	1	$\bar{1}$	⋮	i	$\bar{i}$	⋮	j	$\bar{j}$	⋮	k	$\bar{k}$
$\bar{1}$	$\bar{1}$	1	⋮	$\bar{i}$	i	⋮	$\bar{j}$	j	⋮	$\bar{k}$	k
	...	...	...	...	...	...	...	...	...	...	...
i	i	$\bar{i}$	⋮	$\bar{1}$	1	⋮	$\bar{k}$	k	⋮	j	$\bar{j}$
$\bar{i}$	$\bar{i}$	i	⋮	1	$\bar{1}$	⋮	k	$\bar{k}$	⋮	$\bar{j}$	j
	...	...	...	...	...	...	...	...	...	...	...
j	j	$\bar{j}$	⋮	k	$\bar{k}$	⋮	$\bar{1}$	1	⋮	$\bar{i}$	i
$\bar{j}$	$\bar{j}$	j	⋮	$\bar{k}$	k	⋮	1	$\bar{1}$	⋮	i	$\bar{i}$
	...	...	...	...	...	...	...	...	...	...	...
k	k	$\bar{k}$	⋮	$\bar{j}$	j	⋮	i	$\bar{i}$	⋮	$\bar{1}$	1
$\bar{k}$	$\bar{k}$	k	⋮	j	$\bar{j}$	⋮	$\bar{i}$	i	⋮	1	$\bar{1}$

gives rise to a row XY that corresponds to the product of the two elements. The inverse corresponds to the permutational inverse.

Example 2.9. *In Table 2.1 all diagonal entries corresponding to elements multiplied by themselves, are equal to the unit. This shows that Z_4 is not isomorphic to the Klein 4-group since Z_4 has an element of order 4 while all the elements of the Klein 4-group are of order 2.*

§ 53. The Quaternion Group. Quaternions generalize complex numbers. While a complex number can be written as $z = a+bi$, a quaternion is of a form $q = a+bi+cj+dk$. Multiplication of quaternions is non-commutative. In order to define multiplications of quaternions we have to specify multiplication of their units. If we write $\bar{i}$ for $-i$,$\bar{j}$ for $-j$,$\bar{k}$ for $-k$, the multiplication closes to a group on 8 elements, called the quaternion group defined by the group table in Table 2.2.

§ 54. Symmetric Group Sym(n). Let X be a finite set, $|X| = n$ and let Sym(n) denote the group of all permutations of the elements of X, i.e. the group elements are bijections $\alpha : X \mapsto X$ and composition is ordinary composition of maps. The order of Sym(n) is $n!$.

§ 55. Permutation Groups. We say that a subgroup G of $Sym(n)$ *acts* on X and talk about the *action* of G on X.

§ 56. Orbits and Transitive Groups. The *orbit*, $[x]$, of an element $x \in X$ is defined as the set of images of x under the action of G:

$$[x] = \{gx | g \in G\}$$

. The action of G on X is called *transitive* if there is only one orbit. Clearly, the action of $\mathrm{Sym}(n)$ is transitive. The rows of a group table for a group G can be considered the set of symbols, X, of a permutation group, isomorphic to G which acts transitively on the rows.

§ 57. Primitive vs. Imprimitive Action. A transitive action of G on X is called *imprimitive* if the elements of X can be partitioned into k sets, $X = \bigcup_{i=1}^{k} X_i$, with $1 < k < |X|$ and each $g \in G$ induces a set-wise permutation of the X_i's.

Example 2.10. *Consider the symmetries of the prism graph Π_n. The action of $Aut\,\Pi_n$ on $V(\Pi_n)$ is imprimitive if and only if $n \neq 4$. For $n = 4$, the prism becomes a cube graph Q_3 with primitive automorphism group.*

§ 58. Quotient Group. As an example of an imprimitive action, consider a group table for to N. These groups correspond to the X_i's and the group table is naturally partitioned into i^2 sub-squares, where i is the index of n in G and these sub-squares form the group table of the quotient group G/N. In Table 2.2 we see the group table of the quaternion group which has S_2 as a normal subgroup, yielding the Klein 4-group as a quotient group.

§ 59. Matrix Representation. A convenient way to represent permutation group is to identify the elements with 0-1 matrices. If the permutation group acts on a set of n elements, we use an $n \times n$ matrix. The identity is represented by the unit matrix and the matrix corresponding to the transposition that exchanges elements i and j is obtained from the identity matrix by exchanging row i with row j. Composition of permutation corresponds to matrix multiplication.

Example 2.11. *For the Klein 4-group, for example, we can use*

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and with this notation and matrix multiplication we get the group table 2.1.

§ 60. Alternating Group Alt(n). Every permutation can be achieved by successive pairwise transposition of elements. This sequence is not unique, but for a given permutation the number of transpositions needed is either even or odd and a permutation is called *even* or *odd* respectively depending on the parity of the number of transpositions in its definition. The set of even permutations forms a subgroup of Sym(n), which is called the *alternating* group and denoted by Alt(n). The following is easy to show:

Theorem 2.12. *The alternating group* Alt(n) *is a normal subgroup of* Sym(n) *and* Alt$(n) = n!/2$.

The proof of the next theorem is left as an exercise, see Exercise 2.7.

Theorem 2.13. *If $n \neq 4$, the* Alt(n) *has no nontrivial normal subgroups. A_4 has the Klein 4-group as a normal subgroup.*

§ 61. Cyclic Group Cyc(n). We have seen that the powers g^m of any element g in a group G form a subgroup Cyc(k) of G and this subgroup is of order k, where k is the smallest positive power such that $g^k = e$. Cyc(k) is called *cyclic*.

For cyclic groups of finite order, every negative exponent can be replaced by a positive one, since $g^{-x} = g^{mk-x}$ for every integer m. For infinite cyclic groups this is not the case, but clearly the infinite set $\text{Cyc}(\infty) = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\}$ satisfies the group axioms, it is called the infinite cyclic group, or *free* group on one generator g, denoted by $\langle g \mid \ \rangle$. The powers which are multiples of a fixed integer m form a normal subgroup H of Cyc(∞) and the quotient $\text{Cyc}(\infty)/H = \langle g \mid g^m = e\rangle$ is the cyclic group of order m. It is isomorphic to Z_n with $\phi : a^k \to k$ as isomorphism.

Given a natural number n we can find a group of order n, namely for example the cyclic group of order n. To a given prime number p there is, up to isomorphism, exactly one group of order p, namely Z_p.

§ 62. Dihedral Group $\mathrm{Dih}(n)$**.** Let us consider the group of symmetries of a regular n-gon in the plane, which contains, together with rotations about its center through integer multiples about $2\pi/n$ the reflections about the symmetry axes and is therefore of order $2n$. It is called the *dihedral group* $\mathrm{Dih}(n)$. The subgroup of all rotations is of index 2 in $\mathrm{Dih}(n)$. It is a normal subgroup. A reflection together with the identity also forms a subgroup of $\mathrm{Dih}(n)$, but such a subgroup is not normal, it has n distinct conjugates.

§ 63. The Free Groups. The free group on two generators, $F(a,b) = \langle a, b \mid \; \rangle$ is defined as follows. Consider all *words*, i.e. concatenations of (positive and negative) powers of the generators. A word is *reduced* if contains no generator next to its inverse. The reduced words are the elements of $F(a,b)$, the unit element is the empty word and the group operation is concatenation of words followed by reduction of the concatenation. It is straightforward to verify that the associative law holds, see Exercise 2.9. This notion of free group on two generators can easily be generalized to any finite set S of generators.

§ 64. Group Presentation. Let F be a free group finitely generated by S and let $R = \{R_1, \dots R_k\}$ be a set of words of F. The formal products of conjugates of words in R form a normal subgroup of F. For the quotient we write $G = F/R = \langle S | R_i = 1, i = 1, \dots, k \rangle$, and say that G is a finitely presented group and the expression between $\langle$ and $\rangle$ is called a *presentation* for G. The elements of the set S are called *generators*, while the words R_i are caller *relators.*

§ 65. Generators. Let G be a finite group and let $S \subset G$ be a set of elements of G. By $\langle S \rangle = H$ we denote the smallest subgroup of G containing S. We say that H is generated by S. In practice this means that H is obtained by repeated multiplication of elements of S and their inverses. That is why we usually assume that $1 \notin S$ and $S = S^{-1}$. In other words, we may assume that S does not contain the group unit and is closed under inverse.

For finite groups we get a naive algorithm for constructing all subgroups. For each set S construct the group $\langle S \rangle$.

Example 2.14. *Let us determine all subgroups of the quaternion group of order 8. In addition to the two trivial ones, we get three isomorphic subgroups of order 4:* $\{1, \bar{1}, i, \bar{i}\}, \{1, \bar{1}, j, \bar{j}\}, \{1, \bar{1}, k, \bar{k}\}$*, that all share a subgroup of order 3:* $\{1, \bar{1}\}$*,*

§ 66. Direct Product of Groups. Given two groups $A = \langle S_A | R_A \rangle$ and $B = \langle S_B | R_B \rangle$, we define the *direct product* of A and B to be the group with the following presentation: $A \times B = \langle S_A \uplus B | R_A \uplus R_B, [s_a, s_b] \rangle$, where $[s_a, s_b] = s_a s_b s_a^{-1} s_b^{-1}$ is the *commutator*of s_a and s_b. The set of generators of $A \times B$ is the disjoint union of generators of A and B, the relations consist of the disjoint union of relations for A and those for B as well as all commutators between generators for A and B. For the special case of finitely generated Abelian groups, we mention the following structure theorem, whose proof can be found for example in [**?**]

Theorem 2.15. *Every finitely generated Abelian group is the direct product of cyclic groups.*

Example 2.16. *The Klein 4-group is isomorphic to the direct product* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.2 Cayley Graphs

Let Γ be a group with identity 1 and let S be a set of generators with the property that $x \in S$ implies $x^{-1} \in S$, and $1 \notin S$.

The graph $G = G(\Gamma, S)$ with vertex set $VG = \Gamma$ and two vertices $g, h \in VG$ adjacent if and only if $g^{-1}h \in S$ is called the *Cayley graph* of Γ and the generating set S.

Theorem 2.17. *The group* Γ *is a subgroup of the automorphism group of any of its Cayley graphs* $G(\Gamma, S)$*.*

Proof. Every $g \in \Gamma$ defines an automorphism of $G(\Gamma)$ by mapping each vertex v of $G(\Gamma)$ to the vertex gv, since an edge (v, vs) is mapped to an edge (gv, gvs) as required. □

There is a special case of interest in the above Theorem 2.17, namely the case when$\Gamma = \operatorname{Aut} G(\Gamma, S)$. In such a case we say that the graph is *GRR (Graphical Regular Representation)* for Γ.

Example 2.18. *The cycle C_n is a Cayley graph for the cyclic group* $\mathrm{Cyc}(n)$*. However, it is not its GRR, since $Aut\, C_n = \mathrm{Dih}(n)$. However, for even values of $n = 2m$ C_{2m} is GRR for* $\mathrm{Dih}(m)$*.*

Given Cayley graph $G = G(\Gamma, S)$ we may look for a suitable presentation for Γ, namely: $\Gamma = \langle S|R\rangle$ for some set of relators R. For instance, in the example above, C_n may be regarded as the Cayley graph associated with the presentation $\langle a|a^n = 1\rangle$ in the cyclic case or $\langle b, c|b^2 = c^2 = (bc)^m = 1\rangle$ for thew dihedral case.

Example 2.19. *Let H be the group of order 16 generated as follows:*

$$H = \langle a, b, r|a^2 = b^2 = r^4 = arbr^{-1} = 1\rangle$$

The corresponding Cayley graph is isomorphic to the hypercube Q_4 and is depicted in Figure 2.2

Theorem 2.20. *Cayley graph $G(\Gamma, S)$ is connected if and only if S does actually generate the whole group Γ.*

Proof. If S generates Γ, then every group element g can be written as word in the generators and such a word corresponds to a path in the Cayley graph from the unit element to g. □

In Chapter I we considered various graphs that are isomorphic to Cayley graphs. Here are some examples.

Example 2.21. *Let us consider the group $\mathbb{Z}_n \times \mathbb{Z}_2, n \geq 3$. It is generated by $S = \{(1,0), (-1,0), (0,1)\}$. The corresponding Cayley graph is isomorphic to the prism graph Π_n.*

Example 2.22. *In Chapter I we considered the truncated tetrahedron as one of the thirteen Archimedean graphs. It has 12 vertices and 4 disjoint triangles. Let r be an element of order 3 that generates each triangle: $r^3 = 1$. The remaining edges represent an involution $s^2 = 1$. If we make a consistent orientation of triangles, we may explain each hexagon as: $(rs)^3 = 1$. This completes the group presentation: $G = \langle r, s|r^3 = s^2 = (rs)^3 = 1\rangle$. We note in passing that G is isomorphic to* $\mathrm{Alt}(4)$*.*

§ 67. Stablizers and Orbits. Given a set X and a group G acting on it, let $x \in X$ and consider the set of all elements of G that map x to itself, $Fix(x) = \{g \in G | g(x) = x\}$. $Fix(x)$ is called the *stabilizer* if x, it is a subgroup of G. Recall that the orbit of x, denoted by $[x]$, is the set of all elements of X which are images of x under the action of G.

Proposition 2.23. *The index of $Fix(x)$ in G is the orbit size of x.*

Proof. $g_1 x = g_2 x$ exactly when $g_2^{-1} g_1 \in Fix(x)$, so we have $|[x]||Fix(x)| = |G|$. □

§ 68. Burnside's Lemma. For a given element $g \in G$ we denote by $Fix(g)$ the set of elements of X which are mapped to themselves by g: $Fix(g) = \{x \in X | g(x) = x\}$. The next theorem enables us to find the number of orbits induced on X by the action of G from the sizes of the sets $Fix(g)$. This theorem is usually called Burnside's lemma and its proof is an example of a standard counting technique in combinatorics which is to count the same set in two different ways to obtain an interesting result.

ADD REFERENCES!!!

Theorem 2.24 (Burnside's Lemma). *The number,t, of orbits of the action of of a group G on the set X is given by $t = 1/|G| \sum_{g \in G} |Fix(g)|$.*

Proof. The set of pairs (g, x), $g \in G$, $x \in X$, so that $g(x) = x$ can be counted by fixing g, calculating $|Fix(g)|$ and then summing over $g \in G$, or equivalently by first fixing x, calculating $|Fix(x)|$ and then summing over $x \in X$, so we have $\sum_{g \in G} |Fix(g)| = \sum_{x \in X} |Fix(x)|$. If $[x_1] = [x_2]$, then the stabilizers of x_1 and x_2 have the same size as they are conjugates. That means that for each element in the same orbit we get the same contribution on the right hand side, so each orbit contributes $|[x]||Fix(x)| = |G|$ and $\sum_{g \in G} |Fix(g)| = t|G|$. □

§ 69. The Escher Problem. As an application of Burnside's lemma we investigate how many different (infinite) patterns one can achieve using one square stamp A with an asymmetric pattern on it and first producing a square B by stamping the pattern of A on it 4 times in up to 4 orientations, and then tiling the plane with translates of B. Since there are 4 possible orientations of A and we can choose any one of them for the 4 spots on B, there are 4^4 different tiles B. But clearly several of these tiles will produce the same pattern on the plane. We call two of the resulting patterns identical if we can rotate or translate the plane so that the two patterns are identical. Clearly, C_4 acts on the set of tiles and we want to call all elements in an orbit of this action identical. We also see that if we tile the plane instead of B with a tile derived from B by switching top and bottom row, left and right column, or by switching pairs of opposite corners, we obtain the same pattern in each case. That means that the Klein 4–group also acts on the set of tiles. Let r denote a 90–degree counterclockwise rotation of a tile, a denote switching the columns and b denote switching rows of a tile. We note that $ar = rb$ and this relation enables us to study the group H generated by $\{a, b, r\}$, in particular we can, for each element h of H calculate $|Fix(h)|$. For example, $|Fix(ar)| = 0$, because the stamped symbol in the right upper corner of a tile in $|Fix(ar)|$ should be identical to its 90–degree rotated version, which is impossible if the stamp-pattern is asymmetric. $|Fix(ar^2)| = 16$ since the tile symbols get pairwise exchanged under ar^2. $|Fix(abr)| = 4$, since the abr permutes the symbols cyclically. Proceeding similarly for all elements of H, we get 6 fixed point sets of size 16, 4 of size 4, 5 empty ones, and of course $Fix(1) = H$. Burnside's lemma gives us $1/16(256 + 6.16 + 4.4) = 23$ for the number of orbits, a number that was computed already by the Dutch graphic artist M.C. Escher, see [68].

The Cayley graph of H is drawn in Figure 2.2.

According to the enumeration, there are 23 classes of patterns, i. e. orbits of H on the set of tiles, which are depicted in Figure 2.3. The figure shows 3×3 tiles which can be completed to the corresponding tiling of the plane by translating any of its 2×2 sub-squares.

2.3 Symmetry and Orbits in Graphs

Some polyhedra, such as Platonic and Archimedean polyhedra are highly symmetric. One measure of their symmetry is that each Archimedean solid has indistinguishable vertices. However, the edges of Archimedean polyhedra are not all alike. This phenomenon can be understood by the concept of *orbits.* We will explain this concept first in graphs.

Let $G = (V, E)$ be a graph with the vertex set V and edge set E. Recall that the permutation $\pi : V \to V$ is an *automorphism* of G if it preserves adjacencies: $(u, v) \in E$ if and only if $(\pi(u), \pi(v)) \in E$. Two vertices u and v of G are *indistinguishable* if one can be mapped to the other by some graph

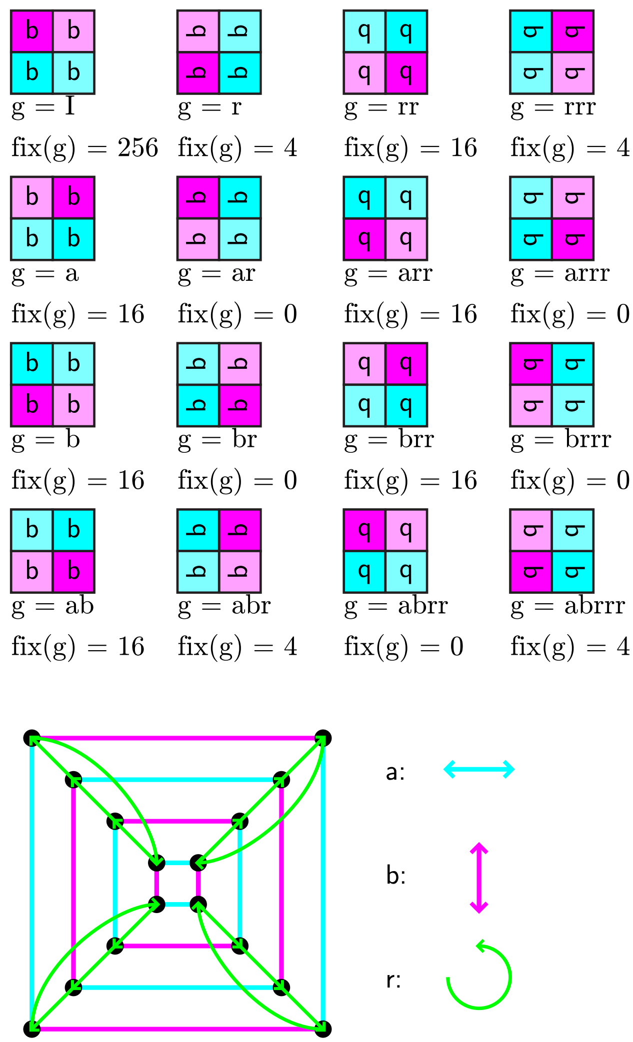


Figure 2.2: The group acting on the tiles.

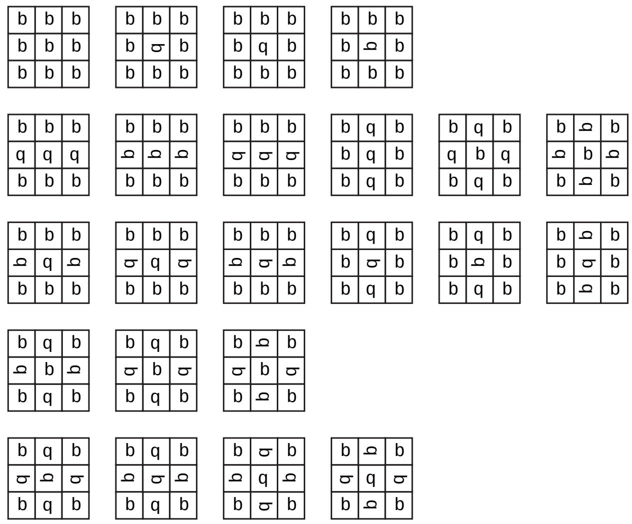


Figure 2.3: The 23 tiled stamp patterns.

automorphism. The set of indistinguishable vertices form a *vertex orbit* of $Aut(G)$.

§ 70. Vertex- and Edge-Transitive Graphs. If all vertices of G are indistinguishable under the action of the automorphism group, we say that G is *vertex-transitive.* Cayley graphs provide a nice class of vertex-transitive graphs.

Theorem 2.25. *Every (connected) Cayley graph* $G(\Gamma)$ *is vertex transitive.*

Proof. Given two vertices g and h of $G(\Gamma)$, there is an automorphism, namely the one induced by $g^{-1}h$ as in Theorem 2.17, mapping g to h, (provided $g^{-1}h$ is expressible as a product of generators,) so both g and h are in the same orbit under the action of the automorphism group. □

We may also view automorphisms of G acting on the edges of G. Analogously we can define *edge-orbits* and *edge-transitive* graphs.

The graph of a regular prism Π_n is obviously vertex-transitive but in general not edge-transitive. The automorphism group does not distinguish among the vertices, but one can tell apart lateral edges from the base edges unless the prism is a cube. For $n \neq 4$ the action of $\operatorname{Aut}\Pi_n$ on the vertex set $V(\Pi_n)$ is imprimitive.

This discussion may help clarify the distinction between abstract group and permutation group. The group of automorphism is a single abstract group but admits different actions. When acting on the set of vertices it forms a permutation group that is not the same as the permutation group acting on the edges.

§ 71. Arc-Transitive Graphs. There is another notion that we will use later; it is called arc-transitivity. A graph G is called *arc-transitive* if for any pair of edges $(u, v) \in E$, $(u', v') \in E$ one can find two automorphisms π and π' such that $\pi(u) = u', \pi(v) = v'$ and $\pi'(u) = v'$ and $\pi'(v) = u'$. Hence the notion of arc-transitivity is stronger than edge-transitivity. It means that we can map any edge to any other edge in an arbitrary direction. Clearly arc-transitivity implies edge-transitivity.

Theorem 2.26. *An arc-transitive graph is vertex-transitive.*

§ 72. s-arc transitivity. Edge transitivity is a refinement of vertex transitivity. By considering the action of the automorphism group on more complex substructures of the graph, we obtain further refinements. We follow the notation of Biggs [9] and define a *t-arc* $[\alpha]$ an a graph Γ to be a sequence $(\alpha_0, \alpha_1, ..., \alpha_t)$ of $t + 1$ vertices of Γ, such that $\alpha_{i-1,i}$ is adjacent to α_i, for $1 \leq i \leq t$ and $\alpha_{i-1} \neq \alpha_{i+1}$, for $1 \leq i < t$. We may identify 0-arcs with vertices and 1-arcs with arcs.

A graph Γ is *t-arc-transitive* if its automorphism group acts transitively on the set of t-arcs and is not transitive on the set of $t + 1$-arcs.

For cubic graphs it is quite easy to determine their arc transitivity. The two ways in which an s-arc may be extended to an $s + 1$ arc are essentially different. The tetrahedron, for example, is 2-arc transitive because adding an edge to a path of length 2, we can either complete a cycle or get a path of length 3.

The cube is also 2-arc transitive. Here a path of length 2 may be extended to a path of length 3 such that its end vertices are either adjacent or not adjacent. The Petersen graph is 3-arc transitive.

From the preceding examples it should be transparent that arc transitivity and girth of the graph are related.

Theorem 2.27. *([77]) Let b be a k-regular graph, $k \geq 3$ which is also s-arc transitive and of girth g. Then $s \leq g/2 + 1$.*

Proof. Consider a cycle C of length g. Since the degree of each vertex is at least 3, there is a vertex not on c, but adjacent to a vertex of C, so there is a path of length g whose endpoints are not incident, i.e. $s < g$. □

Since g is s-arc transitive, and C contains s-arcs, every s-arc must be in a cycle of length g. Moreover, each $(s-1)$ arc may be extended to an s-arc in at least two distinct ways, yielding two 3-arcs intersecting in an $(s-1)$ arc, so there are two circuits of length g intersecting $(s-1)$ edges. Their symmetric difference contains another circuit of length at least g, so

$$2g - 2(s-1) \geq g \text{ or } s \leq g/2 + 1$$

§ 73. 1/2-arc-transitivity. Furthermore, a graph Γ is 1/2-*arc-transitive* if it is 0-arc-transitive (= vertex-transitive), edge-transitive but not 1-arc-transitive.

Theorem 2.28. *A 1/2-arc-transitive graph G is regular of even valence.*

Proof. Regularity of G follows from vertex transitivity. Let $G = (V, S, i, r)$ with $E(G)$ equal to the orbits of r. If there is an automorphism $\alpha \in Aut(G)$ with $\alpha(s) = x$ and an automorphism $\beta \in Aut(G)$ with $\beta(s) = r(x)$, then $\gamma = \alpha\beta^{-1}$ fixes the edge $s, r(s)$ as a set, but interchanges its endpoints. Edge transitivity of the automorphism group means that $(s, r(s))$ may be mapped to any other edge e and following that automorphism by γ also to $r(e)$, so $Aut(G)$ is arc-transitive, contrary to our assumptions. We conclude that a given semi-edge s sweeps out exactly half of all semi-edges under the action of the automorphism group. The number $|\alpha \mid i(s) = i(\alpha(s))|$ is the same for each vertex by vertex transitivity, so by the handshaking lemma this number is exactly half of the vertex valence, which means that the vertex valence must be even. □

§ 74. Regular Actions. If the action of the automorphism group produces only one orbit and is moreover of the same size as the set it acts upon, the action is called *regular*. A graph is 1-*regular* if the automorphism group acts regularly on the set of arcs.

Sabidussi characterized Cayley graph in terms of regular actions.

Theorem 2.29. *Graph G is a Cayley graph for some group Γ if and only if there is a subgroup Γ_0 in Γ acting regularly on $V(G)$.*

By the well known result of Tutte [77], a cubic arc-transitive graph is at most 5-arc-transitive, with the degree of transitivity having a reflection in the corresponding vertex stabilizers. For example if a cubic graph is 1-arc-transitive (but not 2-arc-transitive), then the corresponding vertex stabilizers are isomorphic to a cyclic group of order 3. In this case the automorphism group is regular on the set of 1-arcs, and the graph is 1-regular. Such graphs are of particular interest to us for their line graphsare tetravalent $\frac{1}{2}$-arc-transitive graphs as is seen by the following result. Recall that the vertices of the line graph $L(X)$ of a graph X are the edges of X, with adjacency corresponding to the incidence of edges in X.

Proposition 2.30. ([55, Proposition 1.1]) *A cubic graph is 1-regular if and only if its line graph is a tetravalent $\frac{1}{2}$-arc-transitive graph.*

§ 75. Zero-symmetric Graphs. Cubic vertex-transitive graphs fall naturally into three classes depending on the number of edge-orbits of the corresponding automorphism group. In case of three orbits, the graphs are called *0-symmetric* (see [19, **?**]). At the other extreme, if there is only one orbit, then the graph is arc-transitive. This follows from the fact that vertex- and edge-transitive graphs of odd valency are necessarily arc-transitive by Theorem **??**, see also [76].

Finally, let us mention another class of graphs that has been studied in the past. It was defined for cubic graphs, although there is no particular reason why it should not be defined for all (regular) graphs.

A vertex-transitive graph is obviously regular, say, of valence d. A d-valent graph is *zero-symmetric* if the automorphism group partitions the edge set into maximum number of d orbits; compare [19] where this notion is defined for cubic graphs only.

The term *zero-symmetric graph* has been used for a cubic graph which is a GRR. The characterization of zero-symmetric Cayley graphs of dihedral groups, was given by Foster and Powers [19]. In 1974 Watkins found the first zero-symmetric Cayley graph of a dihedral group whose generating set consists of three irredundant involutions (see [19]).

§ 76. Semisymmetric graphs There are graphs that have all edges indistinguishable but do not have all the vertices alike. The simplest family of this type is the family of complete bipartite graphs $K_{n,m}, n \neq m$. It is much more difficult to find regular graphs with this property. That is why they

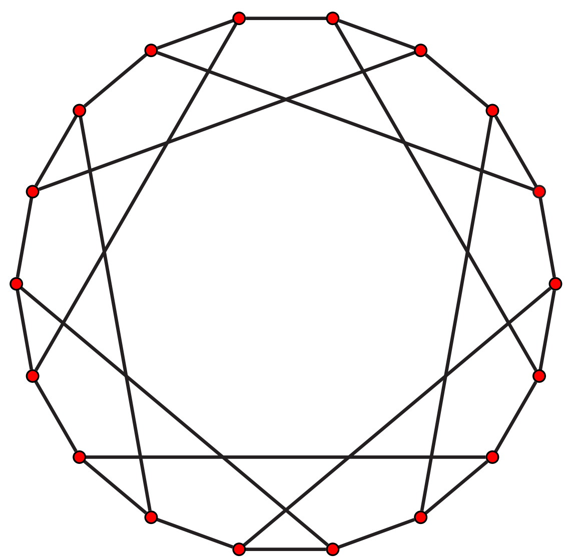


Figure 2.4: The smallest 0-symmetric graph has 18 vertices.

deserve a special name. Regular edge-transitive but not vertex-transitive graphs are called *semi-symmetric* graphs.

Proposition 2.31. *Each vertex-transitive graph is regular.*

Theorem 2.32. *If graph is edge-transitive but not vertex-transitive then it is bipartite.*

One of the semi-symmetric graphs is the so-called Folkman graph. The *Folkman graph* on 20 vertices is 4-valent and is therefore edge-transitive but not vertex-transitive. See Figure 2.5.

§ 77. The Gray Graph. The smallest known cubic edge- but not vertex-transitive graph has 54 vertices and is known as the *Gray graph*, denoted hereafter by $\mathcal{G}$. The first published account on the Gray graph is due to Bouwer [13] who mentioned that this graph had in fact been discovered by Marion C. Gray in 1932, thus explaining its name. Bouwer [13] gives two ways

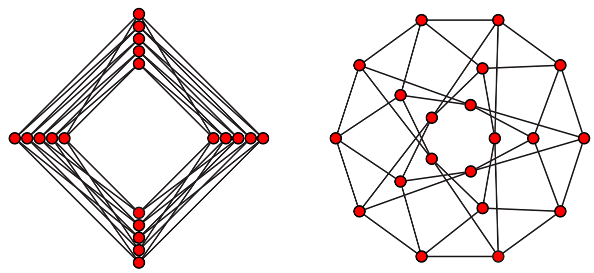


Figure 2.5: The standard drawing of the Folkman graph and another drawing of the same graph.

of constructing $\mathcal{G}$. First, three copies of the complete bipartite graph $K_{3,3}$ are taken, and to a particular edge e of $K_{3,3}$ a vertex is inserted in the interior of e in each of the three copies of $K_{3,3}$, and the resulting three vertices are then joined to a new vertex. The second construction identifies a particular Hamilton cycle in $\mathcal{G}$ and the corresponding 27 chords (see Figure 2.6).

The Gray graph $\mathcal{G}$ is a cubic, bipartite, and edge- but not vertex-transitive graph. Its automorphism group $\operatorname{Aut}\mathcal{G}$ acts transitively on each of the bipartition sets.

There is a simple reason for intransitivity of $\operatorname{Aut}\mathcal{G}$ on the vertex set of $\mathcal{G}$. Two vertices have the same distance sequence if and only if they belong to the same bipartition set.

§ 78. Symmetry of generalized Petersen Graphs. It is interesting to consider symmetry of generalized Petersen graphs. Here is a theorem of Frucht, Graver and Watkins [31].

Theorem 2.33. *A generalized Petersen graph $G(n, k)$ is vertex-transitive if and only if $(n, k) = (10, 2)$ or*

$$k^2 \equiv \pm 1 \pmod{n}.$$

In 1995 Nedela and Škoviera refined this theorem as follows.

Theorem 2.34. *A generalized Petersen graph $G(n, k)$ is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$.*

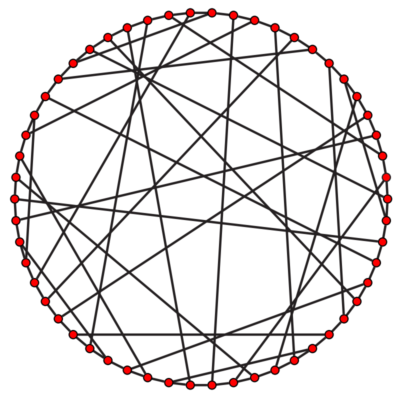


Figure 2.6: The Gray Graph with an identified Hamilton cycle as in [13].

It is perhaps interesting to note that only 7 among the generalized Petersen graphs are arc-transitive.

Theorem 2.35. *A generalized Petersen graph $G(n, k)$ is arc-transitive if and only if*

$$(n, k) \in \{(4, 1), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}.$$

We obviously have the following consequence.

Corollary 2.36. *A generalized Petersen graph $G(n, k)$ is arc-transitive Cayley graph if and only if* $(n, k) \in \{(4, 1), (8, 3), (12, 5), (24, 5)\}$.

2.4 Voltage Graphs and Covering Graphs.

§ 79. Pregraphs Revisited A *pregraph* G is is quadruple $G = (V, S, i, r)$ where V is the set of vertices, S is the set of arcs (or semi-edges, darts, sides, ...), i is a mapping $i : S \to V$, specifying the origin or *initial vertex* for each arc, while r is the *reversal involution*: $r : S \to S, r^2 = 1$. We may also define $t : S \to V$ as $t(s) := i(r(s))$, specifying the terminal vertex for each arc. An arc s with $r(s) \neq s$ forms an *edge* $e = \{s, r(s)\}$, which is called *proper* if $|e| = 2$ and is called a *half-edge* if $|e| = 1$. Define $\partial(e) = \{i(s), t(s)\}$. A pregraph without half-edges is called *a (general) graph.* Note that G is a graph if and only if the involution r has no fixed points. A proper edge e with $|\partial(e)| = 1$ is called a *loop* and two edges e, e' are *parallel* if $\partial(e) = \partial(e')$. A graph without loops and parallel edges is called *simple*. The *valence* of a vertex v is defined as $val(v) = |\{s \in S | i(s) = v\}|$. A pregraph of valence 3 is called *cubic*.

§ 80. Pregraphs on a Single Vertex. There is obivously a single pregraph $K_1 = B(0;0)$ of valence 0 on a single vertex u. Since it contains isolated vertices it is of lesser importance and will not be discussed and further. There is also a single pregraph $B(0;1)$ of valence 1. Both graphs can be seen in Figure 2.9. $B(0;1)$ can be described as follows:

$B(0;1)$	a
i	u
r	a

Figure 2.7: $K_1 = B(0;0)$ and $B(0;1)$

There are two pregraphs of valence two on a single vertex, $B(1;0)$ with a loop and $B(0;2)$ one with two half-edges.

$B(1;0)$	b	c
i	u	u
r	c	c

$B(0;2)$	b	c
i	u	u
r	b	c

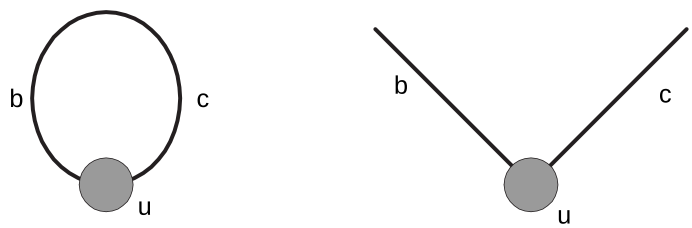


Figure 2.8: $B(1;0)$ and $B(0;2)$

There are two cubic pregraphs on a single vertex $B(1;1)$ and $B(0;3)$.

$B(1;1)$	a	b	c
i	u	u	u
r	a	c	b

$B(0;3)$	a	b	c
i	u	u	u
r	a	b	c

We may define a family of generalized bouquets of circles: $B(k;p)$ of valence $2k+p$ having k circles and p half-edges and a single vertex. In general there are $1 + \lfloor d/2 \rfloor$ single vertex d-valent pregraphs.

§ 81. Voltage Graphs and Regular Coverings. Now we shall briefly introduce voltage graphs. Voltage graphs are obtained from pregraphs by assigning group elements to arcs. More precisely, a *voltage graph* X is a 6-tuple $X = (V, S, i, r, \Gamma, \alpha)$ where (V, S, i, r) is the underlying pregraph, Γ is a group and α is a mapping $\alpha : S \to \Gamma$) satisfying the following axiom:

For each $s \in S$ we have $\alpha(r(s)) = \alpha^{-1}(s)$.

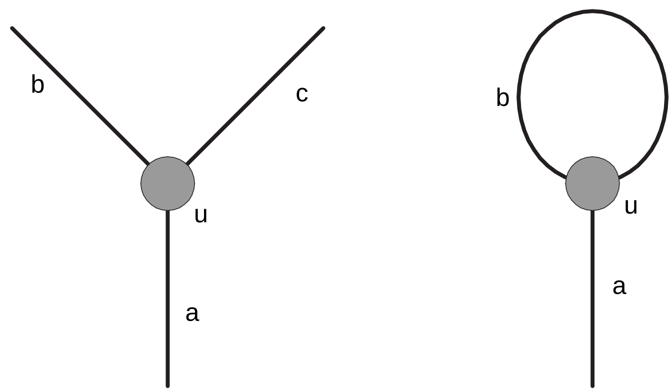


Figure 2.9: $B(0;3)$ and $B(1;1)$

Any voltage graph X defines the so-called *derived graph* or *regular covering graph* Y as follows:

$$V(Y) := V \times \Gamma$$

$$S(Y) := S \times \Gamma$$

$$i(s,\gamma) := (i(s),\gamma), \text{for any}(s,\gamma) \in S(Y)$$

$$r(s,\gamma) := (r(s),\alpha(s) \circ \gamma), \text{for any}(s,\gamma) \in S(Y)$$

If $s \in S$ is a half-edge, i.e. if $r(s) = s$ its voltage $\alpha(s)$ is of order at most two: $\alpha(s) = \alpha^{-1}(s)$. Hence $\alpha = \alpha^{-1}$ or equivalently, $\alpha^2 = 1$. In the case when the group Γ is cyclic $\Gamma = \mathbb{Z}_n$, the only voltages that can be assigned to a half-edge are 0 and $n/2$. A voltage 0 on a half-edge implies that the derived graph remains a pregraph having half-edges. If we are interested in simple covering graphs then the only admissible voltage for a half-edge remains $n/2$; furthermore n must be even.

Note that voltage graphs enable us to develop a combinatorial analog of the well-known theory of covering spaces in algebraic topology. The reader may find more on the theory of voltage graphs and covering graphs in [10, **?**, 61]. We note in passing that there is a convention in drawing voltage graphs. Usually the voltage on only one of the pair of reversing arcs is specified. The choice of the arc is denoted by placing an appropriate arrow. When the voltage is an involution (in our case a voltage from $\{0, n/2\}$) no arrow is drawn.

§ 82. The Handcuff Graph. The *handcuff graph* is not simple. It has two loops attached to the endpoints of a single edge. If we relax a bit the conditions for the parameters of a generalized Petersen graph one may describe the handcuff graph as the generalized Petersen graph $G(1,1)$. We may also describe it by the following table.

$G(1,1)$	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_3	s_2	s_1	s_6	s_5

The handcuff graph $G(1,1)$ is a twofold regular covering over $B(1;1)$. In general there are four possible $\mathbb{Z}_2$-voltage assignments over $B(1;1)$.

The trivial voltage assignment

$B(1;1) - - - \mathbb{Z}_2$	a	b	c
i	u	u	u
r	a	c	b
α	0	0	0

gives rise to $2B(1;1)$.

The voltage assignment that assigns 1 to each arc

$B(1;1) - - - \mathbb{Z}_2$	a	b	c
i	u	u	u
r	a	c	b
α	1	1	1

gives rise to the so-called canonical double cover or Kronecker cover. In this case, the canonical double cover is the theta graph θ_3, known also as the cubic dipole.

The voltage assignment that assigns 0 to the half-edge and 1 to the loop

$B(1;1) - - - \mathbb{Z}_2$	a	b	c
i	u	u	u
r	a	c	b
α	0	1	1

gives rise to a two-valent theta graph θ_2, augmented to a cubic pregraph by attaching a half-edge at each vertex.

Finally, the voltage assignment that assigns 1 to the half-edge and 0 to the loop

$B(1;1) - - - \mathbb{Z}_2$	a	b	c
i	u	u	u
r	a	c	b
α	1	0	0

gives rise to $G(1,1)$.

§ 83. Generalized Petersen Graphs as Coverings over $G(1,1)$. By assigning voltages from $\mathbb{Z}_n$ to $G(1,1)$ as follows

$G(1,1) - - - \mathbb{Z}_n$	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_3	s_2	s_1	s_6	s_5
α	0	1	-1	0	r	$-r$

the derived graph is isomorphic to the generalized Petersen graph $G(n,r)$. The construction defines a graph mapping from $G(n,r)$ to $G(1,1)$. The mapping is called the *covering projection* and is a local isomorphism.

The outer circle is projected to one loop, the inner star-polygon is projected to the second loop and the rims are projected to the edge between the loops.

There is a general rule: in any voltage graph one may change the direction of any arrow and replace the corresponding voltage by its inverse without changing the covering graph. In particular, this means one can always choose p^{-1} instead of p on any loop. This gives a short argument why $G(n,r)$ and $G(n,n-r)$ are isomorphic. In particular, $G(5,3)$ is the same graph as $G(5,2)$.

§ 84. Group Actions and Permutation Groups Revisited. Originally we defined a permutation group in a seemingly simplistic way as a subgroup of $\mathrm{Sym}(n)$. Here we re-define it in a slightly more general way. The triple (Γ, A, h) is called a group action if Γ is an abstract group, A is a set and $h : \Gamma \to \mathrm{Sym}(A)$ is a group homomorphism. We say that group Γ acts on the set A by group action h. We are only interested in the case when h is an isomorphism from Γ to $h(\Gamma) \leq \mathrm{Sym}(A)$. In such a case we say that the group action is *faithful* and we may identify Γ with the group of permutations $h(\Gamma)$. The group action $\Gamma, \Gamma, r)$ defined by $r(\gamma)(\alpha) := \gamma \circ \alpha$ is faithful and is called *right multiplication*, while $\Gamma, \Gamma, l)$ defined by $l(\gamma)(\alpha) := \alpha \circ \gamma$ is also faithful and is called *left multiplication*.

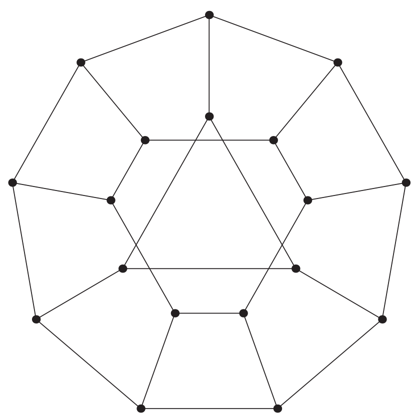


Figure 2.10: Generalized generalized Petersen graph $G(9, [4, 3, 5, 7, 6, 8, 1, 9, 2])$.

§ 85. Permutation Voltage Assignments and Ordinary Coverings. The theory of covering graphs can be extended from regular coverings to ordinary coverings. For this purpose we modify the definition of voltage assignment and the definition of the derived graph. Instead of voltage group Γ we use voltage permutation group (Γ, A, h), i.e. the group Γ acts on the set A. The (permutation) derived graph is a $|A|$-fold covering over the base graph.

A permutation voltage graph $(X, \Gamma, A, h, \alpha)$ is a pregraph X, together with a group action $\Gamma, A, h)$ and a mapping $\alpha : S(X) \to \Gamma$ such that $\alpha(r(s)) = \alpha^{-1}(s)$ for any $s \in S(X)$. Any permutation voltage graph defines the so-called *permutation derived graph* or *ordinary covering graph* Y as follows:

$$V(Y) := V \times A$$
$$S(Y) := S \times A$$
$$i(s, a) := (i(s), a), \text{for any}(s, a) \in S(Y)$$
$$r(s, a) := (r(s), h(\alpha(s))(a)), \text{for any}(s, a) \in S(Y)$$

Since we are considering only faithful actions we may replace Γ by $h(\Gamma)$. If needed we may always choose A to be the set $\{1, 2, \cdots, n\}$. On the other hand if we consider right action our ordinary covering graphs become regular covering graphs.

Example 2.37. *Let*

$$id = [1,2,3,4,5,6,7,8,9], \delta = [2,3,4,5,6,7,8,9,1], \pi = [4,3,5,7,6,8,1,9,2]$$

be permutations of 9 *elements written in the positional form. The permutation voltage assignment on the handcuff graph given here*

$G(1,1)---9$	s_1	s_2	s_3	s_4	s_5	s_6
i	u	u	u	v	v	v
r	s_4	s_3	s_2	s_1	s_6	s_5
α	id	δ	δ^{-1}	id	π	π^{-1}

has the permutation derived graph isomorphic to generalized generalized Petersen graph $G(9,\pi)$ *depicted in Figure 2.10.*

The assignment of the permutations on the edges of the base graph is called the *permutation voltage assignment*. Now we will show how to construct the permutation derived graph. Above each vertex of the handcuff graph we place five vertices one on each layer. Above each edge of the handcuff graph we place five edges. However, the edge that was directed from a to b and having voltage p runs from the vertex above a on layer i to the vertex above b on layer $p(i)$. The resulting graph is clearly the Petersen graph. In this case we have a shortcut. Instead of permutations we can assign group elements from Z_5. First loop gets 1, second loop 3 and the edge 0. This is called the *ordinary voltage assignment*.

Now we label the layers by group elements and perform the operation again, getting the same result. The *voltage graph* is also called the *base graph* and the derived graph is called the *covering graph*. The terminology follows the one of covering spaces in algebraic topology. The reader is referred to the book by Gross and Tucker [43] for further information about graph coverings. The Petersen graph obtained in this way can be labeled as $G(5,3)$. This means that we are working in the group Z_5 and that the voltage on the second loop is 3. Note that this process can be obviously generalized to the generalized Petersen graphs $G(n,r)$ with group Z_n and voltage r.

If the covering graph is trivalent then the base graph must be trivalent as well. It may have loops and parallel edges. Even though we assigned voltages to directed edges, we get the same result if we change the direction at an edge. We only have to assign the inverse voltage to the reversely directed edge. We may also have to allow *halfedges*, sometimes called semiedges, in the base graph. The only condition is the half-edge must have an involutory voltage.

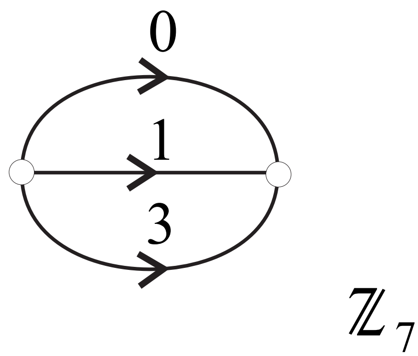


Figure 2.11: The theta graph, equipped with voltages from Z_7 for the 6-cage.

§ 86. Dipoles or Theta Graphs. A *dipole* is a general graph on two vertices with k parallel edges between them. The so-called *theta graph* Θ, is a dipole with three parallel edges. In general we will denote by Θ_k a dipole with k parallel edges.

§ 87. Cages as Covering Graphs. The *2-cage* is the *theta graph* Θ. As we know, we can describe it as a 2-cover over $B(1;1)$.

The *3-cage* K_4 is a $\mathbb{Z}_4$-covering graph over $B(1;1)$. The loop gets voltage 1 and the half-edge gets the voltage 2.

The *4-cage* $K_{3,3}$ is the 3-fold covering graph over the theta graph Θ (with the voltages 0,1,2 in $\mathbb{Z}_3$.

The *5-cage* As we already know, $G(5,2)$ is a covering graph over $G(1,1)$.

Taking the theta graph and the voltages 0, 1, and 3 in $\mathbb{Z}_4$ we get the cube graph Q_3, but the same voltages in $\mathbb{Z}_7$ (see Figure 2.11) define the *6-cage*, i.e. the Heawood graph.

The *7-cage* is an 8-fold covering graph over the voltage graph on 3 vertices; see Figure 2.12.

The *8-cage* can be represented as a 5-fold covering graph over a graph on 6 vertices; see Figure 2.13.

§ 88. The Kronecker Cover. We should mention a special construction that is called the *Kronecker double cover*. For an arbitrary graph X we may assign to each edge voltage 1, the non-trivial element of the two-element group $\mathbb{Z}_2$. This defines a canonical double cover graph, $X(2)$ also known as

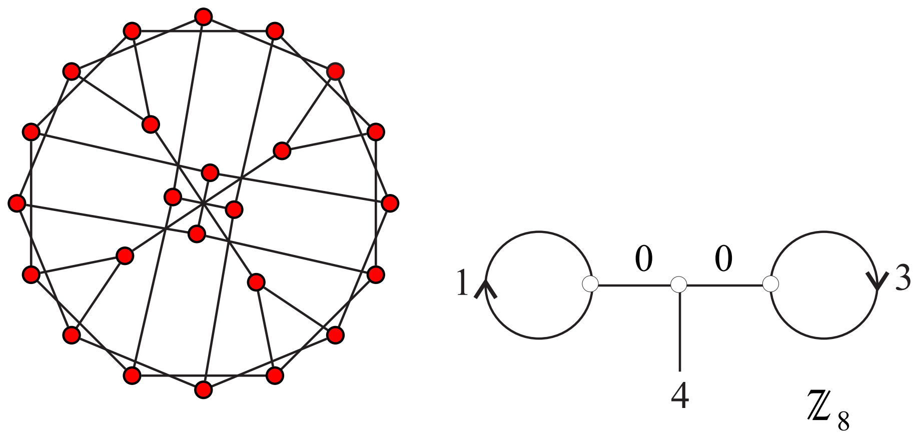


Figure 2.12: The 7-cage with its voltage graph. This is also known as the McGee graph.

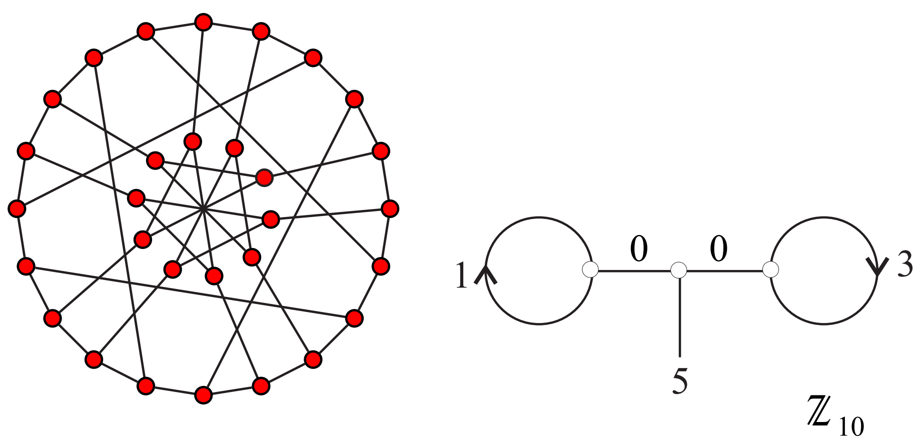


Figure 2.13: The 8-cage with its voltage graph. This is also known as the Cremona-Richmond graph.

the tensor product of X by K_2: $X(2) = X \times K_2$. The Table **??** gives some examples.

Recall that $G(4,1)$ is the cube Q_3. It turns out that $G(8,3)$, $G(12,5)$ and $G(24,5)$ are its covers, [27]. On the other hand, $G(5,2)$ is the Petersen graph whose canonical (Kronecker) double cover is $G(10,3)$, as clearly seen on Figures 4.17 and 4.18.

Obviously only those base graphs that are not bipartite are of interest here.

§ 89. Rotagraphs. The simplest voltage graph G has the voltage group Z_n and has only two types of voltage labels: the trivial 0 and the non-trivial 1. In this case the covering graph is called a *rotagraph*. The subgraph M of G composed of those edges of G with trivial voltages is called a *monograph*; see [63] for more information on rotagraphs.

Many well-known graphs can be described as rotagraphs, for example the prisms and antiprisms. However, the Petersen graph is not a rotagraph although it has a rotational symmetry. In this case we need jumps of size one as well as jumps of size 2. Hence, we will call the covering graph obtained by the cyclic group Z_n a generalized rotagraph. The main idea about generalized rotagraphs is that we may easily keep the voltages and vary n. So we always get not only one graph but an infinite family of graphs of the same type. Viewing the Petersen graph as a generalized rotagraph, we immediately get the whole family $G(n,2)$.

§ 90. General Cages. The notion of cages can be generalized to other regular graphs. A (d,g)-*cage* is the smallest regular d-valent graph with girth g.

In Figures 1.9,2.14,2.15, and 2.16 are given some known examples, perhaps drawn in a new way.

Recently Markus Meringer showed by computer that there are exactly 4 $(5,5)$-cages. His computer program `genreg`, which can generate all connected k-regular graphs with a given minimum girth, found the fourth $(5,5)$-cage and proved that there are no other ones, [56].

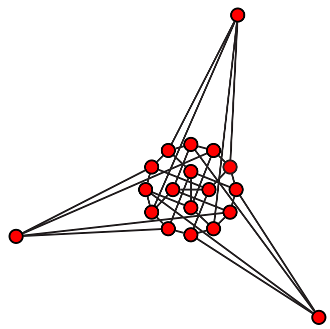


Figure 2.14: The (4,5)-cage.

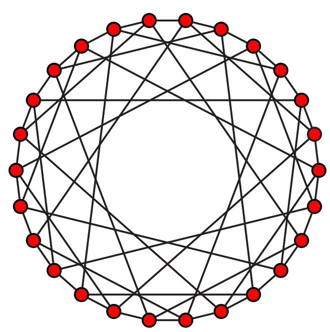


Figure 2.15: The (4,6)-cage.

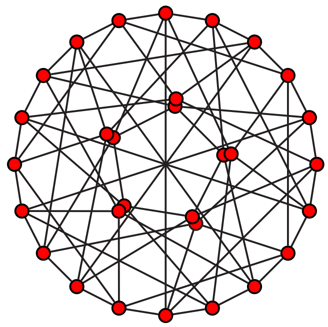


Figure 2.16: The fourth (5,5)-cage.

2.5 Maps

§ 91. Polyhedra and Graphs. There is an easy way of getting a graph from a polyhedron, namely by taking the one-skeleton, i.e. the graph composed of vertices and edges of the polyhedron. We keep the vertices and the edges and forget all the other information (e.g. the facial structure or metric structure.) This route is quite interesting and gives among other things a very important family of graphs. A graph G is *3-connected* if deletion of any pair of vertices results in a connected subgraph. It is planar if it can be drawn in the plane without crossings. Let us call a graph *polyhedral* if it is a one-skeleton of a convex polyhedron.

Theorem 2.38. *(Steinitz [83].) The one-skeleton of an arbitrary convex polyhedron is a planar 3-connected graph and each planar 3-connected graph is polyhedral.*

Example 2.39. *The one-skeleton of the octahedron is the complete tripartite graph $K_{2,2,2}$ on 6 vertices. One way of drawing this graph is to take the so-called Schlegel diagram of the octahedron. A* Schlegel diagram *of a convex polyhedron is obtained by first selecting a face and then using a stereographic projection from the center of that face onto a plane. Thus the selected face becomes the infinite face of the plane graph, which is the Schlegel diagram. For aesthetic reasons the resulting drawing is then homeomorphically changed in such a way that the faces are not accumulated too much in the center. However, we should always bear in mind that the graph G does not carry*

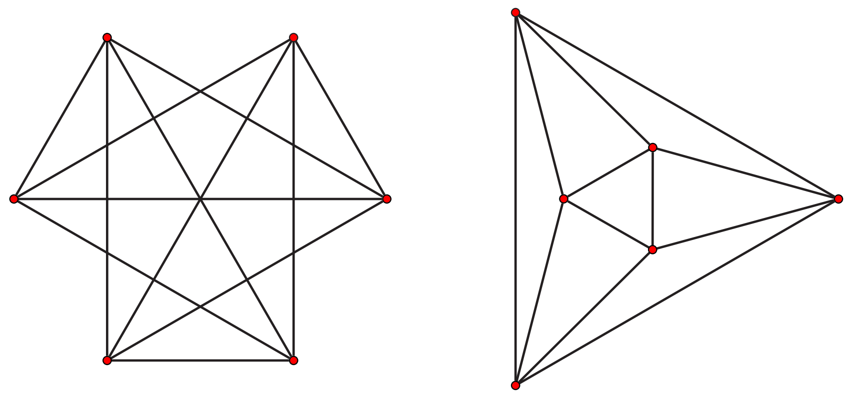


Figure 2.17: Two alternative drawings of $K_{2,2,2}$ (the graph of the octahedron).

explicit information about the position of its vertices. We can represent the same graph by different drawings; see Figure 2.17.

The route back from a graph to a polyhedron is not so obvious. Recovering hidden or missing information is never as easy and obvious as throwing away information.

If we allow non-convex polyhedra we may get different polyhedra giving the same one-skeleton. The *uniform polyhedron* of Figure 2.18 is a model of a projective plane. In [81] it is called a *heptahedron*. However, it is better known as a *tetrahemihexahedron*; see [53], where one can find more information about and illustrations of uniform polyhedra. It has 6 vertices, 12 edges, 4 triangles and 3 squares. Its one-skeleton is again $K_{2,2,2}$; see Figure 2.17.

Figure 2.20 shows its embedding in the (topological) projective plane. The antipodal points of the disk are identified.

Here we are interested only in topological properites of the usual real projective plane. The well-known classification of closed surfaces namely states that each closed surface, ie. each compact surface without boundary is homeomorphic to one member of two infinite families of pairwise non-homeomorhpic surfaces, namely the orientable surfaces $S_g, g = 0, 1, 2, ...$ and the non-orientable surfaces $N_k, k = 1, 2, 3, ...$ The parameter g is called the *genus* and k is called the *non-orientable genus* or *the crosscap number.*

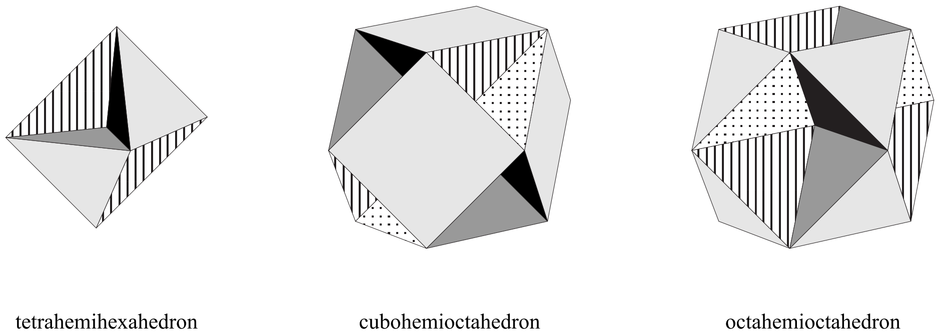


Figure 2.18: The heptahedron or tetrahemihexahedron is not convex. It is self-intersecting and even non-orientable as a map in the projective plane. Its one-skeleton is the graph on Figure 2.17. Octahemioctahedron and cubohemioctahedron share with cuboctahedron the same one-skeleton shown on Figure 2.45.

The simplest non-orientable surface N_1 is homeomorphic to the standard real projective plane endowed with the obvious topology. That is why it is called the *topological projective plane.* The simplest model of a topological projective plane is obtained from a disk by pairwise identifying antipodal points on th disk boundary.

If we start with a *cuboctahedron*, we can obtain two different polyhedra depending on what type of faces are replaced by hexagons.

By keeping all triangles and replacing the quadrilaterals by main hexagons, we get an *octahemioctahedron* whose map on torus is shown in Figure 2.21.

Our interest in the torus is again purely topological. The simplest orientable surface of genus 0 is the sphere S_0 while the next non-trivial surface of S_1 genus 1 is the *torus.* The standard model for the torus is a quadrilateral whose antipodal sides are pairwise identified in the natural way. If we change the direction on one side at the identification the resulting surface is the non-orientable N_2, the *Klein bottle.* On the other hand one may keep quadrilaterals and remove triangles. The polyhedron is called a *cubohemioctahedron* [53]. It consists of 6 quadrilaterals and 4 hexagons.

By keeping information about faces, we get the so-called *2-skeleton* of a polyhedron. It has vertices, edges, and faces and all the information about how to glue the pieces together.

Informally, a *map* is a collection of fused polygons. More rigorously, a *map*

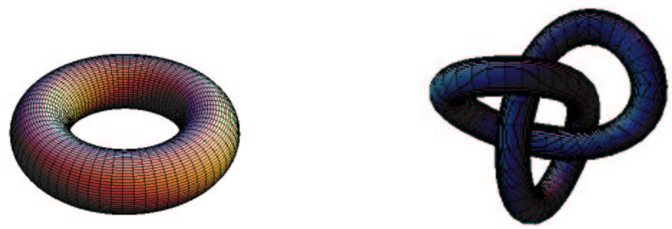


Figure 2.19: Topological torus may have different geometric realizations.

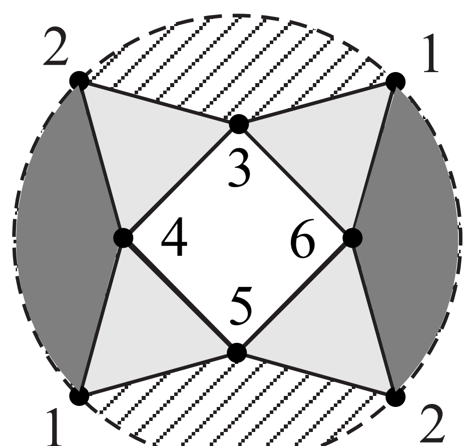


Figure 2.20: $K_{2,2,2}$ embedded in the projective plane with 4 triangles and 3 squares. The antipodal points on the dotted border are identified.

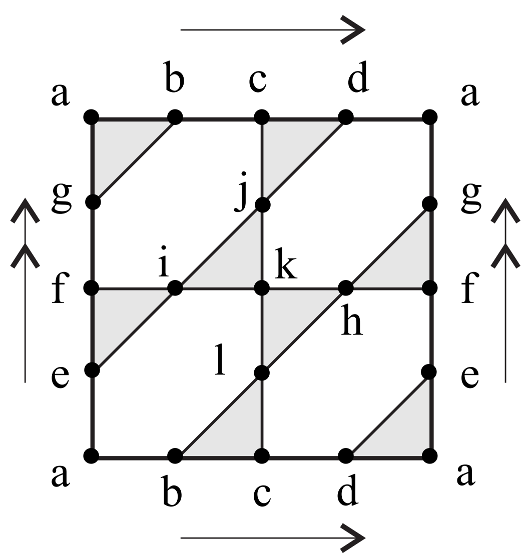
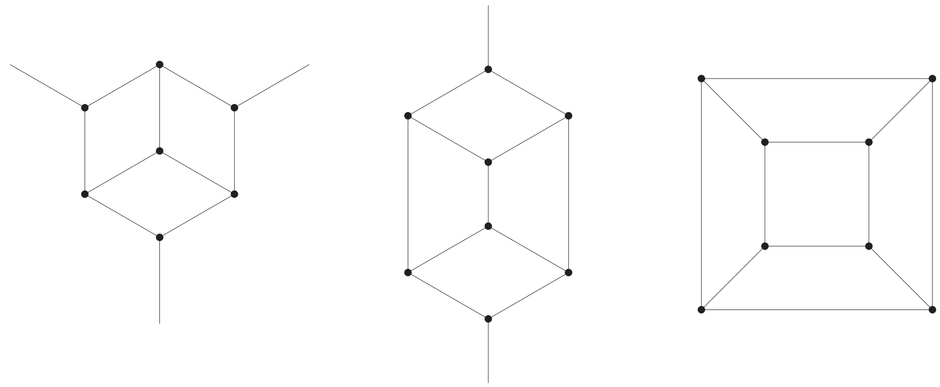


Figure 2.21: The map of octahemioctahedron in torus with 8 triangles and 4 hexagons. The arrows and double arrows show how to identify the sides of a rectangle in order to form the torus.



(a) (b) (c)

Figure 2.22: Schlegel diagrams of cube (a) centered at a vertex, (b) centered at an edge, (c) centered at a face.

is a collection of polygons with directed sides such that the total number of sides is even together with an involution on the set of sides without fixed points which determines how the sides are pairwise glued (respecting the orientation.) If the resulting complex is connected, we get a surface. (For exact definition and more examples, see the book by Ringel [67]; see also [82].)

A projection of a sphere-like polyhedron on a plane is sometimes called a *Schlegel diagram.* Such a projection is similar to the so-called stereographic projection in which exaclty one point on a sphere, called the *center*, is mapped to infinity. In a polyhedron the center is usually taken either at a vertex, at the center of an edge or at a center of a face, see Figure 2.22.

§ 92. Topological Approach to Maps. Now we turn to maps that can be regarded as topological polyhedra where all incidence properties are retained while metric properties are ignored. In topology maps are equivalent to cellular embedding of graphs in surfaces. A surface, in turn, is a Hausdorff space in which each point contains a neighborhood homeomorphic to open disk. It is a special case of manifold. However, our approach to maps and surfaces will be purely combinatorial. We want to reassure the reader that the combinatorial approach works nicely in this case. Things become much more involved if higher-dimensional manifolds are considered.

The combinatorial approach to maps and surfaces was started by Levi, continued by Ringel and culminated by Tutte. Several independent axiomatizations were proposed by various researchers. We follow the approach by Tutte and show that it is equivalent to the one, found, for instance in Godsil and Royle.

§ 93. Flags. The idea behind flags is simple. Suppose we want to embed G on some surface, say draw it on a piece of paper representing the edges by line segments (or curves) and the vertices by points, imposing the additional requirement that the edges only intersect at their endpoints. Such a drawing, if it can be produced, contains information not contained in the graph. Each edge drawn on the surface has a left and a right side, where left and right depend on the direction of edge traversal. We can think of the local picture of an edge as in Figure 2.23. This extra information will be encoded in the flags. Each flag, say x is represented by a triangle.

Given a connected graph $G = (V, E)$, we associate to each edge e a set $e' = \{x, \theta x, \phi x, \theta\phi x\}$ of four elements called *flags*, making the eight flags on any two distinct edges all distinct. We obtain a set Φ of flags and we have $|\Phi| = 4|E|$.

We consider ϕ and θ as permutations on Φ satisfying

1. $\phi^2 = \theta^2 = \mathrm{Id}$,

2. $\theta\phi = \phi\theta$.

For each edge we partition its flags $\{x, \theta x\} \cup \{\phi x, \theta\phi x\}$ and assign one of these pairs to each to each of the endpoints of the edge, so each vertex has associated to it a θ invariant pair of flags for each edge incident to it.

The local picture of a drawing of G at a vertex v consists of the set of semi-edges with endpoint v drawn in a particular clockwise order and labelled on each side by a flag. Using this local picture, we may associate to each flag x at v the flag Px opposite θx.

This induces a permutation P on the set of flags, namely P rotates each flag x one place in the cyclic order around v, with half the flags rotating clockwise, and the other half counterclockwise. See Figure 2.24.

We require that the orbits of the flags x and θx under P are distinct and of the same size, i.e.

3. $P\theta = \theta P^{-1}$ and

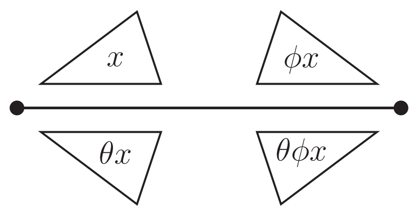


Figure 2.23:

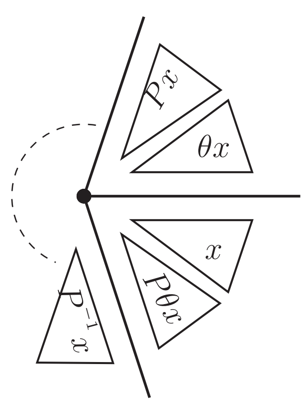


Figure 2.24:

4. $\{P^i x\} \cap \{P^i \theta x\} = \emptyset$

must hold.

§ 94. The Groups A and Θ. The permutations $\{\theta, \phi, P\} \subset \mathrm{Sym}(\Phi)$ generate a group A of permutations of Φ. Since G is connected, A acts transitively on Φ.

The permutations $\theta\phi$ and P generate a subgroup Θ of A. Consider two flags x and y. Since A acts transitively there is an $\alpha \in A$ such that $\alpha(x) = y$.

§ 95. Combinatorial Definition of a Map. Now we may set aside the intuitive picture of graphs drawn on surfaces and define a map in purely combinatorial way.

We call $M = M(\theta, \phi, P)$ a *map* if Axiom 1–4 are satisfied and if A acts transitively on Φ.

§ 96. The one-skeleton. Each map M gives rise to a graph G, the so-called one-skeleton of M. The arcs (semi-edges) of G are the orbits generated by θ. The vertices or G are the orbits generated by $\{\theta, P\}$. The reversal of arcs r is defined by ψ, while i is given by $\supseteq$. Finally, the edges are the orbits generated by $\{\phi, \theta\}$.

§ 97. Orientable and Non-Orientable Maps. Since ϕ and θ commute and $\theta P = P^{-1}\theta$, we can write $\alpha \in A$ either as ωP, for some $\omega \in \Theta$, or α itself is an element of Θ. We conclude that for each pair of flags x and y there is an element of Θ mapping x into y or θy. This means that Θ partitions Φ into either one or two equal equivalence classes (orbits), the *orientation classes* of M.

If there are two orientation classes and the flag x is in one of them, then the flag θx must be in the other, as well as $\phi x = (\theta\phi)\theta x$. We call the map *orientable* if $\Theta = \{\theta\phi, P\}$ generates two orientation classes.

M is *non-orientable* if Θ only generates one orientation class.

§ 98. The Dual Map. Given a map $M = (\theta, \phi, P)$ on a set Φ of flags, we can construct other maps from it. For example, we could replace P by P^2, or by P^{-1}, and all axioms would be satisfied.

Let us study the permutation $P^* = P\theta\phi$. In order for P^* to define a map on Φ, we need P^* to satisfy Axiom 3, not necessarily together with θ. We have

$$P^*\phi = P\theta\phi^2 = P\theta = \theta P^{-1} = \phi(\phi\theta P^{-1}) = \phi(P^*)^{-1}.$$

This means that P^* satisfies Axiom 3 with the role of θ replaced by ϕ.

Interchanging the roles of ϕ and θ means interchanging the roles of the endpoints of an edge with that of its left and right side. We need to prove Axiom 4, namely that the orbits of P^* through x and ϕx are distinct. Assume to the contrary that $(P^*)^m x = \phi x$ and assume that x is chosen so that m is minimal. If $m = 1$, we have $P^*x = P\theta(\phi x) = \phi x$, contradicting Axiom 4 for $M = (\theta, \phi, P)$. If $m \geq 2$, we calculate as follows:

$$\begin{aligned}
(P^*)^m x &= \phi x \\
(P^*)^{-1}(P^*)^m x &= (P^*)^{-1}\phi x \\
(P^*)^{m-1} x &= \phi P^* x \\
(P^*)^{m-2}(P^* x) &= \phi(P^* x)
\end{aligned}$$

So we have found a flag, namely P^*x, whose exponent is strictly smaller than m. contradicting our choice of x and m. The map $M^* = (\phi, \phi, P^*)$ on Φ is called the *dual map* of $M = (\theta, \phi, P)$.

This construction has an intuitive interpretation. Given our connected graph $G = (V, E)$, together with local information of a cyclical order of edge incident to a vertex, as well as local information information concerning and edge, we obtained the map $M(\theta, \phi, P)$. We can recover our graph G from it by observing that each vertex corresponds to a pair of conjugate orbits of $\theta\phi$. We call the orbits of P^* the *faces* of G. The map $M^* = (\phi, \theta, P^*)$ defines the graph G^*, with the conjugate orbits of P^* as the vertex set and the orbits of $\phi\theta$ as the edge set. We call the G^* the *geometric dual* of G with respect to the map M. Note that G and G^* have the same edge set. The faces of G are the vertices of G^*. Moreover, $(G^*)^* = G$.

If G is connected, then G^* is connected as well because the permutation group generated by $\{\theta, \phi, P\}$ is the same as that generated by $\{\phi, \theta, P\}$.

Similarly, the permutations $\phi\theta$ and P^* generate the same group Θ as do the permutations $\theta\phi$ and P, implying that both M and M^* are either both orientable or both non-orientable.

§ 99. The Flag Graph. We are now in a position to define the flag graph F associated with the map M. The vertex of F is the set of flags,

$V(F) = \Phi$. We distinguish between three kinds of edges.

Type 1 Every flag x is adjacent to θx.

Type 2 Every flag x is adjacent to ϕx.

Type 3 Every flag x is adjacent to $P\theta x$.

Since by Axiom 3, $\theta P = P^{-1}\theta$, we see that $(\theta P)^2 = \theta P P^{-1}\theta = \theta^2 = \mathrm{Id}$, so θP is also an involution that we denote by $\lambda = P\theta$. Note that the map is completely determined by its flag graph (with a given edge-coloring.)

Altogether, we have four involutions, namely ϕ, θ, $\theta\phi$ and $\theta P = \lambda$. With the help of these four involutions it is easy to describe, from the flag graph, the vertices, edges, faces, and the Petrie walks of the map M. The edges are the orbits generated by $\{\phi, \theta\}$. The vertices are the orbits generated by $\{\theta, \lambda\}$. The faces are the orbits generated by $\{\phi, \lambda\}$. The Petrie walks are the orbits generated by $\{\theta\phi, \lambda\}$.

Note that Godsil and Royle define the map by three involutions that correspond to our θ, ϕ and λ.

§ 100. Map projections. To depict a map obtained from a graph by choosing a particular clockwise order of semi-edges around each vertex, we may start out by drawing each vertex, on a piece of paper, together with its half edges in the correct cyclic order with each side of a semi-edge marked with a flag. Then connect corresponding semi-edges by line segments, disregarding the perhaps unavoidable line crossings. If x and θx appear in clockwise order around v, but $\theta\phi x$ and ϕx appear in counterclockwise order around w, mark the edge corresponding to $\{x, \theta x, \phi x, \theta\phi x\}$ with a little cross in the middle. The resulting drawing is called a *map projection.* It allows us in small cases to easily deduce the set of faces from the diagram: We start at one side of an edge, staying on the same side of the edge until we reach a vertex where we turn to follow the next edge as prescribed by the corner. However, if an edge is marked by a cross, we use the cross to switch to the other side and proceed as before. We keep walking on the same side (using all crosses as they come along) until we reach the starting point again. The edges we have travelled along are the boundary edges of the corresponding face.

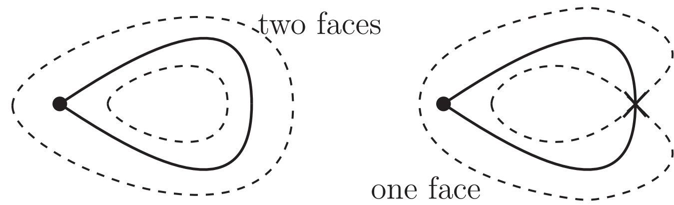


Figure 2.25: On the left the loop is embeddind on the sphere, on the right the same grap is embedded in the projective plane.

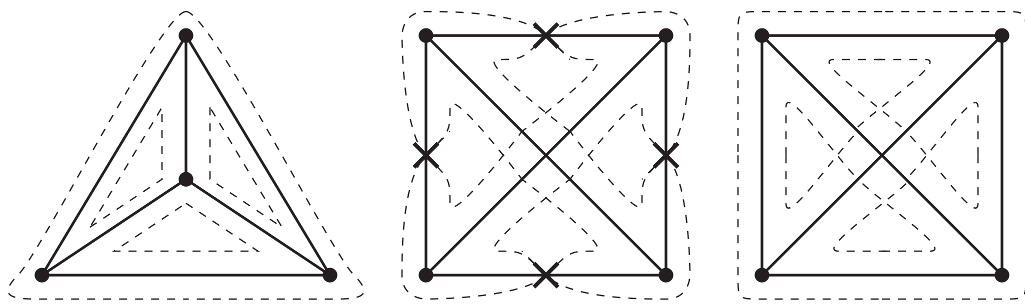


Figure 2.26: Three map projections of the same graph K_4. The left one and the central one correspond to the embedding of K_4 in the sphere with 4 triangular faces, the right one corresponds to the embedding of K_4 in the torus.

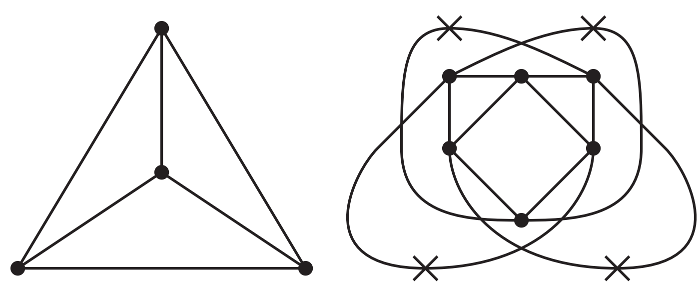


Figure 2.27: An octahedron (8 faces) and a tetra-semi-hexahedron (four triangles and three squares.

§ 101. Euler Characteristics. Given a map $M = (\theta, \phi, P)$ with v vertices, e edges and f faces, we call the alternating sum $v - e + f$ the *Euler characteristics* of M.

§ 102. Vertex Splitting and Edge Contraction. We now informally describe an operation and its inverse on maps which will allow us combinatorially describe the different types of surfaces (canonical normal forms) on which our graphs are embedded.

Let M be a map on a set Φ of flags, $|\Phi| = 4n$. We want to construct from M a map M_1 on Φ_1, $|\Phi_1| = 4(n+1)$, by choosing a vertex v of M, separating it into two vertices and inserting a new edge as described by the following local picture, see Figure 2.28. Nothing else of M is changed.

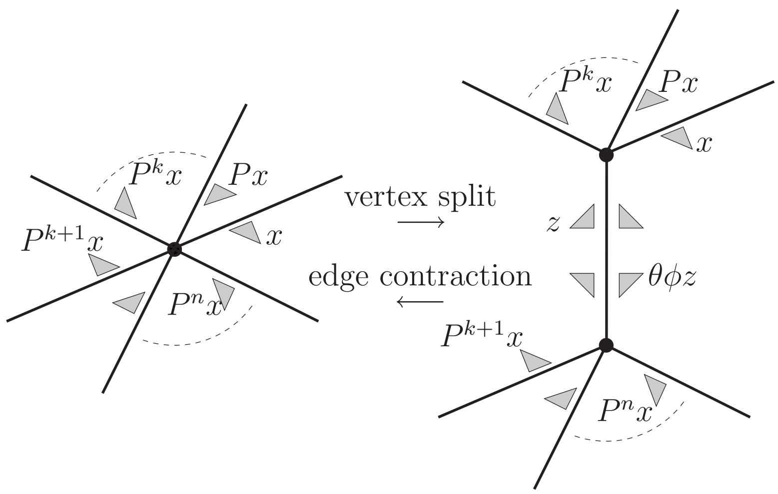


Figure 2.28: Vertex Split/Edge Contraction.

As indicated in Figure 2.28, $\Phi_1 = \Phi \cup \{z, \phi z, \theta z, \theta\phi z\}$ and θ and ϕ are extended to θ_1 and ϕ_1.

The two conjugate orbits of P corresponding to v are split into two pairs of conjugate orbits of P_1 with z, ϕz, θz, and $\theta\phi z$ inserted into exactly one of these four orbits as indicated by the drawing.

M_1 has one more vertex than M and one more edge than M, but the number of faces of M_1 equals the number of faces of M. Therefore M And M_1 have the same Euler characteristic.

The inverse operation of vertex splitting is called *edge contraction*. Keeping in mind that a map has at least one edge, edge contraction is not defined for the link map.

§ 103. Combinatorial Surface. Clearly, vertex splitting or edge contraction do not change the orientability character of the map. We define a *combinatorial surface* to be the (non-empty) class of all maps with a given Euler characteristic and a given orientability character. We say that a map is on the corresponding combinatorial surface.

§ 104. Unitary Map. Any map on a combinatorial surface can be transformed, by a finite sequence of edge contractions and dual replacements, to a map, on the same surface, which is either the link or the loop map, or a map consisting of one vertex and one face (and one or more edges.) A map with exactly one vertex and one face is called *unitary.*[1]

§ 105. Crosscap. The permutation P of a unitary map consists of two conjugate orbits of equal length. If the pair $\{x, \phi x\}$ of flags is in the same orbit, $\{x, \phi x\}$ is called a *crosscap*. If the pair $\{x, \theta\phi x\}$ is contained in the same orbit of P, then there must be some other pair $\{y, \phi y\}$ contained in that orbit, and the orbit must be of the form $(x, R_1, y, R_2, \theta\phi x, R_3, \theta\phi y, R_4)$, where the R_i's are (possibly empty) sequences of flags.

To prove this important observation, let us assume to the contrary that the orbit of P through x is of the form $(x, R, \theta\phi x, S)$ and that with any flag y in R also $\theta\phi y$ is in R. Now for each flag y in R, consider P^*y. Since $P^*y = P\theta\phi y$ and both y and $\theta\phi y$ are in R, P^*y is a flag in $\{R, \theta\phi x\}$. Therefore one orbit of P^* contains only flags of $\{R, \theta\phi x\}$, but it does not contain x, contrary to the assumption that the map under consideration is unitary, i.e., P^* has only one vertex.

§ 106. Handle. We call a quadruple $(x, y, \theta\phi x, \theta\phi y)$ a *handle* if there are integers $i < j < k$ such that $P^i x = y$, $P^j x = \theta\phi x$, $P^k x = \theta\phi y$. The handle is called *assembled* if $\{i, j, k\} = \{1, 2, 3\}$. Likewise, a crosscap $\{x, \phi x\}$ is assembled if $P(x) = \phi x$.

Let M be a unitary map with some unassembled crosscap $\{x, \phi x\}$. Figure 2.29 shows that M can be transformed, by a vertex splitting and an edge

[1]Topologists call the 1 skeleton a unitary map *bouquet of circles*.

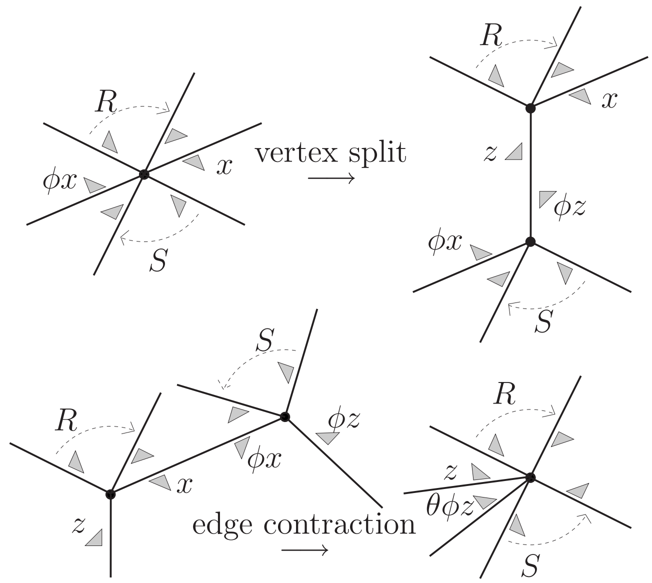


Figure 2.29: Assembling a crosscap.

contraction, into a map where $\{x, \phi x\}$ is replaced by an assembled crosscap denoted $\{z, \phi z\}$. The resulting map is unitary and no other crosscaps of M are disturbed. By induction, we can show that any unitary map can be transformed into a unitary map where all crosscaps are are assembled by a finite sequence of vertex splittings and edge contractions. Likewise, we can assemble all handles, see Figure 2.30.

However, assembling crosscaps and handles is not enough to canonically classify unitary maps. Figure 2.31 shows a unitary map with one assembled handle and one assembled cross cap which is transformed, using only vertex splittings and edge contractions, into a unitary map containing three crosscaps. Such a transformation is possible on a unitary map M as long as M contains both a crosscap and a handle.

We see that, as in Figure 2.31, we can transform a given unitary map into one containing only assembled crosscaps or only assembled handles. So we can describe canonical forms of combinatorial unitary maps as follows:

Either M is orientable and consists of g assembled handles and has Euler characteristic $2 - 2g$, in which case g is called the *orientable genus* of M, or M is unorientable and consists of g assembled crosscaps, hence has Euler characteristic $2 - g$ and g is called the *nonorientable genus* of M. For a

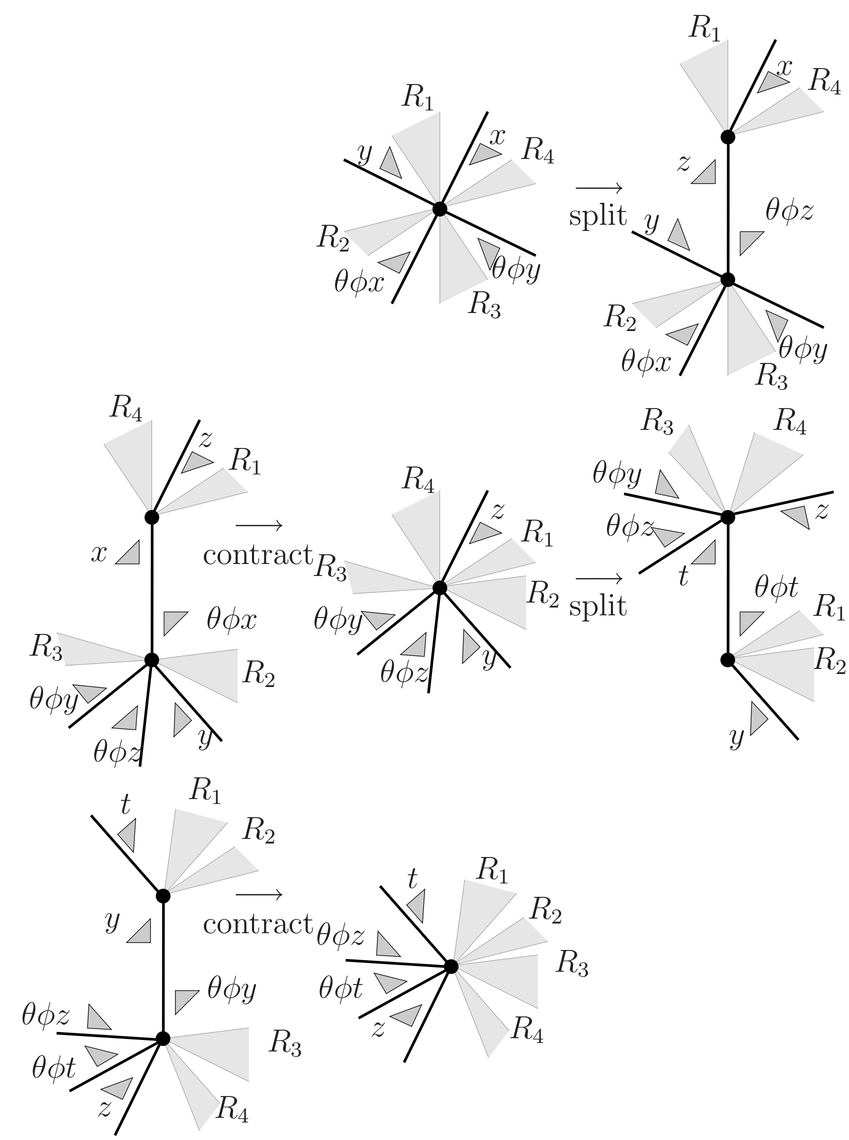


Figure 2.30: Assembling a handle.

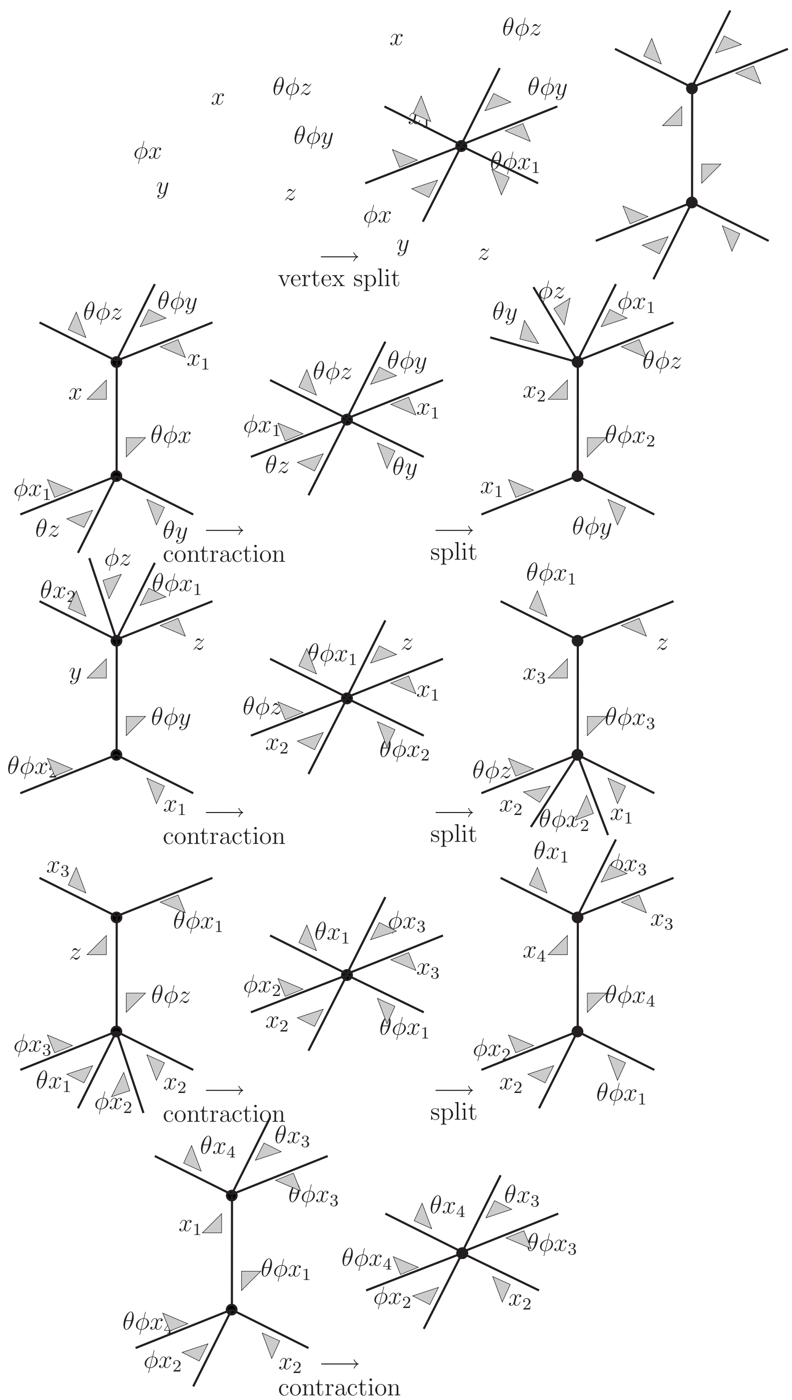


Figure 2.31: Map with a handle and crosscap is transformed into three crosscaps.

unitary map, the genus is positive. We define the genus of the link and loop map to be zero. A combinatorial surface of genus zero is called a *sphere*. The surface of orientable genus 1 is called a *torus*, and a *double torus* if the orientable genus is two. The surfaces of nonorientable genus 1 and 2 are called *projective planes* and *Klein bottles,* respectively. Map projections of unitary maps on these surfaces are drawn in Figure 2.32. It is customary and useful to draw instead of the map projection, the single face of the unitary map as a polygon, where the corners of the polygon all correspond to the unique vertex of the map and are to be identified. Likewise every edge occurs twice on the boundary of the polygon and edges are to be identified pairwise, respecting the orientation of face traversal. Equivalently, one could represent the unitary map as a vertex in the interior of the polygon and edges to the boundary edges of the polygon, where these boundary edges indicate how the ends of the edge are attached to the vertex, twisted (in the nonorientable case) or conforming (orientable case,) yielding our standard local picture of the map. See Figure 2.32.

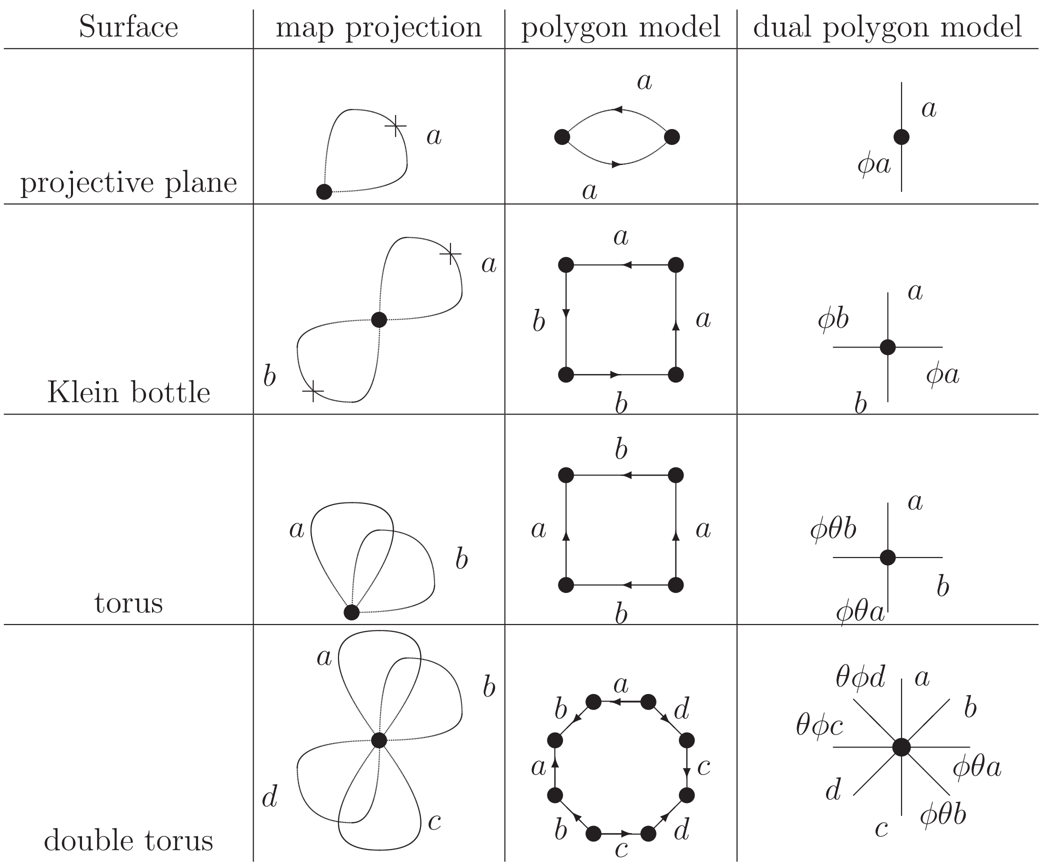
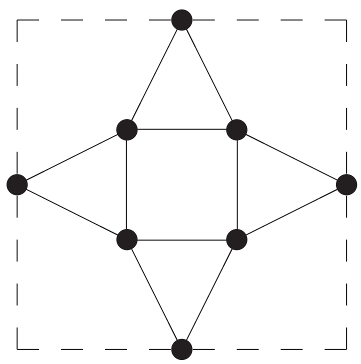
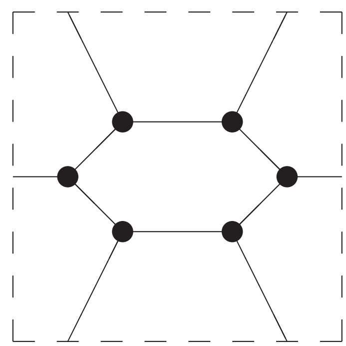


Figure 2.32: Elementary unitary maps.

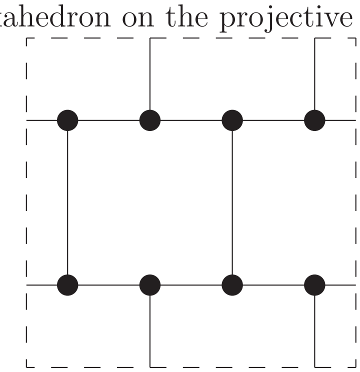
Any map M can be reduced to one of the canonical forms by a finite sequence of edge contractions, vertex splittings and dual replacements and M is said to be on that combinatorial surface. We can draw M on the polygon corresponding to the canonical form. Such a drawing can be produced without edge crossings, which can be shown by using the unitary map as a base case, and applying inducting on the number of edges, using the fact that the operations of vertex splitting, edge contraction and dual replacement can each be performed without introducing edge crossings. In 2.33, we



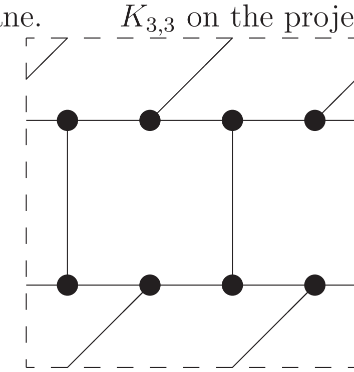
Tetrahexahedron on the projective plane.



$K_{3,3}$ on the projective plane



Cube on the torus



Cube on the torus (2)

Figure 2.33:

give examples of maps on the projective plane and the torus, which we have met before.

Given a map $M = (\theta, \phi, P)$ on the set of flags Φ, it is easy to construct the flag graph, which in turn is useful to determine whether or not M is orientable.

Theorem 2.40. *A map M is orientable if and only if its flag graph is bipar-*

tite.

Proof. We use induction on the number of edges. For the base case we construct flag graphs for the normal forms, see Figure 2.34

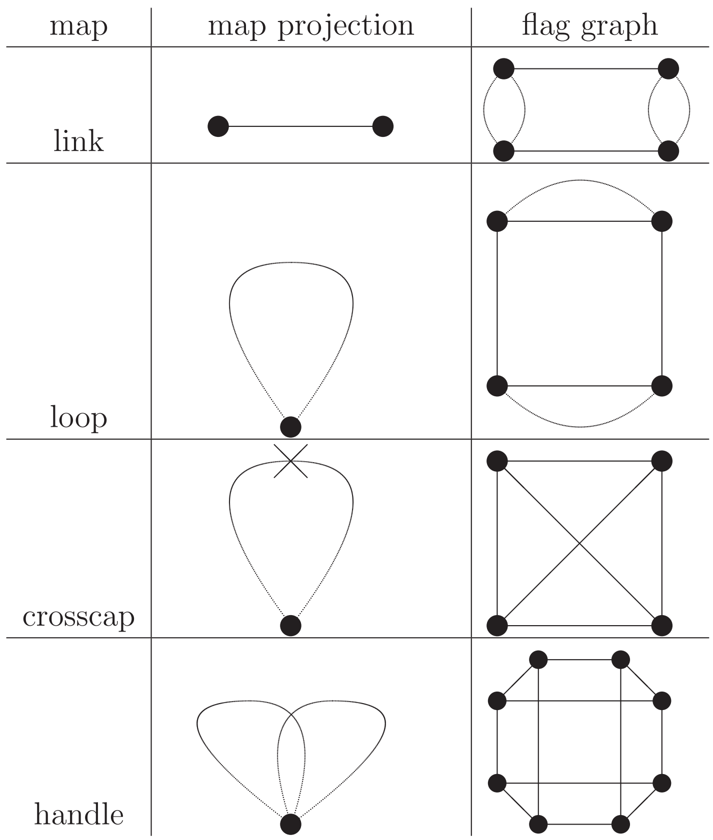


Figure 2.34:

For a unitary map consisting of two cross caps, the flag graph has vertex set $\{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\}$. The edge set consists of the edges of the flag graphs for two single cross caps on the vertex sets $\{a_i, b_i, c_i, d_i\}$, $i \in \{1, 2\}$, except for the edges (a_i, d_i), which are replaced by (a_1, a_2) and (b_1, b_2).

Every odd cycle in either of the two flag graphs corresponds to an odd cycle in the merged flag graph, where the edge (a_1, d_1) is replaced by the path $(a_1, a_2, b_2, c_2, d_2, d_1)$, containing an odd number of edges. Similarly for (a_2, d_2).

This argument can be extended to an arbitrary number of crosscaps. Therefore the flag graphs of unorientable unitary maps are not bipartite.

By a similar argument, (see exercises), the flag graph graphs of the orientable unitary normal forms are bipartite. Now we first observe that the flag graphs of a map M and its dual M^* are isomorphic. Any flag x of $M*$ is adjacent to θx, ϕx and $P\theta x$. A flag x of M^* is adjacent to ϕx, θx, and $P^*\phi x = P\theta\phi\phi x = P\theta x$, so the edge sets of the flag graphs for M and M^* are identical - only the coloring is different.

Figure 2.35 shows the effect of an edge contraction on the flag graph. We

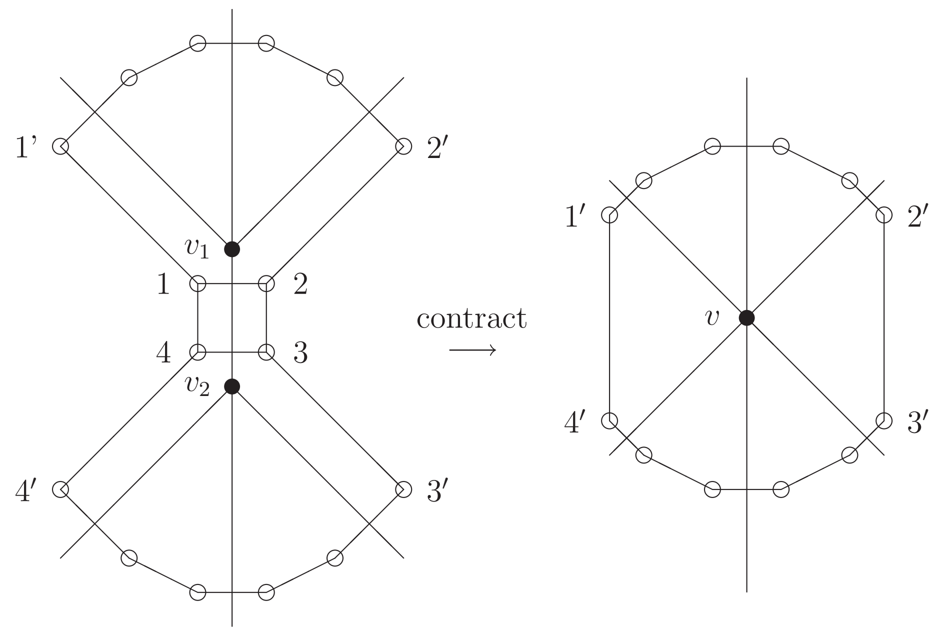


Figure 2.35:

note that cycles about a vertex of any flag graph are always even, and the other cycles that change in length get changed by two edges each, so the parity of the length is preserved, which completes the proof of the theorem. □

§ 107. Maps with Symmetry. Given map M on the set of flags Φ we may define an automorphism of M to be a permutation of Φ that commutes with θ, ϕ, and P. Clearly there is at most one automorphism that maps a given flag x to any other flag y. This means that $|\mathrm{Aut}\, M| \leq |\Phi|$. If $\mathrm{Aut}\, M$ acts transitively on Φ it must act regularly. Maps M with $|\mathrm{Aut}\, M| = |\Phi|$. are called em regular maps.

Maps can be viewed as graphs with additional information. Therefore one can clearly speak about vertex- or edge-transitive maps. For instance, Grünbaum and Shephard have shown that there are only nine finite 3-connected planar edge-transitive maps.

2.5.1 Operations on Maps

§ 108. Operations on Maps. Using formal definition of a map we can now formally define several transformations or operations on maps. The most important among them are the dual, the truncation, the medial and the one dimensional subdivision.

The medial map. For a definition of the medial, see for instance [**?**, 33]. First, an intuitive description: If M is realized as a polyhedron, then think of cutting away each vertex v by a separate plane Π_v in such a way, that for each edge e with the endvertices u and v the corresponding planes Π_u and Π_v intersect at the midpoint of e. The resulting map is fourvalent, it resides in the same surface as M and has two types of faces. The ones, corresponding to the original faces and the ones, corresponding to the original vertices. In particular, a vertex v of valence k gives rise to a $k-$gonal face. For example, the medial of a tetrahedron is an octahedron, and the medial of an octahedron is a cuboctahedron.

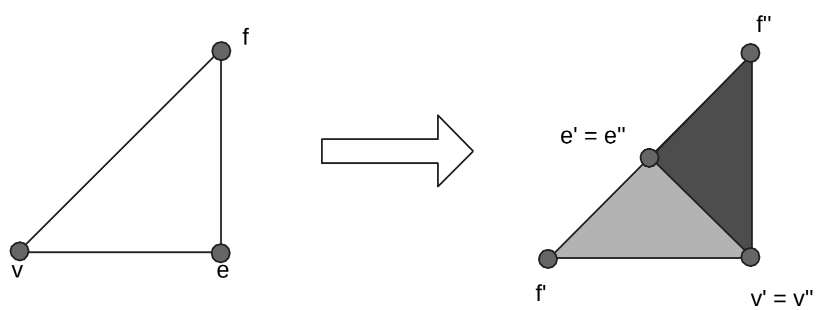


Figure 2.36: The medial operation Me can be defined as a subdivision of flag triangles.

Now we define the medial map Me using flags. Given $M(\theta, \psi, P)$ on Φ, its medial $(Me)(M)$ is a map on the disjoint union $\Phi' \cup \Phi''$ of two copies of Φ. The three involutions are defined as: $\lambda x' := (\theta x)', \lambda x'' := (\phi x)'', \phi x' := (\lambda x)', \phi x'' := (\lambda x)'', \theta x' := x'', \theta x'' := x'$.

Given the flag graph of M we may readily construct the flag graph of $(Me)(M)$ using the information above. The flag graph together with the corresponding edge-coloring defines an embedding in the same surface as M. The dual of the flag map is a triangulation in which each flag corresponds to a triangle. The vertices of such a triangle correspond to the three types of orbits: vertex orbit, edge orbit and face orbit. The described construction had a pseudo-geometric interpretation as a subdivision of the triangle.

Let (v, e, f) be the vertices of a triangle, corresponding to a flag x. We subdivide this triangle into two triangles: (v', e', f') corresponding to x' and (v'', e'', f''), corresponding to x'' where $v' = v'' = e, e' = e'', f' = v, f'' = f$. The new vertex $e' = e''$ is located on the edge vf of the original triangle. If placed in the barycenter, one can write this in the form: $e' = e'' = (v + f)/2$. There is a shorter description of this subdivision.

$$\begin{bmatrix} v' \\ e' \\ f' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ e \\ f \end{bmatrix}, \begin{bmatrix} v'' \\ e'' \\ f'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ e \\ f \end{bmatrix}$$

If we define the two matrices:

$$M' = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, M'' = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

then the medial operation admits a description by a set of two matrices $\{M', M''\}$.

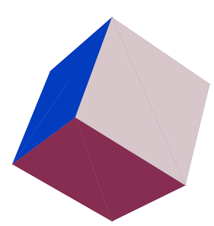


Figure 2.37: Id(cube).

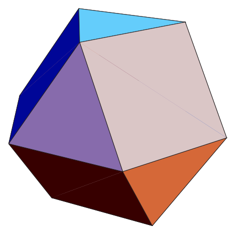


Figure 2.38: Me(cube).

§ 109. Identity – Id. $M' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

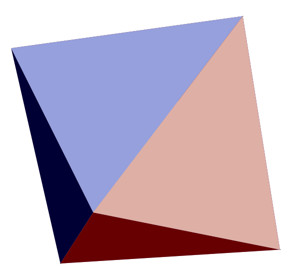


Figure 2.39: Du(cube).

§ 110. Dual – Du. $M' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

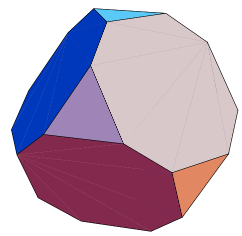


Figure 2.40: Tr(cube).

§ 111. Truncation – Tr. $M' = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, M'' = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}, M''' = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

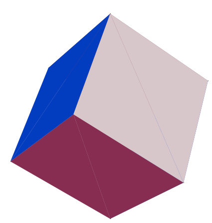


Figure 2.41: Su1(cube).

§ 112. Onedimensional Subdivision – Su1. $M' = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M'' = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

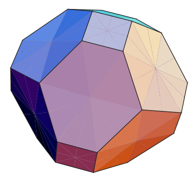


Figure 2.42: Cha(cube).

§ 113. Chamfering – Cha. $M' = \begin{bmatrix} 1 & 0 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$, $M'' = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$, $M''' = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$, $M'''' = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

§ 114. Combined operations. If we combine some of these transformations some other interesting transformations result. We mentioned already that $\mathrm{Du}(\mathrm{Du}(M)) = M$.

§ 115. Leapfrog – Le. The operation $\mathrm{Le}(M) := \mathrm{Tr}(\mathrm{Du}(M))$ is sometimes called the *leapfrog transformation* in chemistry. For instance Le(*Dodecahedron*) is the well-known Buckminsterfullerene, a model for the pure carbon molecule discovered in the 1980's.

§ 116. Angle transformation – An. The operation $\mathrm{Du}(\mathrm{Me}(M))$ was studied in [58] under the name of the *angle transformation* and will be denoted here by $\mathrm{An}(M) := \mathrm{Du}(\mathrm{Me}(M))$.

§ 117. Twodimensional subdivision. – Su2. An interesting operation is the *two dimensional subdivision* $\mathrm{Su2}(M) := \mathrm{Du}(\mathrm{Tr}(\mathrm{Du}(M)))$.

§ 118. Barycentric subdivision – BS and Combinatorial Map – Co. The well-known *barycentric subdivision* plays an important role in mathematics. It can be defined as follows. $\mathrm{BS}(M) := \mathrm{Su2}(\mathrm{Su1}(M))$. Its dual $\mathrm{Co}(M) := \mathrm{Du}(\mathrm{BS}(M))$ was used in [60]. We would like to stress the fact that $\mathrm{Co}(M)$ is in fact the flag graph of M. Note that $\mathrm{Co}(M)$ has a quadrilateral 2-factor.

The following result is quite useful for automatic determination of the set of matrices for combined operations.

Theorem 2.41. *Let $\{M_1, ..., M_m\}$ be the matrices defining operation S and let $\{N_1, ..., N_n\}$ define operation T. Then the combined operation ST is defined by the mn product matrices $\{M_1N_1, M_1N_2, ..., M_1N_n, M_2N_1, ..., M_mN_n\}$.*

Using this result it can be easily shown that $\mathrm{Du}(\mathrm{Du}(M)) = M$ and that $\mathrm{Me}(M) = \mathrm{Me}(\mathrm{Du}(M))$.

§ 119. Maps and Covering Graphs. Given a map M and its one-skeleton G we may consider a covering graph K over G. There is a unique way to extend K to a map that is a branched covering over M. If x is a flag of Φ then there are n flags (x, i) in the covering flag graph. Define $\theta(x, i) := (\theta x, i), P(x, i) := (Px, i)$ and $\phi(x, i) := (\phi x, \alpha(i))$, where α is the voltage.

This observation is the key instrument in the topological graph theory since we may completely determined the structure of faces of the map whose skeleton is K.

However, since we defined coverings of pregraphs, it is therefore a natural question how do we have to extend the maps to premaps whose one-skeleta are pregraphs in order to keep the above technique.

WE HAVE TO FIND AND DESCRIBE THE CORRECT GENERALIZATION.

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USE THE REST OF THIS FILE IN EXERCISES AND IN FURTHER EXPLANATIONS.

§ 120. Operations on Maps. A map is a combinatorial representation of a closed surface. There are several equivalent combinatorial descriptions available for maps. In the computer package Vega [**?**], we implemented a series of operations on maps. This enables us to produce new maps from old ones. In turn, we can get new polyhedra or new graphs.

Here we present some operations on maps; all of them are explained via examples in the figures. We consider only connected maps.

§ 121. **Du:** ***Dual.*** This operation is well-known for planar graphs. However, it can be generalized for maps in other surfaces. It is also known as the *Poincaré dual.* The dual map $Du(M)$ is built from the original map M as follows: we put a vertex of $Du(M)$ in the center of each face of M. For each edge e of M we produce its dual $Du(e)$ so that $Du(e)$ connects the vertices corresponding to the faces of M that have e on the common boundary. We place the vertex $Du(M)$ in the same surface as M in such a way that the faces of $Du(M)$ correspond to the vertices of M. This means that the dual edges are traversed along faces in the same cyclic order as the original edges are traversed cyclically around a vertex. It can be shown that $Du(Du(M)) = M$.

Let us consider three examples that will serve us also for other operations.

1. The cube Q_3 in the sphere: $v = 8$, $e = 12$, $f = 6$. The dual is the octahedral graph $K_{2,2,2}$ in the sphere: $v = 6$, $e = 12$, $f = 8$; see Figure 2.43.

2. The *bouquet* B_n of n circles is a graph with n loops attached to a single vertex. The bouquet of one circle B_1 in the projective plane: $v = 1$, $e = 1$, $f = 1$. It is self-dual: $v = 1$, $e = 1$, $f = 1$.

3. Another example of a self-dual map is the tetrahedron K_4 in the sphere.

4. K_4 in the torus: $v = 4$, $e = 6$, $f = 2$. One face is a quadrilateral and the other one is a hexagon. The dual graph has two vertices; one vertex has a double loop and there are four parallel edges between the two vertices: $v = 2$, $e = 6$, $f = 4$. All faces are triangles.

If v, e, f are the parameters of the original and v', e', f' are the parameters of the transformed map we have the relations:

$$v' = f$$
$$e' = e$$
$$f' = v.$$

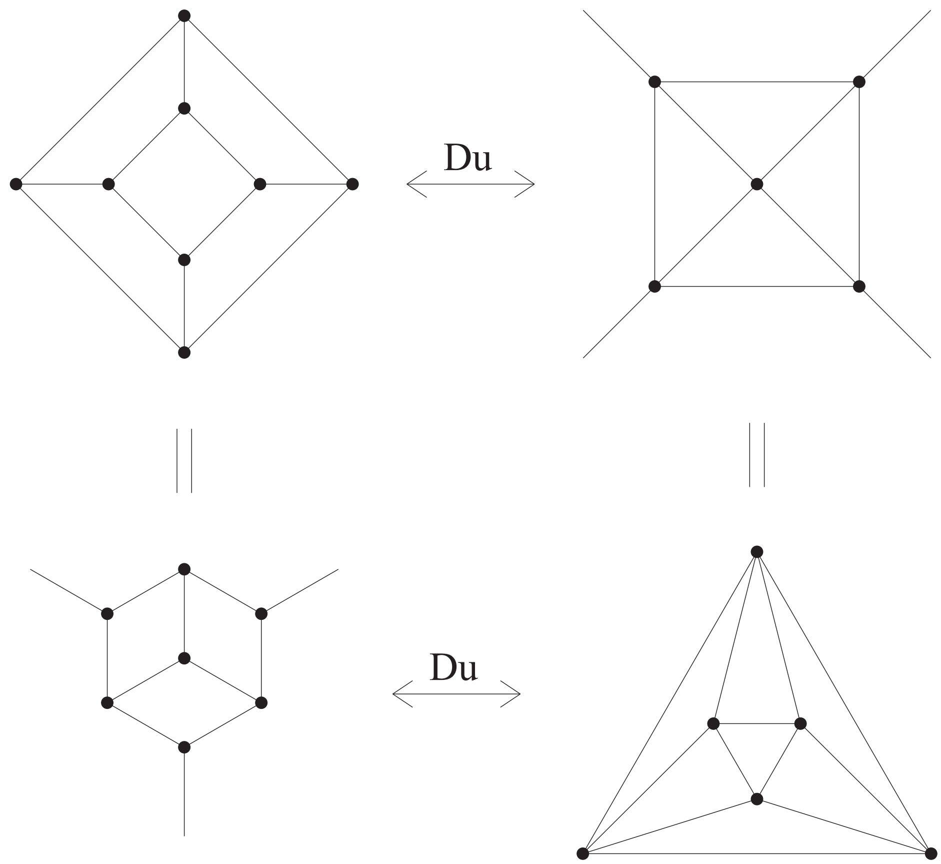

Figure 2.43: The cube and its dual, the octahedron.

§ 122. Su1: *1-dimensional subdivision.* This is the simplest of all. We subdivide each edge by inserting a vertex at the midpoint of each edge thus splitting the edge in two; see Figure 2.44.

1. The cube Q_3 in the sphere: $v = 8$, $e = 12$, $f = 6$. The 1-dimensional subdivision has the parameters: $v = 20$, $e = 24$, $f = 6$.
2. The bouquet of one circle B_1 in the projective plane: $v = 1$, $e = 1$, $f = 1$. Its subdivision Su1 is the cycle C_2 in the same surface: $v = 2$, $e = 2$, $f = 1$.
3. K_4 in the torus: $v = 4$, $e = 6$, $f = 2$. The subdivision graph has parameters $v = 10$, $e = 12$, $f = 2$. All faces have twice as many edges as before.

If v, e, f are the parameters of the original and v', e', f' are the parameters of the transformed map we have the relations:

$$v' = v + e$$
$$e' = 2e$$
$$f' = f.$$

§ 123. Pa : *Parallelization.* Here we replace each edge by a pair of parallel edges forming a digon on the surface. Instead of giving this description we could define Parallelization formally as: $Pa(M) := Du(Su1(Du(M)))$

$$v' = v$$
$$e' = 2e$$
$$f' = e + f$$

§ 124. Si : *Simplification.* This operation is in a sense the inverse to the operation of Su1. It removes all vertices of valence 2 but leaves the graph (and the map) topologically the same. The only exception is the bouquet B_1 which does not allow further reductions.

§ 125. PSi : *Parallel Simplification.* This operation is in a sense the inverse to the operation of Pa. It removes all digons by changing them into a single edge. $PSi(M) := Du(Si(Du(M)))$

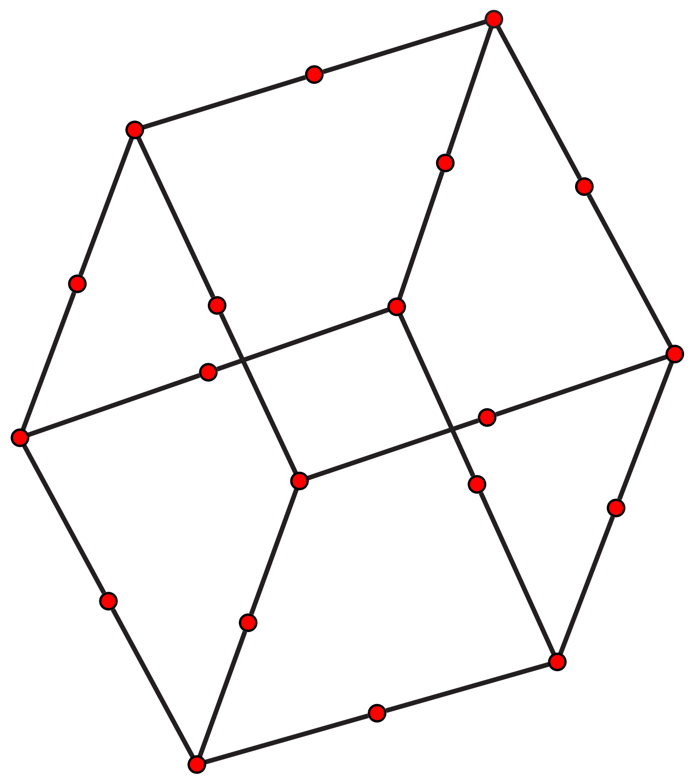

Figure 2.44: The subdivided cube.

§ 126. Me : *Medial.* This is probably the most important transformation of a map and is not so simple. The new vertices are the midpoints of the original edges. New vertices are adjacent if and only if the two original edges span an angle; i.e. the two edges must be incident and consecutive when traversing the rotation about their common vertex in the map. The medial graph is thus a subgraph of the *line-graph*. For the definition of line-graph, see for instance, [**?**]. In the line-graph each original vertex gives rise to a complete graph; in the medial graph it only gives rise to a cycle.

If v, e, f are the parameters of the original and v', e', f' are the parameters of the transformed map we have the relations:

$$v' = e$$
$$e' = 2e$$
$$f' = v + f$$

1. The cube Q_3 in the sphere: $v = 8$, $e = 12$, $f = 6$. Its medial is the cuboctahedron and has the parameters: $v = 12$, $e = 24$, $f = 14$; see Figure 2.45.

2. The bouquet of one circle B_1 in the projective plane: $v = 1$, $e = 1$, $f = 1$. Its medial is B_2 in the same surface: $v = 1$, $e = 2$, $f = 2$.

3. K_4 in the torus: $v = 4$, $e = 6$, $f = 2$. The medial graph Me is obtained from $K_{2,2,2}$ by doubling 8 edges with one endpoint in the same color class with parameters: $v = 6$, $e = 12$, $f = 6$. There are 4 triangles, one quadrilateral and one octagon.

Medials have interesting properties:

- Each one is isomorphic to the medial of the dual:

$$Me(G) = Me(Du(G))$$

- All are 4-valent and their duals are bipartite.
- The structure of the map and its dual are visible in the medial.
- Face lengths and vertex valencies are readily visible in the medial.
- The map and its dual occur symmetrically in the medial.

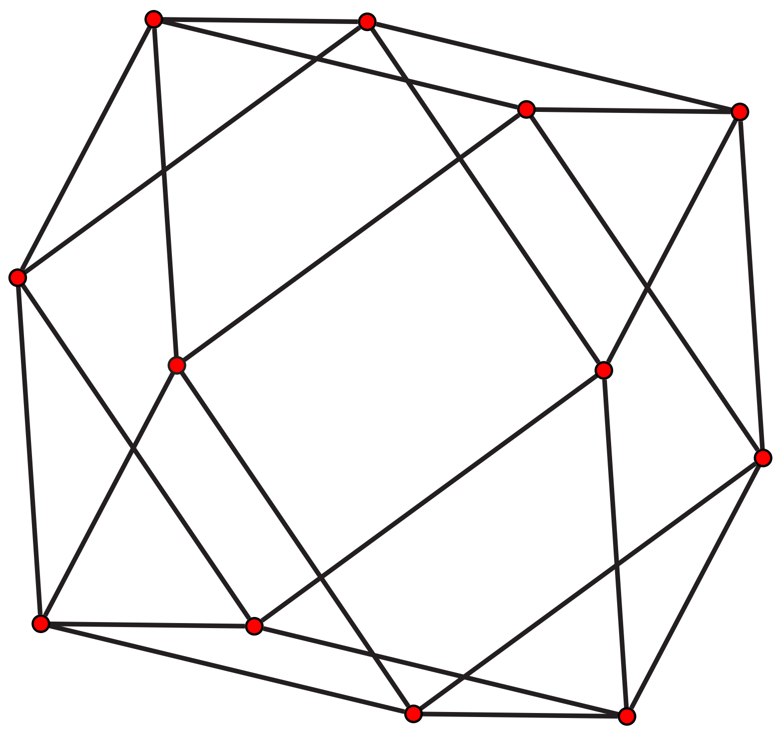

Figure 2.45: The cuboctahedron, the medial of the cube.

§ 127. Tr: *Truncation.* Truncation can be first described intuitively. We cut off the neighborhood of each vertex by a plane "close" to the vertex that meets each edge incident to the vertex. Using transformations from above we may define truncation Tr to be $Tr(G) = PSi(Me(Su1(G)))$. It is the medial of a 1-subdivided map. That would introduce parallel edges and digons. That is why we insist that the operation is followed by parallel simplification. Compare [29].

§ 128. Su2: *2-dimensional subdivision.* The 2-dimensional subdivision of a graph is obtained by adding a vertex in the center of each face and joining it by edges to the vertices of the original face; see Figure 2.46.

There is also an interesting connection between truncation and the 2-dimensional subdivision, namely, $Tr(Du(G)) = Du(Su2(G))$; see for instance [50]. Therefore, $Su2$ can be defined in terms of truncation:

$$Su2(G) = Du(Tr(Du(G)))$$
$$= Du(PSi(Me(Su1(Du(G))))).$$

§ 129. BS: *Barycentric subdivision.* The barycentric subdivision is a composite operation: It is a 2-dimensional subdivision of the 1-dimensional subdivison.

$$BS(G) = Su2(Su1(G))$$
$$= Du(PSi(Me(Su1(Du(Su1(G)))))).$$

The barycentric subdivision has an important role in topology and its dual is also very interesting, as it is a cubic graph and thus much easier to visualize.

§ 130. Co: *Combinatorial map.* Combinatorial map is a composite operation: it is the dual of a barycentric subdivision.

$$Co(G) = Du(BS(G)) = Du(Su2(Su1(G)))$$
$$= PSi(Me(Su1(Du(Su1(G))))).$$

The name comes from the fact that $Co(G)$ represents the description of G by the three involutions.

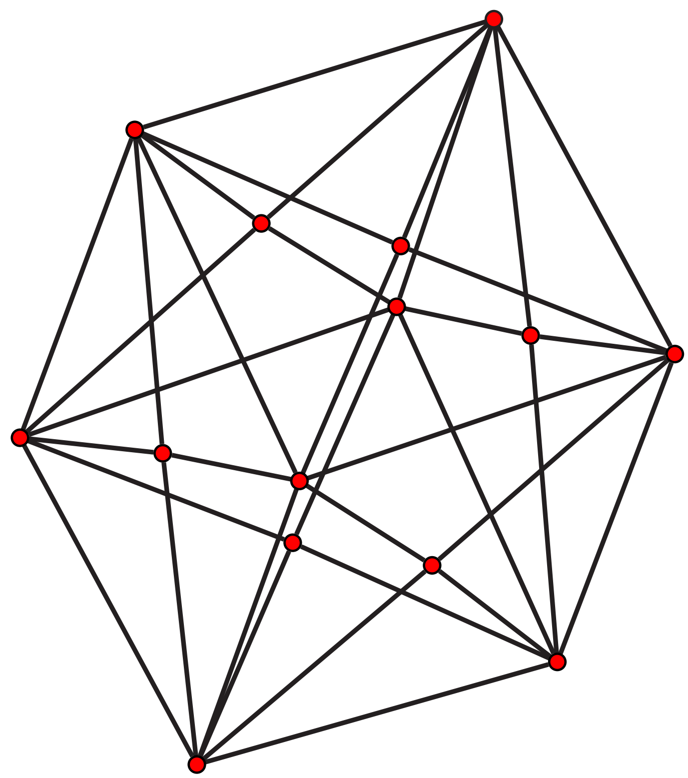

Figure 2.46: The two-dimensional subdivision of the cube.

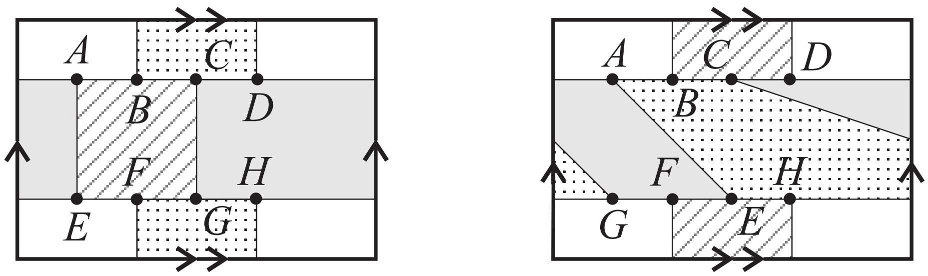

Figure 2.47: Two all-hexagonal drawings of the cube Q_3 on the torus.

§ 131. Pe: *Petrie Dual.* The Petrie dual is the first operation that keeps the 1-skeleton fixed and only changes the faces of the map. New faces are composed of all left-right walks on the surface: every two consecutive edges of the new faces are also consecutive edges of the old faces. However, no three consecutive edges of new faces form consecutive edges of the old faces. It can be shown that it is indeed a “dual” in the sense that $Pe(Pe(G)) = G$. The Petrie dual is a powerful operation as it can turn an orientable surface into a nonorientable one and vice versa.

For instance, the Petrie dual of the ordinary cube Q_3 gives its embedding in the torus with all hexagonal faces. However, there is another, different hexagonal embedding of Q_3 in torus; see Figure 2.47. This may run counter our intuition, since Q_3 in particular has only one drawing in the sphere.

The following is true:

$$Du(Pe(Du(G))) = Pe(Du(Pe(G)))$$

The dual and the Petrie dual form an interesting hexagon. By alternating duals and Pertie duals one obtains the sequence:

$$G, Du(G), Pe(Du(G)), Pe(Du(G)),$$

$$Du(Pe(Du(G))) = Pe(Du(Pe(G))), Du(Pe(G)), Pe(G).$$

§ 132. An: *Antipodal Dual.* The definition is simple:

$$\begin{aligned} An(G) &= Du(Pe(Du(G))) \\ &= Pe(Du(Pe(G))) \end{aligned}$$

§ 133. R2: *Rotation Square.* The rotation square changes faces. It is well-defined only for graphs that have all vertices of odd valence. The map is completely defined if we specify its *rotation scheme*, i.e. at each vertex we give the cyclic order of incident edges. Rotation square takes the square of the overall permutation. If $(e_1, e_2, \ldots, e_s)$ is the local rotation at a vertex, then we have in the rotation square the following sequence $(e_1, e_3, e_5, \ldots, e_2, e_4, \ldots)$. That is why we need s to be odd.

§ 134. S2: *Embedded Square.* The idea behind the embedded square is similar to the idea of the rotation square; however, the construction is quite different and the resulting graph is quite different from the original. The graph is obtained by keeping the vertices of the original and adding an edge at each angle. If we have a face in the original that has $(v_1, v_2, \ldots, v_s)$ as a sequence of vertices then the new edges are $v_1 - v_3, v_2 - v_4, v_3 - v_5, \ldots$

§ 135. Sn1 and Sn2: *Snub.* There are two snub operations. First we take two consecutive medials $Me(Me(G))$. The resulting map is 4-valent and also equipped with a collection of quadrilaterals, arising from vertices of the first medial operation $Me(G)$, such that each vertex of $Me(Me(G))$ belongs to two quadrilaterals. By inscribing a diagonal in one of the quadrilaterals one induces diagonals in all remaining quadrilaterals. If the other initial diagonal is selected the resulting map is $Sn2(G)$ instead of $Sn1(G)$. When Sn1 and Sn2 are isomorphic we simply have a snub Sn. For instance, for a tetrahedron T, $Sn(T) = Sn1(T) = Sn2(T)$ is topologically equivalent to the icosahedron I but $Sn1(I) \neq Sn2(I)$.

§ 136. Le: *Leapfrog.* Leapfrog [29] is a term coined by chemists. It represents a composite operation. It is the truncation of the dual:

$$Le(G) = Tr(Du(G)) \\ = PSi(Me(Su1(Du(G)))).$$

It can be described in an intuitive way. Recall how we envision truncation. The process of truncation involves a collection of planes, one for each vertex, that cut off parts of the polyhedron close to each vertex. If we "move" these planes towards the center of polyhedron, each original edge is diminished and is at a certain moment reduced to a single point, the midpoint. The

Symbol	Polyhedron	Operation	Formula
T	Tetrahedron	Primitive	—
O	Octahedron	Medial	Me(T)
C	Cube(Hexahedron)	Dual	Du(O) = Du(Me(T))
I	Icosahedron	Snub	Sn(T)
D	Dodecahedron	Dual	Du(Sn(T))
TT	Truncated tetrahedron	Truncation	Tr(T)
TO	Truncated octahedron	Truncation	Tr(O) = Tr(Me(T))
TC	Truncated cube	Truncation	Tr(C) = Tr(Du(Me(T)))
TI	Truncated icosahedron	Truncation	Tr(I) = Tr(Sn(T))
TD	Truncated dodecahedron	Truncation	Tr(D) = Tr(Du(Sn(T)))
CO	Cuboctahedron	Medial	Me(C) = Me(Me(T))
ID	Icosidodecahedron	Medial	Me(I) = Me(Sn(T))
TCO	Truncated cuboctahedron	Truncation	Tr(CO) = Tr(Me(Me(T)))
TID	Truncated icosidodecahedron	Truncation	Tr(ID) = Tr(Me(Sn(T)))
RCO	Rhombicuboctahedron	Medial	Me(CO) = Me(Me(Me(T)))
RID	Rhombicosidodecahedron	Medial	Me(ID) = Me(Me(Sn(T)))
SC	Snub cube	Snub	Sn(C) = Sn(Du(Me(T)))
SD	Snub dodecahedron	Snub	Sn(D) = Sn(Du(Sn(T)))

Figure 2.48: Derivation of the Platonic and Archimedean polyhedra from the tetrahedron.

polyhedron obtained at that moment is the medial. If we continue the process beyond that point, we obtain the leapfrog. Similar intuitive processes can be applied to an arbitrary map yielding the same result.

§ 137. Chamfering and other operations. There are other useful operations known. One of them is the so-called *chamfering operation*; [23] which is onein the series of the so-called *Goldberg operations*.

§ 138. Platonic and Archimedean Polyhedra. There are five Platonic polyhedra, with all vertices, all edges, and all faces mutually equivalent. There are 13 Archimedean polyhedra in which all vertices are equivalent but neither the edges nor the faces are. All Platonic and Archimedean polyhedra can be obtained from the tetrahedron T by some sequence of operations that we have introduced; see Figures 2.48 and 2.49.

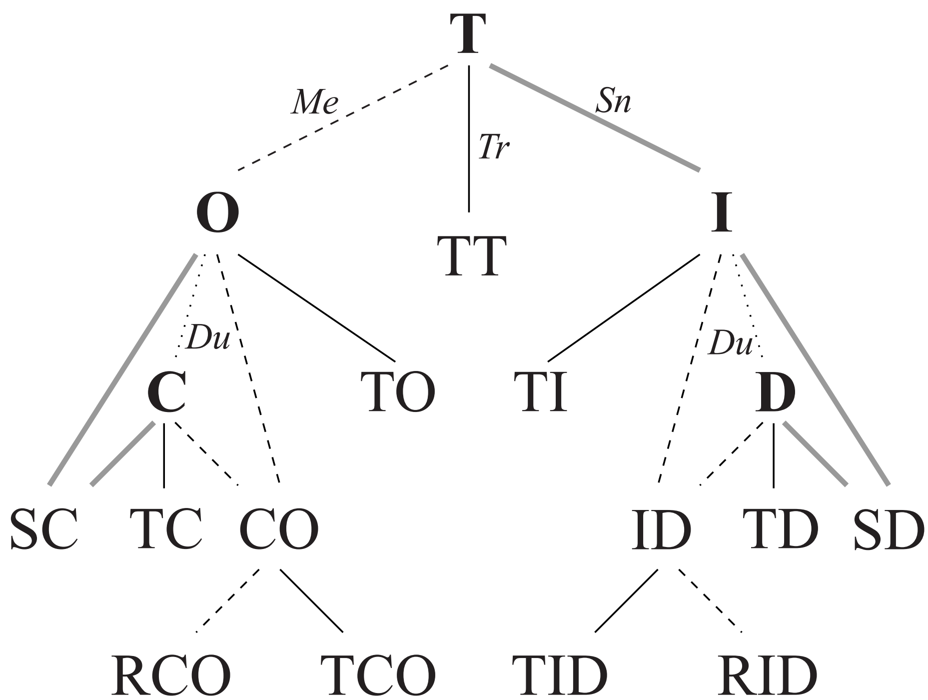

Figure 2.49: Evolution of the Platonic and Archimedean polyhedra from the tetrahedron.

2.6 Exercises

Exercise 2.1. *Show that the first group axiom implies that if* $g_i \centerdot g_j = g_i \centerdot x$*, then* $x = g_j$ *(left cancellation), as well as* $g_i \centerdot g_j = x \centerdot g_j$ *implies* $x = g_i$ *(right cancellation). Deduce the existence of a unique unit element* e*, satisfying* $g \centerdot e = e \centerdot g = g$ *for all* $g \in G$*.*

Exercise 2.2. *Let* g *be an element of order* k *in a group of order* n*. Show that* k *divides* n*.*

Exercise 2.3. *Show that the first group axiom implies that if* $g_i \centerdot g_j = g_i \centerdot x$*, then* $x = g_j$ *(left cancellation), as well as* $g_i \centerdot g_j = x \centerdot g_j$ *implies* $x = g_i$ *(right cancellation). Deduce the existence of a unique unit element* e*, satisfying* $g \centerdot e = e \centerdot g = g$ *for all* $g \in G$*.*

Exercise 2.4. *Let* g *be an element of a group* G*. Deduce, from the group axioms the existence of a unique inverse* g^{-1} *and prove that every group element commutes with its inverse.*

Exercise 2.5. *Give an example of an infinite group and an infinite subset which is closed under multiplication but does not form a subgroup.*

Exercise 2.6. *Show that the elements of* A_n *can be written as products of three cycles, i.e. permutations that cyclically permute three elements and leave the other ones fixed. In fact* A_n *consists of all three-cycles and their products.*

Exercise 2.7. *Prove Theorem 2.13 by showing that if a normal subgroup* N *of* A_n *contains a permutation which cyclically permutes three elements and leaves the others fixed, then it contains all such permutations and hence is the alternating group itself, see Exercise 2.6. Then show that from any element of* N*, unless* $n = 4$*, one can derive an element in* N *which is the product of disjoint three cycles and use induction.*

Exercise 2.8. *Show that for* $n \neq 6$ *the group of automorphisms of* S_n *is isomorphic to* S_n*. What is the group of automorphisms of* S_6*?*

Exercise 2.9. *Let* g *denote a generator and* $\bar{g}$ *its inverse. An element of the free group on these generators is a word* $w = g_1 g_2 \ldots g_w$*, with* $g_i \neq \bar{g_{i+1}}$ *for all subscripts. The product of two elements* $a = a_1 \ldots a_i c_1 \ldots c_k$ *and* $b = \bar{c_k} \ldots \bar{c_1} b_1 \ldots b_j$ *where* $a_i \neq \bar{b_1}$ *is defined as* $ab = a_1 \ldots a_i b_1 \ldots b_j$*. Show that the associative law holds for this product.*

Exercise 2.10. *Show that the kernel of a group homomorphism* $\phi : G \to\to H$ *is a normal subgroup of* H*.*

Exercise 2.11. *Show that for each natural number* n *there exists at least one group of order* n*.*

Exercise 2.12. *Let* x *be a group element of order* k *in a finite group* G *of order* n*. Show that* n *is divisible by* k*.*

Exercise 2.13. *Let* G *be a finite group of order* n *that has only two subgroups. Prove that* n *is prime.*

Exercise 2.14. *Show that for a prime* p*, all groups of order* p *are isomorphic. Hence up to isomorphism, there is only one group of order* p*.*

Exercise 2.15. *Draw all nine finite 3-connected planar edge-transitive maps.*

Exercise 2.16. *Show that for any map* M *the one-dimensional subdivision map* $Su1(M)$ *is bipartite.*

Project 2.17. *Implement Schlegel diagram centered in a vertex and in an edge.*

Exercise 2.18. *Is it possible to derive octahemioctahedron from the tetrahedron by a series of map operations described in this chapter?*

Exercise 2.19. *Using the Euler formula for the sphere prove that any fullerene has exactly 12 pentagons.*

Project 2.20. *Generalize the approach we took for Archimedean polyhedra in order to construct all uniform polyhedra from the tetrahedron by applying a series of operations on maps.*

Project 2.21. *Implement the chamfering operation.*

Exercise 2.22. *True or false: Smallest* n *such that* $L(n) = L(H(n))$ *4-valent,* $\frac{1}{2}$*-arc-transitive graph is* $n = 4105$*?*

Exercise 2.23. *Determine all vertex-transitive fullerenes. Which of them are edge-transitive?*

Exercise 2.24. *Determine the group of automorphisms of order 336 of the Heawood graph.*

Exercise 2.25. *Can you find an example of a torusene that is not isomorphic to a cyclic Haar graph?*

Exercise 2.26. *Note that there is a simple criterion for checking whether a vertex- and edge-transitive graph is* $\frac{1}{2}$*-arc-transitive; namely, that is the case if and only if no automorphism of the graph interchanges the endvertices of some edge.*

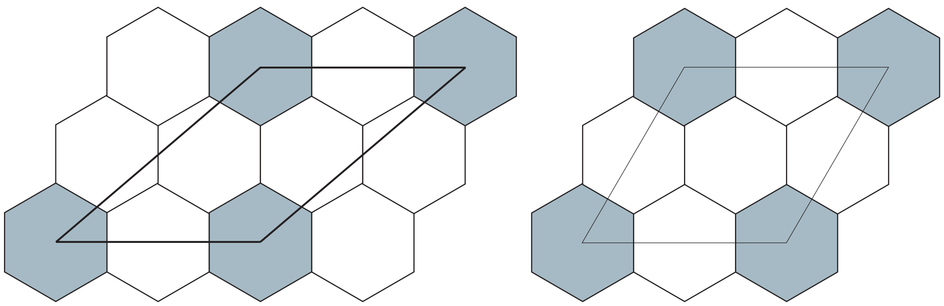

Figure 2.50: The cube Q_3 (Two nonequivalent embeddings $H(2,2,1)$ and $H(2,2,0)$.).

Exercise 2.27. *Show that Whitney's theorem does not hold for torus. Hint: check the embeddings on Figure 2.50.*

IS THIS THE RIGHT PLACE

Exercise 2.28. *Show that the Heawod graph admits a descrption via LCF notation. Show that it is isomorphic to the graph* $[(5,-5)^7]$.

Exercise 2.29. *Show that the Tutte 8 cage is isomorphic to the graph*

$$[(-7,9,13,-13,-9,7)^5].$$

Exercise 2.30. *Show that the Balaban's 10-cage is hamiltonian and find an LCF notation for it.*

Exercise 2.31. *Describe the operations of vertex splitting and edge contraction in* M *as operations in* M^*.

Exercise 2.32. *Write the permutation* M *in cycle notation on the set of flags for the tetrahexahedron map. Do the same for the octahedron, and find* P^* *in each case.*

Exercise 2.33. *Find an embedding of the octahedron on the torus. Is the embedding unique.*

Exercise 2.34. *Find the duals of the cube embedded on the torus as in Figure 2.33. Are these duals isomorphic as maps? As graphs?*

Exercise 2.35. *Construct the Levi graph of a unitary map consisting of* k *handles and show that it is bipartite.*

Exercise 2.36. *Using the face the Petrie walks are the orbits generated by* $\{\theta\phi, P\theta\}$*, list and trace all Petrie walks for the maps in Figure 2.33 and all other maps presented in this section.*

Exercise 2.37. *Show that the girth of the Gray graph is 8.*

Exercise 2.38. *Show that the Gray graph is the Levi graph of two dual, triangle-free, point-, line- and flag-transitive, non-self-dual* 27_3*-configurations.*

Exercise 2.39. *Show that the two distance sequences of the Gray graph are a* $(1, 3, 6, 12, 12, 12, 8)$ *and* $(1, 3, 6, 12, 16, 12, 4)$.

Exercise 2.40. *Prove that the diameter of the Gray graph is 6.*

Exercise 2.41. *Prove that the order of the automorphism group of the Gray graph is* $1296 = 2^4 3^4$.

Exercise 2.42. *Prove that there are a total of 81 octagons, that is induced cycles of length 8, in the Gray graph and that each of its vertices is contained in 12 octagons and each of its edges is contained in 8 octagons.*

Exercise 2.43. *Let* $S(3)$ *denote an arbitrary Sylow 3-subgroup of the authomorphism group of the Gray graph. Show that* $S(3) \cong \mathbb{Z}_3^3 \rtimes \mathbb{Z}_3$ *and that* $S(3)$ *acts transitively on the edge set of* $\mathcal{G}$ *as well as on the sets of black and white vertices.*

Exercise 2.44. *Show that the line graph of the Gray graph is isomorphic to a a Cayley graph for the group* $S(3) = \mathbb{Z}_3^3 \rtimes \mathbb{Z}_3$.

Exercise 2.45. *[This construction is due to Randić (but see also [**?**])]. Show that* $\mathcal{G}$ *can be given in the LCF notation [30], as the graph with code* $[7, -7, 13, -13, 25, -25]^9$ *(see Figure 2.52).*

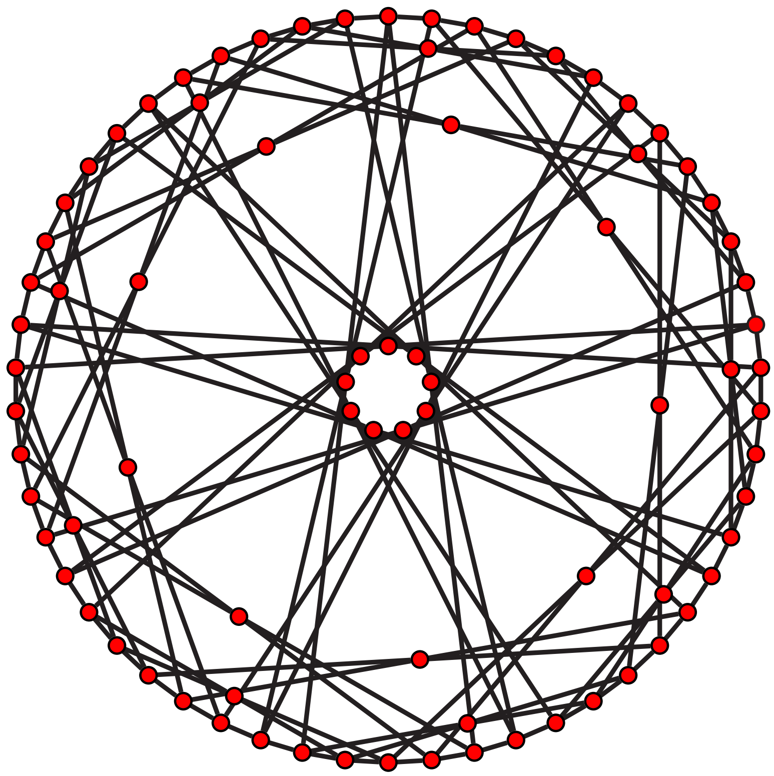

Figure 2.51: The line graph of the Gray graph.

Exercise 2.46. *Prove that $\mathcal{G}$ is a regular cover of $K_{3,3}$ with $\mathbb{Z}_3^2$ as the group of covering transformations (see [**?**] for notation and terminology). More precisely, letting $\sigma = (1,0)$ and $\tau = (0,1)$ be the two generators of $\mathbb{Z}_3^2$, then Figure 2.53 gives the corresponding voltage graph $K_{3,3}$.*

Exercise 2.47. *Describe the operations of vertex splitting and edge contraction in M as operations in M^*.*

Exercise 2.48. *Write the permutation M in cycle notation on the set of flags for the tetrahexahedron map. Do the same for the octahedron, and find*

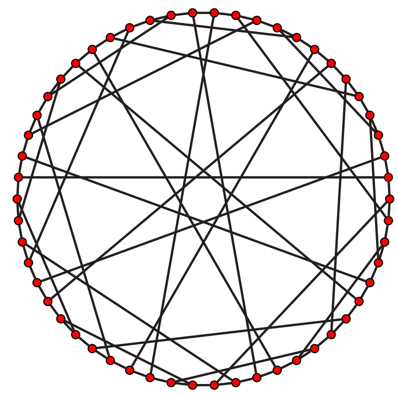

Figure 2.52: The Gray graph with an identified Hamilton cycle admitting a $\mathbb{Z}_9$.

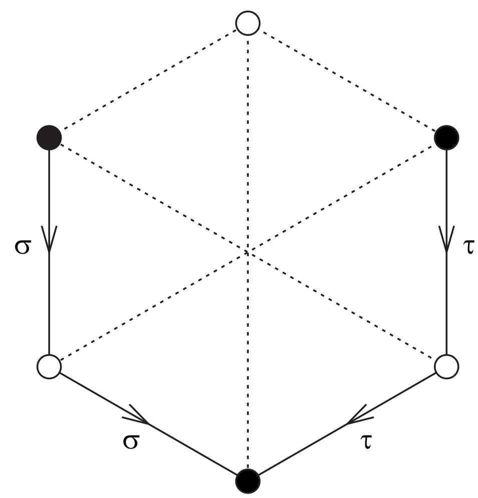

Figure 2.53: The Gray graph is a $\mathbb{Z}_3^2$-regular cover of $K_{3,3}$. The broken lines carry identity voltages $(0,0)$, whereas $\sigma = (1,0), \tau = (0,1)$. Black and white vertices of $K_{3,3}$ lift to the vertices with distance sequences $(1,3,6,12,12,12,8)$ and $(1,3,6,12,16,12,4)$ respectively.

P^* in each case.

Exercise 2.49. *Find an embedding of the octahedron on the torus. Is the*

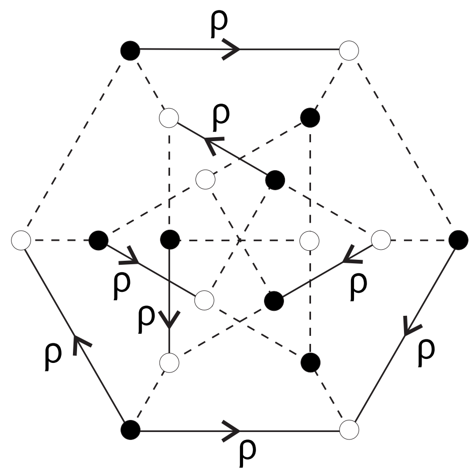

Figure 2.54: A $\mathbb{Z}_3$-vlotage graph over the Pappus graph $\mathcal{P}$. The broken lines carry identity voltages 0, whereas $\rho = 1 \in ZZ_3$.

embedding unique.

Exercise 2.50. *Find the duals of the cube embedded on the torus as in Figure 2.33. Are these duals isomorphic as maps? As graphs?*

Exercise 2.51. *Construct the flag graph of a unitary map consisting of k handles and show that it is bipartite.*

Exercise 2.52. *Using the face the Petrie walks are the orbits generated by $\{\theta\phi, P\theta\}$, list and trace all Petrie walks for the maps in Figure 2.33 and all other maps presented in this section.*

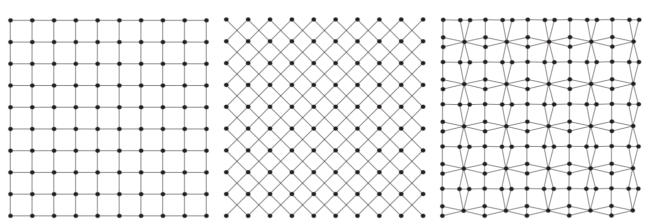

Figure 2.55: The grid and its mysterious transformation.

Exercise 2.53. *Describe the operation on maps that transforms the squares of the grid on the left of Figure 2.55 in order to obtain the map that looks locally like the one on the right of the same figure. How many matrices are needed for the description of this mysterious operation.*

Exercise 2.54. *Repeat the previous exercise by replacing the right map of Figure 2.55 with the middle one.*

Exercise 2.55. *Let* M *be a map. Define* $Go(M) := Du(Me(Tr(M)))$. *Draw* $Go(M)$ *if* M *is locally a square grid.*

General Index

References

[1] B. Alspach, C. C. Chen, and K. McAvane. On a class of hamilton laceable 3-regular graphs. *Discrete Math.*, 151:19–38, 1996.

[2] B. Alspach and C. Q. Zhang. Hamilton cycles in cubic cayley graphs on dihedral groups. *Ars. Combin.*, 28:101–108, 1989.

[3] E. Merlin (and E. Steinitz). Configurations. *Encyclopédie des Sciences Mathematiques, Édition française, Tom III. Vol. 2, 1913, 144-160.*

[4] M. Aschbacher. Finite Group Theory. *Cambridge studies in advanced mathematics 10 Cambridge University Press, 1993.*

[5] H. F. Baker. Principles of Geometry; Volume II Plane Geometry - Conics, Circles, Non-Euclidean Geometry. *Cambrige, 1922.*

[6] L. M. Batten. Combinatorics of Finite Geometries, Second ed. *Cambridge University Press, 1997.*

[7] A. Betten, G. Brinkmann, and T. Pisanski. Counting symmetric configurations v_3. *Discrete Appl. Math., 99 (2000) 331–338.*

[8] Anton Betten and Dieter Betten:. Regular linear spaces. *Beiträge zur Algebra und Geometrie 38 (1997), No. 1, 111–124.*

[9] N. Biggs. Algebraic Graph Theory, second edition. *Cambridge Univ. Press, Cambridge, 1993.*

[10] M. Boben, T. Pisanski, and Coordinatizations of the configurations arising from the 10-Cages. `http://www.ijp.si/balaban/balweb.nb`.

[11] M. Boben, T. Pisanski, Polycyclic configurations, submitted.

[12] J. Bokowski and B. Strumfels. Computational Synthetic Geometry. *LNM 1355, Springer-Verlag, 1989.*

[13] I. Z. Bouwer. An edge but not vertex transitive cubic graph. *Bull. Can. Math. Soc. 11 (1968), 533–535.*

[14] I. Z. Bouwer. Vertex and edge-transitive but not 1-transitive graphs. *Canad. Math. Bull., 13 (1970), 231–237.*

[15] G. Brinkmann, B. D. McKay, , and C. Saager. The smallest cubic graph of girth 9. *Combinatorics, Probability and Computing, 5 (1995) 1–13.*

[16] F. Buekenhout. Foundations of incidence geometry. *in Handbook of Incidence Geometry, (F. Buekenhout, ed.) Elsevier, Amsterdam, (1995) 63–105.*

[17] P.J. Cameron and J.H. van Lint:. Designs, Graphs, Codes and their Links,. *Cambridge Univ. Press, Cambridge, 1991.*

[18] H. S. M. Coxeter. Projective Geometry. *Blaisdell Publishing Company, New York, 1964.*

[19] H. S. M. Coxeter, R. Frucht, , and D. L. Powers. Zero-Symmetric Graphs. *Academic Press, New York, 1981.*

[20] H. S. M. Coxeter and W. O. J. Moser. Generators and Relators for Discrete Groups, fourth edition. *Springer-Verlag, 1980.*

[21] H.S.M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc. 56, (1950) 413–455.*

[22] P. Dembowski. Finite Geometries. *Ergebnisse der Mathematik und ihre Greenzgebiete, Bd. 44 Springer-Verlag, New York, 1968.*

[23] A. Deza, M. Deza, and V. Grishukin. Fullerenes and coornidation polyhedra versus half-cube embeddings. *Discrete Math., 192 (1998) 41–80.*

[24] S. F. Du, D. Marušič, , and A. O. Waller. On 2-arc-transitive covers of complete graphs. *J. Combin. Theory B, to appear.*

[25] F. Buekenhout (ed.). Handbook of Incidence Geometry. *Elsevier, Amsterdam, 1995.*

[26] T. Pisanski (ed.):. Vega Version 0.2; Quick Reference Manual and Vega Graph Gallery,. *IMFM, Ljubljana, 1995.*`http://vega.ijp.si/`.

[27] I. Z. Bouwer et al., editor. *The Foster Census.* The Charles Babbage Research Centre, Winnipeg, Canada, 1988.

[28] J. Folkman. Regular line-symmetric graphs. *J. Combin. Theory, 3 (1967), 215–232.*

[29] P. Fowler and T. Pisanski. Leapfrog transformation and polyhedra of clar type. *J. Chem. Soc. Faraday Trans., 90 (1994) 2865–2871.*

[30] R. Frucht. A canonical representation of trivalent hamiltonian graphs. *J. Graph Theory, 1 (1977) 45–60.*

[31] R. Frucht, J. E. Graver, , and M. E. Watkins. The groups of the generalized petersen graphs. *Proc. Cambridge Philos. Soc., 70 (1971) 211–218.*

[32] David G. Glynn. On the anti-Pappian 10_3 and its construction. *Geom. Dedicata*, 77(1):71–75, 1999.

[33] Chris Godsil and Gordon Royle. Algebraic Graph Theory. *Springer, New York, 2001.*

[34] H. Gropp. Blocking set free configurations and their relations to digraphs and hypergraphs. *Discr. Math. 165/166 (1997), 359-370.*

[35] H. Gropp. Configurations. *In C. J. Colburn and J. H. Dinitz, editors The CRC Handbook of Combinatorial Designs CRC Press LLC, Boca Raton (1996), 253–255.*

[36] H. Gropp. Configurations and graphs. *Discr. Math. 111 (1993), 269–276.*

[37] H. Gropp. Configurations and graphs—ii. *Discr. Math. 164 (1997), 155–163.*

[38] H. Gropp. Configurations and their realizations. *Discrete Math. 174:137–151, 1997.*

[39] H. Gropp. The construction of all configurations $(12_4, 16_3)$. *In J. Nešetřil and M. Fiedler, editors, Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity, Elsevier, (1992) 85–91.*

[40] H. Gropp. The drawing of configurations. *Lecture Notes in Computer Science 1027 , F.J. Brandenburg (ed). Graph Drawing GD'95 Proceedings, Springer Verlag (1995), 267–276.*

[41] H. Gropp. On the existence and nonexistence of configurations n_k. *Graphs, designs and combinatorial geometries (Catania, 1989) J. Combin. Inform.System Sci.15 (1990), no.1-4, 34–48.*

[42] H. Gropp. On the history of configurations. *In A. Deza, J. Echeverria and A. Ibarra, editors Internat. Symposium on Structures in Math. Theories, Bilbao, (1990) 263–268.*

[43] J. L. Gross and T. W. Tucker:. Topological Graph Thoery,. *Wiley Intersicence, 1987.*

[44] B. Grünbaum. Astral (n_k) configrations. *Geombinatorics 3 (1993) 32–37.*

[45] B. Grünbaum. Configurations. *Special Topics in Geometry, Math 553B University of Washington, Spring 1999.*

[46] B. Grünbaum. Connected (n_4) configrations exist for almost all", journal = "n. geombinatorics 9 (2000) 24–29.

[47] M. Hladnik. Schur norms of bicirculant matrices. *Linear Alg. Appl. 286 (1999) 261–272.*

[48] M. Hladnik and T. Pisanski. Schur norms of haar graphs. *work in progress.*

[49] Wilfried Imrich and Sandi Klavžar. Product Graphs. *2000.*

[50] D. J. Klein. Graph geometry, graph metrics, and wiener. *MATCH, 35 (1997) 7–27.*

[51] F. Levi. Geometrische konfigurationen. *Leipzig, 1929.*

[52] M. Lovrečič-Saražin. A note on the generalized petersen graphs that are also cayley graphs. *J. Combin. Theory (B), 69 (1997) 226–229.*

[53] R. E. Maeder. Uniform polyhedra. *The Mathematica Journal, 3(4) (1993) 48–57.*

[54] D. Marušič. Half-transitive group actions on finite graphs of valency 4. *J. Combin. Theory, B 73 (1998) 41–76.*

[55] D. Marušič and M.-Y. Xu. A 1/2-transitive graphs of valency 4 with a nonsolvable group of automorphisms. *J. Graph Theory 25 (1997), 133–138.*

[56] M. Meringer. Fast generation of regular graphs and construction of cages. *J. Graph Theory 30 (1999), 137–146.*

[57] K. Metsch:. Linear Spaces with Few Lines,. *LNM 1490, Springer Verlag, 1991.*

[58] B. Mohar and P. Rosenstiehl. Tessellation and visibility representations of maps on the torus. *Discrete Comput. Geom., 19(2):249–263, 1998.*

[59] R. Nedela and M. Škoviera. Which generalized petersen graphs are cayley graphs. *J. Graph Theory, 19 (1995) 1–11.*

[60] T. Pisanski and A. Malnič. The diagonal construction and graph embeddings. *In Proceedings of the Fourth Yugoslav Seminar on Graph Theory pages 271–290. Novi Sad, 1983.*

[61] T. Pisanski and M. Randić. Bridges between geometry and graph theory. *to appear.*

[62] T. Pisanski, J. Shawe-Taylor, and J. Vrabec. Edge-coloring of graph bundles. *J. Combin. Theory B 35(198?)12–19.*

[63] T. Pisanski, A. Žitnik, A. Graovac, , and A. Baumgartner. Rotagraphs and their generalizations. *J. Chem. Inf. Comp. Sci. 34 (1994) 1090-1093.*

[64] B. Polster:. A geometrical picture book. *Springer, 1998.*

[65] D. L. Powers. Exceptional trivalent cayley graphs for dihedral groups. *J. Graph Theory, 6 (1982) 43–55.*

[66] Th. Reye.

[67] G. Ringel. Map Color Theorem. *Springer-Verlag, Berlin, 1974.*

[68] Doris Schattschneider. Escher's combinatorial patterns. *Electronic Journal of Combinatorics, 4 (1997), R#17.*

[69] A. Schroth. How to draw a hexagon. *Discrete Math. 199 (1999) 161 - 171.*

[70] E. Steinitz. Konfigurationen der projektiven geometrie. *Encyklop. Math. Wiss. 3 (Geometrie) (1910) 481–516.*

[71] E. Steinitz. *Über die Construction der Configurationen* n_3. PhD thesis, Kgl. Universität Breslau, 1894.

[72] B. Sturmfels and N. White. All 11_3 and 12_3 configurations are rational. *Aequationes Mathematicae, 39 (1990) 254-260.*

[73] D. Surowski. The Möbius-Kantor Regular Map of Genus Two and Regular Ramified Coverings. *presented at SIGMAC 98, Flagstaff, AZ, July 20-24, 1998* `http://odin.math.nau.edu:80/s̃ew/sigmac.html`.

[74] J. Tits. Buildings of spherical type and finite bn-pairs. *Lecture Notes in Math.382, Berlin, 1974. Springer–Verlag.*

[75] T. W. Tucker. There is only one group of genus two. *J. Combin. Theory B, 36 (1984) 269–275.*

[76] W. T. Tutte. Connectivity in graphs. *University of Toronto Press, Toronto, 1966.*

[77] W. T. Tutte. A family of cubical graphs. *Proc. Camb. Phil. Soc., 43 (1948), 459–474.*

[78] R. Daublebsky von Sterneck. Die configurationen 11_3. *Monatsh. Math. Physik, 5 (1894) 325–330.*

[79] R. Daublebsky von Sterneck. Die configurationen 12_3. *Monatsh. Math. Physik, 6 (1895) 223–255.*

[80] M. Škoviera and J. Širáň. Regular maps from cayley graphs, part 1: Balanced cayley maps. *Discrete Math., 109 (1992) 265–276.*

[81] D. Wells. The Penguin Dictionary of Curious and Interesting Geometry. *Penguin Books, London, 1991.*

[82] A. White. Graphs, Groups, and Surfaces. *North-Holland Pub. Co., Amsterdam, 1973*.

[83] E. Steinitz (with H. Rademacher). Vorlesungen über die Theorie der Polyeder. *Springer-Verlag, Berlin, 1934*.

[84] P. K. Wong. Cages—a survey. *J. Graph Theory, 6:1–22, 1982*.