CONFIGURATIONS
From a graphical viewpoint

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The first author used the notes as a reading material for students of a graduate course *Konfiguracije in grafi* (Configurations and Graphs) at the University of Ljubljana, Slovenia, 1999–2000.

Several people contributed to these notes. *Aleksandar Jurišić* suggested several important references. Several co-authors allowed him to include parts of our joint work verbatim in the earlier versions of the manuscript. We are grateful to *Anton Betten, Marko Boben, Gunnar Brinkmann, Marston Conder, Milan Hladnik, Ante Graovac*, *Harald Gropp*, *Aleksander Malnič*, *Dragan Marušič, Alen Orbanić*, *Marko Petkovšek*, *Primož Potočnik, Milan Randić, Steve Wilson,* and *Arjana Žitnik*, for that. We would like to thank *Marko Boben*, *Jürgen Bokowski*, *Chris Godsil*, *Josef Leydold*, *Alen Orbanić*, *Marko Petkovšek*, *Bor Plestenjak*, *John Shawe-Taylor*, for reading several versions of the manuscript or prats of the manuscript, giving some very interesting suggestions.

After finishing teaching the course T.P. ceased working on the manuscript for about seven months. When he started preparing these notes he was unaware of some of the most recent work on configurations of Branko Grünbaum. We would like to thank him for sharing his work with us. After receiving his notes for a graduate course on configurations [45] T.P. resumed his work on the manuscript in January 2001 and began adjusting terminology with that of Grünbaum in all feasible cases.

In 2001-2003 he used part of these notes for another graduate course on configurartions for mathematics teachers.

In 2002-2003 Brigitte Servatius spent her sabbatical in Ljubljana. We decided to turn these notes into a book.

The Fall semester of 2003 T.P. spent at Colgate University which enabled further communication.

The work was greatly simplified by using crossplatform *cvs* system as proposed by *Andrej Bauer. Herman Servatius* helped us a lot not only by drawing a number of figures but also with his expertise as a mathematician and as expert latex user.

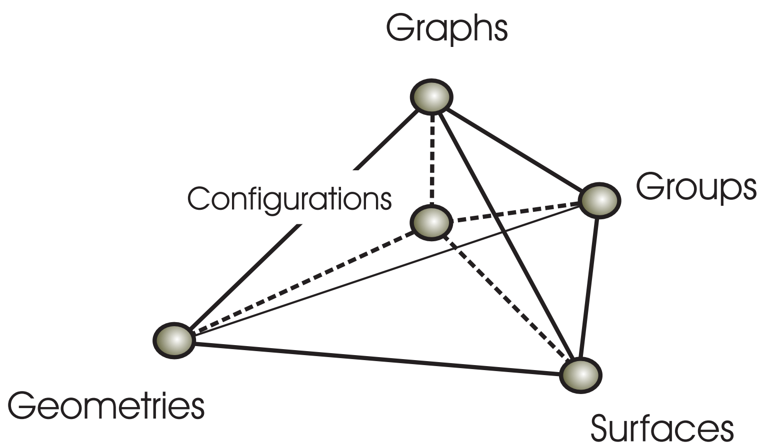

Figure 1: Configurations lie in the heart of graphs, groups, sutfaces and geometries.

Foreword

When I first learned about sets in high school my view of a static mathematics collapsed. In elementary school I was convinced that most mathematics was known to the old Greeks and was sure that anything important was discovered long time ago.

I was thrilled when I was told that sets are at the basis of modern mathematics and was shocked to learn that the set of all sets does not exist. Later in college we were informed that categories are about to replace sets in the foundations of mathematics. When I discovered for myself that graphs are more general than categories I was glad that I have chosen such an important concept which is related to the roots of mathematics as the prime object of my studies.

Roughly and very informally speaking one can divide graphs just like real numbers or most other mathematical concepts into two classes: there is a large volume of *dull* graphs and a small portion of *interesting* graphs. Let us try to clarify the classes of *dull* and *intersting objects* first with real numbers. How to distinguish interesting reals from the dull ones? Maybe we should define interesting reals as those real numbers that admit finite description. Since there are at most countably many finite descriptions, it is not hard to see that the vast majority of reals are dull. For graphs we have to modify this definition, otherwise all finite graphs would qualify as interesting. Any graph on n vertices can be described by its adjacency matrix using about n^2 bits. One can say that a graph is interesting if it admits a short description; for intstance, if it can be described by at most $f(n)$ bits of information where f is an increasing function but the ratio $f(n)/n^2$ tends to 0 when n tends to infinity.

The research in graph theory can be classified into two classes:

- Class I. Find interesting properties of (dull) graphs.

- Class II. Study families of interesting graphs.

A typical result of Class I is the fact that there is no disconnected graph whose complement is also disconnected. On the other hand a Class II result is the fact that there are exactly seven arc-transitive generalized Petersen graphs. These notes would qualify under Class II philosophy. We not only look at some finite families of graphs but consider individual graphs with unique properties.

The final question. Why would one like to revive the interest in configurations, a subject that used to be very much alive in the second half of the nineteenth century? A partial answer lies in the following analogy. Graphs are used to model other mathematical structures. Such graphs reflect properties of the structures they model and this makes them sometimes very interesting. For instance groups are modeled by their Cayley graphs. In a similar way geometries and configurations can be modeled by their Levi graphs. That is why we may expect that several configurations will give rise to some very interesting graphs too.

The more I wrote these notes the more it became obvious to me that we can paraphrase the key moto of Art White's book [82] by putting configurations in the centre of a tetrahedron whose four vertices are:

1. Graphs - Basic Graph Theory
2. Groups - Algebraic Graph Theory
3. Surfaces - Topological Graph Theory
4. Geometries - Geometrical Graph Theory

see Figure 1.

This text can be regarded as an ad hoc introduction to the interesting interplay of these topics. The reader will soon find out that configurations play a catalytic role in this book and though considered as the principal actor they do not appear in every scene.

Tomaž Pisanski, Ljubljana, 2002.

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INTRODUCTION

§ 1. Hexagrammum mysticum. In 1640, two years before inventing the first mechanical calculator, Blaise Pascal (1623 – 1662) was 16 years old when he published a small pamphlet entitled *Essai pour les coniques* which contains the following remarkable theorem.

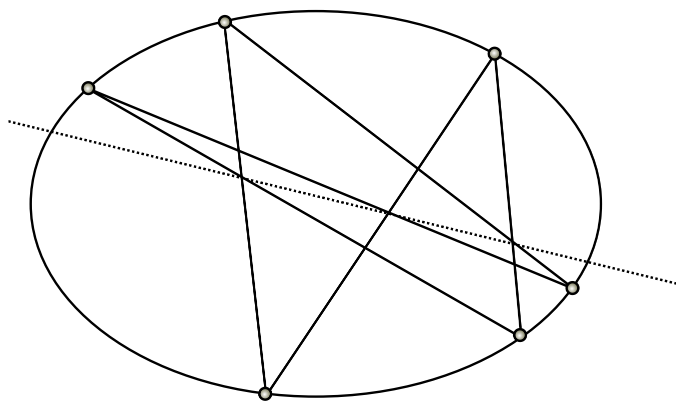


Figure 2: Hexagrammum mysticum.

Theorem 0.1 (Blaise Pascal, 1639). *If a hexagon is inscribed in a conic, then the three points in which pairs of opposite sides meet will lie on a straight line.*

It seems that Pascal proved the theorem in June 1639. According to Coxeter [18] G. W. Leibniz (1646 – 1716) admired Pascal's proof when visiting Paris. Unfortunately the proof was later lost.

The line whose existence is asserted by the above theorem is sometimes called *Pascal line.* Six distinct objects can be arranged in circular order in 60 different ways, so for a given set of 6 points on a conic there are 60 hexagons, which in turn give rise to 60 Pascal lines. Several geometers were attracted by these 60 lines. Kirkman, Cayley, Steiner, Plücker, and Salmon have studied this *figure* or *configuration* which is called *Pascal configuration* by David Wells on page 172 of [81]. H. F. Baker, Sc.D., LL.D., F.R.S., Lowdean Professor of Astronomy and Geometry, and Fellow of St. John's College, at the University of Cambridge provides a list of references to this matter in the second volume of his 6-volume opus [5]. The list includes the following contributors: Pascal(1640), Brianchon (1806), Steiner (1828), Plücker (1830), Hesse (1842), Cayley (1846), Grossman (1861), Von Staudt (1863), Bauer (1874), Kirkman (1849), Veronese (1877), Cremona (1877), Sylvester(18??), Salmon (1879), Klug (1898), Castelnuovo (1887), Caporali (1888), and Richmond (1894). Baker calls the Pascal configuration also *Hexagrammum mysticum.* In addition to 60 Pascal lines it contains 20 Steiner points, 60 Kirkman points, 20 Cayley lines, 15 Plücker lines and 15 Salmon points. It is possible to draw a picture of *Hexagrammum mysticum,* however it is very hard to see the relationships that hold among its various types of points and lines.

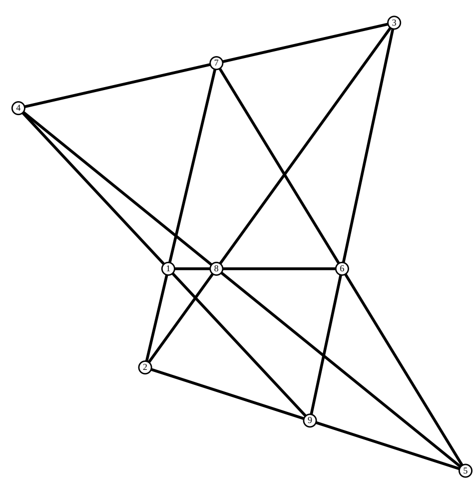
One of the purposes of these notes is to enable the reader to see the Pascal configuration from a different view-point using graphs. The emphasis is thus shifted from geometry to combinatorics. In the first book on configurations, [51], Friederich Levi uses about 50 pages of Chapter 5 in order to study the properties of Pascal's configuration.

§ 2. Why configurations? The topic of configurations seems to be worth studying for various reasons. Many great mathematicians contributed to the development of the theory. Configurations lie at the intersection of various mathematical disciplines. Nowadays they form a part of combinatorics but their origin lies in geometry of the 19$^{\text{th}}$ century. In particular, curves and surfaces of the projective geometry give rise to some important historical configurations. Configurations can be studied from a graph-theoretical viewpoint via the so-called Levi graphs. At the same time they bring geometrical flavor to graphs. Symmetries play an important role in the study of configurations. Recently several new problems have arisen that put configurations in the domain of theoretical computer science. The theory has evolved from various sources and viewpoints which makes the objects of our interest visible in quite diverse lights.

The central theme of modern combinatorics is probably the concept of *combinatorial design* which is in our view too narrow a structure. If we want to embrace all configurations that were studied in the past we have to consider more general combinatorial structures. Perhaps the most general structure that is available is the idea of geometry in the sense of Tits [74] .

It covers all structures that are needed in these notes. Although it represents a deep and powerful tool in connection with groups it is by itself nothing but a graph endowed with a vertex coloring and is readily available to anyone with basic knowledge of combinatorics and graph theory.

The fundamental question about configurations, that is, the question about their existence was posed by T. Reye [66]. This question has several refinements. For instance one can ask not only which *combinatorial configurations* exist but also which *geometric configurations* of points and lines exist. A more complete answer is provided with the list of all non-isomorphic configurations of a given type. Quite recently Branko Grünbaum [46] gave a surprising answer to the problem of configurations for the symmetric case, by proving that for any set of large parameters one can construct a symmetric geometric configuration with those parameters.

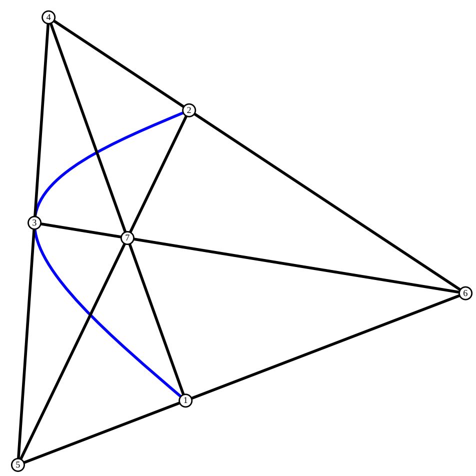


1	2	3	4	5	6	7	8	9
8	7	2	1	6	9	3	4	5
6	1	8	9	7	3	4	5	2

Figure 3: Configuration table for the Pappus configuration.

§ 3. Examples. Let us give several examples of configurations in order to motivate the reader.

Example 0.2. *Figure 3 depicts a* configuration *of* points *and* lines*, known as the* Pappus configuration*.*

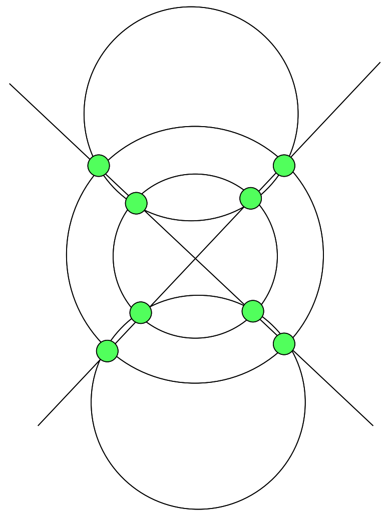


1	2	3	4	5	6	7
5	7	2	1	3	4	6
6	5	1	7	4	2	3

Figure 4: Combinatorial configuration (7_3), the Fano plane.

Example 0.3. *Figure 4 depicts a* configuration *known as the* Fano configuration*. It is defined by its configuration table and can be represented in the ordinary Euclidean plane by 7 points, 6 lines and one curve.*

Example 0.4. *Figure 5 depicts a* configuration *known as the* Miquel configuration*. It consists of 8 points, 6 circles (and lines).*



1	3	1	5	1	2
2	4	2	6	3	4
5	7	3	7	5	6
6	8	4	8	7	8

Figure 5: Miquel configuration of points and circles.

Although Miquel configuration is "geometric" in the sense that is defined by geometric objects and their incidence, it differs from the previous two examples: it cannot be defined as a configuration of points and lines since two circles may intersect more than once! Its designation is $(8_3, 6_4)_2$.

Example 0.5. *Finally, consider the tetrahedron depicted in Figure 6 with vertices denoted by* A, B, C, D. *We may denote the edges by* $1 = AB, 2 = AC, 3 = AD, 4 = BC, 5 = BD, 6 = CD$ *and faces by* $\alpha = BCD, \beta = ACD, \gamma = ABD, \delta = ABC$.

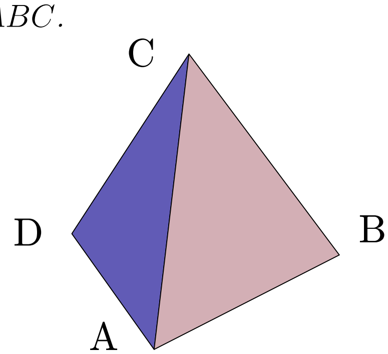


Figure 6: The tetrahedron gives rise to six configurations.

We can consider configurations defined by

- *vertices and edges,*
- *vertices and faces,*
- *edges and faces,*
- *edges and vertices,*
- *faces and vertices,*
- *faces and edges.*

The situation can be described in a tabular form.

	$v = 4$	$e = 6$	$f = 4$
$v = 4$	0	3	3
$e = 6$	2	0	2
$f = 4$	3	3	0

For instance, entry 2 in the second row ($e = 6$) and first column ($v = 2$) tells us that there are 2 vertices at each edge.

Here is the configuration table for the configuration of points and faces

α	β	γ	δ
B	A	A	A
C	C	B	B
D	D	D	C

0.1 Permutations.

We would like to recall a few facts about permutations. A *permutation* on the set V is a bijection $\pi : V \to V$ of V on itself. The set of all permutations on V is denoted by $\mathrm{Sym}(V)$. Usually we consider the "standard set" $V = \{1, 2, ..., n\}$. In this case we write $\mathrm{Sym}(V) = \mathrm{Sym}(n) = S_n$. Usually we write a permutation in the form

$$\pi = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ \pi(v_1) & \pi(v_2) & \cdots & \pi(v_n) \end{pmatrix}.$$

If we have the standard set, the notation can be shortened and only the second row is given.

$$\pi = \begin{bmatrix} \pi(1) & \pi(2) & \cdots & \pi(n) \end{bmatrix}.$$

Example 0.6. *Each row of the configuration table of Pappus configuration can be viewed as a permutation.*

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 7 & 2 & 1 & 6 & 9 & 3 & 4 & 5 \end{pmatrix}.$$

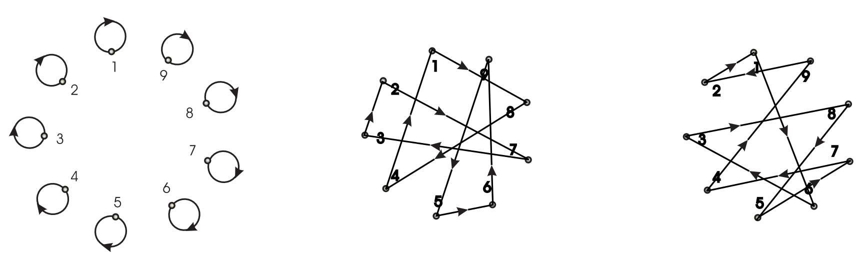
$$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 1 & 8 & 9 & 7 & 3 & 4 & 5 & 2 \end{pmatrix}.$$

Recall that each permutation can be written in a unique way as a product of disjoint cycles:

$$\alpha_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9)$$
$$\alpha_2 = (184)(273)(569)$$
$$\alpha_3 = (16385749)$$

Permutation consisting of a single cycle is called cyclic permutation. *Hence α_3 is cyclic permutation. If all cycles are of the same length, the permutation is called* polycyclic. *If all cycles are of length 1, the permutation is called the* identity permutation. *An element $x \in V$ for which $\pi(x) = x$ is called a* fixed point *of π. Let $Fix(\pi)$ denote: $Fix(\pi) = \{x \in V | \pi(x) = x\}$ and let $fix(\pi) = |Fix(\pi)|$. Hence $fix(\alpha_1) = 9, fix(\alpha_2) = fix(\alpha_3) = 0$. A fixed-point free permutation is also called a* derangement. *The set of derangements over V is denoted by $Der(V)$. If seen as a subset of S_n it is denoted by D_n. A permutation whose longest cycle has length 2 is called an* involution. *If, in addition, it has no fixed points, it is called a* fixed-point free involution.

Each permutation can be depicted in graphic form. Each element $x \in V$ is represented by a little circle and we draw an arrow from x to $\pi(x)$.



α_1 α_2 α_3

Figure 7: Permutations $\alpha_1, \alpha_2, \alpha_3$.

0.2 Exercises

For a warm-up we give the following problems. Problems denoted by (*) are more difficult and usually require material that was not covered up to this point. Problems denoted by (**) represent research problems or very difficult problems.

Exercise 0.1. *How many points and lines does the Pappus configuration have?*

Exercise 0.2. *How many lines pass through a given point of the Pappus configuration?*

Exercise 0.3. *How many points lie on a given line of the Pappus configuration?*

Exercise 0.4. *What is the minimum and what is the maximum number of lines passing through a pair of distinct points of the Pappus configuration?*

Exercise 0.5 (*). *Plant 9 trees in 10 rows in such a way that there are 3 trees in each of the 10 rows.*

Exercise 0.6 (*). *Prove Pascal's Theorem for the case when the conic is circle.*

Exercise 0.7 (*). *Derive the Pappus configuration from Pascal's Theorem by considering two lines as a degenerate conic.*

Exercise 0.8. *Determine the lines that pass through the point labelled 3 of the Fano configuration.*

Exercise 0.9. *How many lines pass through points 4 and 6 of the Fano configuration?*

Exercise 0.10. *How many lines pass through points 1,2, and 3 of the Fano configuration?*

Exercise 0.11 (*). *Is it possible to draw Fano's configuration with straight lines?*

Exercise 0.12. *Write down the configuration tables for the six configurations determined by the tetrahedron. Verify that the point-line and face-line configurations have essentially the same configuration table. Verify that the point-face and face-point configuration tables are essentially the same.*

Exercise 0.13. *In a presidential election in some small country there are* 8 *presidential candidates. Schedule a series of* 8 *TV debates in such a way, that there are three presidential candidates in each debate and no two candidates meet in the studio more than once. Show that the solution is essentially unique.*

Exercise 0.14. *Write down all permutations from* S_3.

Exercise 0.15. *Write down all involutions from* S_4.

Exercise 0.16. *Write down all fixed-point free permutations from* S_5. *In other words, determine the set* D_5.

Exercise 0.17. *Write down all fixed-point free involutions from* S_4 *and from* S_5.

Exercise 0.18. *A permutation is called* semi-regular, *if it is a product of cycles of the same size. Determine the number of semi-regular permutations from* S_{2005}.

Exercise 0.19. *Let* p *be a prime. Show that the number of semi-regular permutations from* S_p *equals* $(p-1)!$

Exercise 0.20. *(*) Determine the number of derangements in* D_n.

Exercise 0.21. *(*) Determine the number of involutions from* S_n.

Exercise 0.22. *(*) Determine the number of fixed-point free involutions from* S_n.

General Index

References

[1] B. Alspach, C. C. Chen, and K. McAvane. On a class of hamilton laceable 3-regular graphs. *Discrete Math.*, 151:19–38, 1996.

[2] B. Alspach and C. Q. Zhang. Hamilton cycles in cubic cayley graphs on dihedral groups. *Ars. Combin.*, 28:101–108, 1989.

[3] E. Merlin (and E. Steinitz). Configurations. *Encyclopédie des Sciences Mathematiques, Édition française, Tom III. Vol. 2, 1913, 144-160.*

[4] M. Aschbacher. Finite Group Theory. *Cambridge studies in advanced mathematics 10 Cambridge University Press, 1993.*

[5] H. F. Baker. Principles of Geometry; Volume II Plane Geometry - Conics, Circles, Non-Euclidean Geometry. *Cambrige, 1922.*

[6] L. M. Batten. Combinatorics of Finite Geometries, Second ed. *Cambridge University Press, 1997.*

[7] A. Betten, G. Brinkmann, and T. Pisanski. Counting symmetric configurations v_3. *Discrete Appl. Math., 99 (2000) 331–338.*

[8] Anton Betten and Dieter Betten:. Regular linear spaces. *Beiträge zur Algebra und Geometrie 38 (1997), No. 1, 111–124.*

[9] N. Biggs. Algebraic Graph Theory, second edition. *Cambridge Univ. Press, Cambridge, 1993.*

[10] M. Boben, T. Pisanski, and Coordinatizations of the configurations arising from the 10-Cages. `http://www.ijp.si/balaban/balweb.nb`.

[11] M. Boben, T. Pisanski, Polycyclic configurations, submitted.

[12] J. Bokowski and B. Strumfels. Computational Synthetic Geometry. *LNM 1355, Springer-Verlag, 1989.*

[13] I. Z. Bouwer. An edge but not vertex transitive cubic graph. *Bull. Can. Math. Soc. 11 (1968), 533–535.*

[14] I. Z. Bouwer. Vertex and edge-transitive but not 1-transitive graphs. *Canad. Math. Bull., 13 (1970), 231–237.*

[15] G. Brinkmann, B. D. McKay, , and C. Saager. The smallest cubic graph of girth 9. *Combinatorics, Probability and Computing, 5 (1995) 1–13.*

[16] F. Buekenhout. Foundations of incidence geometry. *in Handbook of Incidence Geometry, (F. Buekenhout, ed.) Elsevier, Amsterdam, (1995) 63–105.*

[17] P.J. Cameron and J.H. van Lint:. Designs, Graphs, Codes and their Links,. *Cambridge Univ. Press, Cambridge, 1991.*

[18] H. S. M. Coxeter. Projective Geometry. *Blaisdell Publishing Company, New York, 1964.*

[19] H. S. M. Coxeter, R. Frucht, , and D. L. Powers. Zero-Symmetric Graphs. *Academic Press, New York, 1981.*

[20] H. S. M. Coxeter and W. O. J. Moser. Generators and Relators for Discrete Groups, fourth edition. *Springer-Verlag, 1980.*

[21] H.S.M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc. 56, (1950) 413–455.*

[22] P. Dembowski. Finite Geometries. *Ergebnisse der Mathematik und ihre Greenzgebiete, Bd. 44 Springer-Verlag, New York, 1968.*

[23] A. Deza, M. Deza, and V. Grishukin. Fullerenes and coornidation polyhedra versus half-cube embeddings. *Discrete Math., 192 (1998) 41–80.*

[24] S. F. Du, D. Marušič, , and A. O. Waller. On 2-arc-transitive covers of complete graphs. *J. Combin. Theory B, to appear.*

[25] F. Buekenhout (ed.). Handbook of Incidence Geometry. *Elsevier, Amsterdam, 1995.*

[26] T. Pisanski (ed.):. Vega Version 0.2; Quick Reference Manual and Vega Graph Gallery,. *IMFM, Ljubljana, 1995.*`http://vega.ijp.si/`.

[27] I. Z. Bouwer et al., editor. *The Foster Census.* The Charles Babbage Research Centre, Winnipeg, Canada, 1988.

[28] J. Folkman. Regular line-symmetric graphs. *J. Combin. Theory, 3 (1967), 215–232.*

[29] P. Fowler and T. Pisanski. Leapfrog transformation and polyhedra of clar type. *J. Chem. Soc. Faraday Trans., 90 (1994) 2865–2871.*

[30] R. Frucht. A canonical representation of trivalent hamiltonian graphs. *J. Graph Theory, 1 (1977) 45–60.*

[31] R. Frucht, J. E. Graver, , and M. E. Watkins. The groups of the generalized petersen graphs. *Proc. Cambridge Philos. Soc., 70 (1971) 211–218.*

[32] David G. Glynn. On the anti-Pappian 10_3 and its construction. *Geom. Dedicata*, 77(1):71–75, 1999.

[33] Chris Godsil and Gordon Royle. Algebraic Graph Theory. *Springer, New York, 2001.*

[34] H. Gropp. Blocking set free configurations and their relations to digraphs and hypergraphs. *Discr. Math. 165/166 (1997), 359-370.*

[35] H. Gropp. Configurations. *In C. J. Colburn and J. H. Dinitz, editors The CRC Handbook of Combinatorial Designs CRC Press LLC, Boca Raton (1996), 253–255.*

[36] H. Gropp. Configurations and graphs. *Discr. Math. 111 (1993), 269–276.*

[37] H. Gropp. Configurations and graphs—ii. *Discr. Math. 164 (1997), 155–163.*

[38] H. Gropp. Configurations and their realizations. *Discrete Math. 174:137–151, 1997.*

[39] H. Gropp. The construction of all configurations $(12_4, 16_3)$. *In J. Nešetřil and M. Fiedler, editors, Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity, Elsevier, (1992) 85–91.*

[40] H. Gropp. The drawing of configurations. *Lecture Notes in Computer Science 1027 , F.J. Brandenburg (ed). Graph Drawing GD'95 Proceedings, Springer Verlag (1995), 267–276.*

[41] H. Gropp. On the existence and nonexistence of configurations n_k. *Graphs, designs and combinatorial geometries (Catania, 1989) J. Combin. Inform.System Sci.15 (1990), no.1-4, 34–48.*

[42] H. Gropp. On the history of configurations. *In A. Deza, J. Echeverria and A. Ibarra, editors Internat. Symposium on Structures in Math. Theories, Bilbao, (1990) 263–268.*

[43] J. L. Gross and T. W. Tucker:. Topological Graph Thoery,. *Wiley Intersicence, 1987.*

[44] B. Grünbaum. Astral (n_k) configrations. *Geombinatorics 3 (1993) 32–37.*

[45] B. Grünbaum. Configurations. *Special Topics in Geometry, Math 553B University of Washington, Spring 1999.*

[46] B. Grünbaum. Connected (n_4) configrations exist for almost all", journal = "n. geombinatorics 9 (2000) 24–29.

[47] M. Hladnik. Schur norms of bicirculant matrices. *Linear Alg. Appl. 286 (1999) 261–272.*

[48] M. Hladnik and T. Pisanski. Schur norms of haar graphs. *work in progress.*

[49] Wilfried Imrich and Sandi Klavžar. Product Graphs. *2000.*

[50] D. J. Klein. Graph geometry, graph metrics, and wiener. *MATCH, 35 (1997) 7–27.*

[51] F. Levi. Geometrische konfigurationen. *Leipzig, 1929.*

[52] M. Lovrečič-Saražin. A note on the generalized petersen graphs that are also cayley graphs. *J. Combin. Theory (B), 69 (1997) 226–229.*

[53] R. E. Maeder. Uniform polyhedra. *The Mathematica Journal, 3(4) (1993) 48–57.*

[54] D. Marušič. Half-transitive group actions on finite graphs of valency 4. *J. Combin. Theory, B 73 (1998) 41–76.*

[55] D. Marušič and M.-Y. Xu. A 1/2-transitive graphs of valency 4 with a nonsolvable group of automorphisms. *J. Graph Theory 25 (1997), 133–138.*

[56] M. Meringer. Fast generation of regular graphs and construction of cages. *J. Graph Theory 30 (1999), 137–146.*

[57] K. Metsch:. Linear Spaces with Few Lines,. *LNM 1490, Springer Verlag, 1991.*

[58] B. Mohar and P. Rosenstiehl. Tessellation and visibility representations of maps on the torus. *Discrete Comput. Geom., 19(2):249–263, 1998.*

[59] R. Nedela and M. Škoviera. Which generalized petersen graphs are cayley graphs. *J. Graph Theory, 19 (1995) 1–11.*

[60] T. Pisanski and A. Malnič. The diagonal construction and graph embeddings. *In Proceedings of the Fourth Yugoslav Seminar on Graph Theory pages 271–290. Novi Sad, 1983.*

[61] T. Pisanski and M. Randić. Bridges between geometry and graph theory. *to appear.*

[62] T. Pisanski, J. Shawe-Taylor, and J. Vrabec. Edge-coloring of graph bundles. *J. Combin. Theory B 35(198?)12–19.*

[63] T. Pisanski, A. Žitnik, A. Graovac, , and A. Baumgartner. Rotagraphs and their generalizations. *J. Chem. Inf. Comp. Sci. 34 (1994) 1090-1093.*

[64] B. Polster:. A geometrical picture book. *Springer, 1998.*

[65] D. L. Powers. Exceptional trivalent cayley graphs for dihedral groups. *J. Graph Theory, 6 (1982) 43–55.*

[66] Th. Reye.

[67] G. Ringel. Map Color Theorem. *Springer-Verlag, Berlin, 1974.*

[68] Doris Schattschneider. Escher's combinatorial patterns. *Electronic Journal of Combinatorics, 4 (1997), R#17.*

[69] A. Schroth. How to draw a hexagon. *Discrete Math. 199 (1999) 161 - 171.*

[70] E. Steinitz. Konfigurationen der projektiven geometrie. *Encyklop. Math. Wiss. 3 (Geometrie) (1910) 481–516.*

[71] E. Steinitz. *Über die Construction der Configurationen* n_3. PhD thesis, Kgl. Universität Breslau, 1894.

[72] B. Sturmfels and N. White. All 11_3 and 12_3 configurations are rational. *Aequationes Mathematicae, 39 (1990) 254-260.*

[73] D. Surowski. The Möbius-Kantor Regular Map of Genus Two and Regular Ramified Coverings. *presented at SIGMAC 98, Flagstaff, AZ, July 20-24, 1998* `http://odin.math.nau.edu:80/s̃ew/sigmac.html`.

[74] J. Tits. Buildings of spherical type and finite bn-pairs. *Lecture Notes in Math.382, Berlin, 1974. Springer–Verlag.*

[75] T. W. Tucker. There is only one group of genus two. *J. Combin. Theory B, 36 (1984) 269–275.*

[76] W. T. Tutte. Connectivity in graphs. *University of Toronto Press, Toronto, 1966.*

[77] W. T. Tutte. A family of cubical graphs. *Proc. Camb. Phil. Soc., 43 (1948), 459–474.*

[78] R. Daublebsky von Sterneck. Die configurationen 11_3. *Monatsh. Math. Physik, 5 (1894) 325–330.*

[79] R. Daublebsky von Sterneck. Die configurationen 12_3. *Monatsh. Math. Physik, 6 (1895) 223–255.*

[80] M. Škoviera and J. Širáň. Regular maps from cayley graphs, part 1: Balanced cayley maps. *Discrete Math., 109 (1992) 265–276.*

[81] D. Wells. The Penguin Dictionary of Curious and Interesting Geometry. *Penguin Books, London, 1991.*

[82] A. White. Graphs, Groups, and Surfaces. *North-Holland Pub. Co., Amsterdam, 1973*.

[83] E. Steinitz (with H. Rademacher). Vorlesungen über die Theorie der Polyeder. *Springer-Verlag, Berlin, 1934*.

[84] P. K. Wong. Cages—a survey. *J. Graph Theory, 6:1–22, 1982*.