Jacobi–Davidson Method for Two-Parameter Eigenvalue Problems

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This is joint work with M. Hochstenbach
Outline

- Two-parameter eigenvalue problem (2EP)
- Jacobi–Davidson type methods for 2EP
- Harmonic Rayleigh–Ritz for 2EP
- Work in progress: singular 2EP
Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

\[
A_1 x = \lambda B_1 x + \mu C_1 x \\
A_2 y = \lambda B_2 y + \mu C_2 y,
\]

where \( A_i, B_i, C_i \) are \( n \times n \) matrices, \( \lambda, \mu \in \mathbb{C}, x, y \in \mathbb{C}^n \).

- Eigenvalue: a pair \((\lambda, \mu)\) that satisfies (2EP) for nonzero \(x\) and \(y\).

- Eigenvector: the tensor product \(x \otimes y\).

- There are \(n^2\) eigenvalues.

- Goal: compute eigenvalues \((\lambda, \mu)\) close to a target \((\sigma, \tau)\) and eigenvectors \(x \otimes y\).
Tensor product approach

\[ A_1 x = \lambda B_1 x + \mu C_1 x \]
\[ A_2 y = \lambda B_2 y + \mu C_2 y \] (2EP)

- On \( \mathbb{C}^n \otimes \mathbb{C}^n \) we define \( n^2 \times n^2 \) matrices
  \[ \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \]
  \[ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \]
  \[ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2. \]

- 2EP is equivalent to a coupled GEP
  \[ \Delta_1 z = \lambda \Delta_0 z \]
  \[ \Delta_2 z = \mu \Delta_0 z, \]
  where \( z = x \otimes y \).

- 2EP is nonsingular \( \iff \) \( \Delta_0 \) is nonsingular
  \[ \Delta_0^{-1} \Delta_1 z = \lambda z \]
  \[ \Delta_0^{-1} \Delta_2 z = \mu z \]

- \( \Delta_0^{-1} \Delta_1 \) and \( \Delta_0^{-1} \Delta_2 \) commute.
Right definite problem

\[
\begin{align*}
(2EP) \quad &A_1 x = \lambda B_1 x + \mu C_1 x \\
&\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\
&A_2 y = \lambda B_2 y + \mu C_2 y \\
&\Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\
&A_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \\
&\Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \\
&\Delta_1 z = \lambda \Delta_0 z \\
&\Delta_2 z = \mu \Delta_0 z
\end{align*}
\]

2EP is right definite when \( A_i, B_i, C_i \) are Hermitian and \( \Delta_0 \) is positive definite.

If 2EP is right definite then

- eigenpairs are real
- there exist \( n^2 \) linearly independent eigenvectors
- eigenvectors of distinct eigenvalues are \( \Delta_0 \)-orthogonal, i.e., \((x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0\)
Numerical methods

First option: standard algorithms for explicitly computed matrices $\Delta$:

$$\begin{align*}
(2\text{EP}) & \quad A_1 x = \lambda B_1 x + \mu C_1 x \quad \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\
A_2 y = \lambda B_2 y + \mu C_2 y & \quad \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\
& \quad \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2
\end{align*}$$

$$\begin{align*}
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z
\end{align*}$$ (\$\Delta$)

Algorithms that work with matrices $A_i, B_i, C_i$:

- Blum, Curtis, Geltner (1978), and Browne, Sleeman (1982): gradient method,
- Bohste (1980): Newton’s method for eigenvalues,
- Continuation method:
  - Shimasaki (1995): for a special class of RD problems,
  - P. (1999): for RD problems, Tensor Rayleigh Quotient Iteration,
- Jacobi-Davidson type methods.
  - Hochstenbach, P. (2002): for RD problems,
  - Hochstenbach, Košir, P. (2005): for general nonsingular 2EP,
**Subspace methods** (Arnoldi, Lanczos, JD, . . .) compute eigenpairs from low dimensional subspaces. The main ingredients are:

- **Extraction**: We compute an approximation to an eigenpair from a given search subspace (Rayleigh-Ritz, harmonic Rayleigh-Ritz, . . .).
- **Expansion**: After each step we expand the subspace by a new direction.

**Jacobi–Davidson method**:

- a new direction to the subspace is orthogonal or oblique to the last chosen Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

JD method can be efficiently generalized for two-parameter eigenvalue problems, while this is not clear for subspace methods based on Krylov subspaces.
**JD-like method for the right definite case**

**Extraction:** Ritz–Galerkin conditions: search spaces = test spaces: \( u \in U_k, v \in V_k \)

\[
\begin{align*}
(A_1 - \sigma B_1 - \tau C_1)u & \perp U_k \\
(A_2 - \sigma B_2 - \tau C_2)v & \perp V_k
\end{align*}
\]

⇒ projected right definite two-parameter eigenvalue problem

\[
\begin{align*}
U_k^T A_1 U_k c &= \sigma U_k^T B_1 U_k c + \tau U_k^T C_1 U_k c \\
V_k^T A_2 V_k d &= \sigma V_k^T B_2 V_k d + \tau V_k^T C_2 V_k d
\end{align*}
\]

**Ritz value:** \((\sigma, \tau)\), **Ritz vectors:** \(u = U_k c, v = V_k d\), where \(c, d \in \mathbb{R}^k\)

**Expansion:** Correction equation for new directions \(s, t\):

\[
\begin{align*}
(I - uu^T)(A_1 - \sigma B_1 - \tau C_1)(I - uu^T)s &= -(A_1 - \sigma B_1 - \tau C_1)u \\
(I - vv^T)(A_2 - \sigma B_2 - \tau C_2)(I - vv^T)t &= -(A_2 - \sigma B_2 - \tau C_2)v.
\end{align*}
\]

Works well for exterior eigenvalues.
Two-sided JD-like method for a general problem

Petrov–Galerkin conditions: search spaces $u_i \in U_{ik}$, test spaces $v_i \in V_{ik}$

\[
(A_1 - \sigma B_1 - \tau C_1) u_1 \perp V_{1k} \\
(A_2 - \sigma B_2 - \tau C_2) u_2 \perp V_{2k},
\]

$\Rightarrow$ projected two-parameter eigenvalue problem

\[
V_{1k}^* A_1 U_{1k} c_1 = \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1 \\
V_{2k}^* A_2 U_{2k} c_2 = \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2,
\]

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov value: $(\sigma, \tau)$, Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, where $c_i, d_i \in \mathbb{C}^k$

Usually performs better than the one-sided method.

Works well for exterior eigenvalues.
Harmonic Rayleigh–Ritz for 2EP

GEP: \[ Ax = \lambda Bx \]
subspace is \( \mathcal{U}_k \), target is \( \tau \)

Rayleigh–Ritz: \[ Au - \theta Bu \perp \mathcal{U}_k \]
Spectral transformation: \[ (A - \tau B)^{-1} Bx = (\lambda - \tau)^{-1} x \]
Harmonic Rayleigh–Ritz: \[ Au - \theta Bu \perp (A - \tau B) \mathcal{U}_k \]

2EP:
\[
\begin{align*}
A_1 x &= \lambda B_1 x + \mu C_1 x \\
A_2 y &= \lambda B_2 y + \mu C_2 y
\end{align*}
\]
subspace is \( \mathcal{U}_k \otimes \mathcal{V}_k \), target is \( \sigma, \tau \)

Rayleigh–Ritz: \[
\begin{align*}
(A_1 - \theta B_1 - \eta C_1) u &\perp \mathcal{U}_k \\
(A_2 - \theta B_2 - \eta C_2) v &\perp \mathcal{V}_k
\end{align*}
\]
Harmonic Rayleigh–Ritz for 2EP

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2EP:
\[ A_1 x = \lambda B_1 x + \mu C_1 x \]
\[ A_2 y = \lambda B_2 y + \mu C_2 y \]
subspace is \( \mathcal{U}_k \otimes \mathcal{V}_k \), target is \( (\sigma, \tau) \)

Rayleigh–Ritz:
\[ (A_1 - \theta B_1 - \eta C_1) u \perp \mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp \mathcal{V}_k \]

Spectral transformation: ???
Harmonic Rayleigh–Ritz for 2EP

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subspace is \( \mathcal{U}_k \), target is \( \tau \)

Rayleigh–Ritz: \[ Au - \theta Bu \perp \mathcal{U}_k \]

Spectral transformation: \( (A - \tau B)^{-1} B x = (\lambda - \tau)^{-1} x \)

Harmonic Rayleigh–Ritz: \[ Au - \theta Bu \perp (A - \tau B) \mathcal{U}_k \]

2EP: \[ A_1 x = \lambda B_1 x + \mu C_1 x \]
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Rayleigh–Ritz: \[ (A_1 - \theta B_1 - \eta C_1) u \perp \mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp \mathcal{V}_k \]

Spectral transformation: \[ ? ? ? \]

Harmonic Rayleigh–Ritz: \[ (A_1 - \theta B_1 - \eta C_1) u \perp (A_1 - \sigma B_1 - \tau C_1) \mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp (A_2 - \sigma B_2 - \tau C_2) \mathcal{V}_k \]
1. \( s = u_1 \) and \( t = v_1 \) (starting vectors), \( U_0 = V_0 = [ ] \)

   for \( k = 1, 2, \ldots \)

2. \( (U_{k-1}, s) \to U_k, \quad (V_{k-1}, t) \to V_k \)

3. Extract appropriate harmonic Ritz pair \( ((\xi_1, \xi_2), c \otimes d) \)

4. Take \( u = U_k c, \quad v = V_k d \) and compute tensor Rayleigh quotient

\[
\theta = \frac{(u \otimes v)^* \Delta_1(u \otimes v)}{(u \otimes v)^* \Delta_0(u \otimes v)} = \frac{(u^* A_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* A_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}
\]

\[
\eta = \frac{(u \otimes v)^* \Delta_2(u \otimes v)}{(u \otimes v)^* \Delta_0(u \otimes v)} = \frac{(u^* B_1 u)(v^* A_2 v) - (u^* A_1 u)(v^* B_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}
\]

5. \( r_1 = (A_1 - \theta B_1 - \eta C_1)u \)

\( r_2 = (A_2 - \theta B_2 - \eta C_2)v \)

6. Stop if \( \|r_1\|^2 + \|r_2\|^2 \leq \varepsilon \)

7. Solve (approximately) an \( s \perp u, \quad t \perp v \) from corr. equation(s)
Numerical example

$n = 1000$, problem is not right definite. We want to compute 50 eigenvalues closest to the origin using at most 2500 outer iterations.

<table>
<thead>
<tr>
<th>GMRES</th>
<th>Two-sided Ritz</th>
<th>Harmonic Ritz</th>
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<td>64</td>
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The convergence graphs for the two-sided Ritz extraction (left) and the harmonic Ritz extraction (right) for the first 40 outer iterations using 8 GMRES steps in the inner iteration.
This is joint work with A. Muhič.

**Model updating** (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find the smallest perturbation of matrix $A$ by a linear combination of matrices $B$ and $C$, such that $A - \lambda B - \mu C$ has the prescribed eigenvalues $\sigma_1$ and $\sigma_2$. 
Model updating as a singular 2EP

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Model updating (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find the smallest perturbation of matrix $A$ by a linear combination of matrices $B$ and $C$, such that $A - \lambda B - \mu C$ has the prescribed eigenvalues $\sigma_1$ and $\sigma_2$.

The problem can be expressed as a two-parameter eigenvalue problem

\[
(A - \sigma_1 I)x = \lambda Bx + \mu Cx,
\]
\[
(A - \sigma_2 I)y = \lambda By + \mu Cy.
\]

\[
\det(B \otimes C - C \otimes B) = 0 \implies \text{this problem is singular}
\]

Eigenvalues are candidates for the best model update.
We consider

\[(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x = 0\]

\[(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y = 0,\]

(Q2EP)

where \(A_i, B_i, \ldots, F_i\) are \(n \times n\) matrices, \((\lambda, \mu)\) is an eigenvalue, and \(x \otimes y\) is the corresponding eigenvector. In the generic case the problem has \(4n^2\) eigenvalues.
We consider

\[(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda \mu E_1 + \mu^2 F_1)x = 0\]
\[(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda \mu E_2 + \mu^2 F_2)y = 0,\]

where \(A_i, B_i, \ldots, F_i\) are \(n \times n\) matrices, \((\lambda, \mu)\) is an eigenvalue, and \(x \otimes y\) is the corresponding eigenvector. In the generic case the problem has \(4n^2\) eigenvalues.

We can linearize Q2EP as a two-parameter eigenvalue problem, one such linearization is

\[
\begin{pmatrix}
A_1 & B_1 & C_1 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
+ \lambda
\begin{pmatrix}
0 & D_1 & \frac{1}{2}E_1 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \mu
\begin{pmatrix}
0 & \frac{1}{2}E_1 & F_1 \\
0 & 0 & 0 \\
-I & 0 & 0
\end{pmatrix}
\begin{bmatrix}
x \\
x \lambda x \\
x \mu x
\end{bmatrix}
= 0
\]

\[
\begin{pmatrix}
A_2 & B_2 & C_2 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
+ \lambda
\begin{pmatrix}
0 & D_2 & \frac{1}{2}E_2 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \mu
\begin{pmatrix}
0 & \frac{1}{2}E_2 & F_2 \\
0 & 0 & 0 \\
-I & 0 & 0
\end{pmatrix}
\begin{bmatrix}
y \\
y \lambda y \\
y \mu y
\end{bmatrix}
= 0,
\]

where matrices are of size \(3n \times 3n\). This problem is singular.
Numerical method for singular 2EP

\( (2\text{EP}) \)

\[
A_1 x = \lambda B_1 x + \mu C_1 x \\
A_2 y = \lambda B_2 y + \mu C_2 y
\]

\[
\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2
\]

\[
\Delta_1 z = \lambda \Delta_0 z \\
\Delta_2 z = \mu \Delta_0 z
\]

Singular 2EP \( \iff \) \( \det(\Delta_0) = 0 \)

For singular 2EP, there are no general results linking the eigenvalues of \((2\text{EP})\) and \((\Delta)\).
Numerical method for singular 2EP

\[
\begin{align*}
A_1 x &= \lambda B_1 x + \mu C_1 x \\
A_2 y &= \lambda B_2 y + \mu C_2 y \\
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \\
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z \\
\end{align*}
\]

Singular 2EP \iff \det(\Delta_0) = 0

For singular 2EP, there are no general results linking the eigenvalues of (2EP) and (\Delta).

Numerical method: we extract the common regular part of matrix pencils (\Delta). Thus we obtain matrices \(\tilde{\Delta}_0\), \(\tilde{\Delta}_1\), and \(\tilde{\Delta}_2\), such that:

- \(\tilde{\Delta}_0\) is nonsingular,
- eigenvalues of

\[
\begin{align*}
\tilde{\Delta}_1 \tilde{z} &= \lambda \tilde{\Delta}_0 \tilde{z} \\
\tilde{\Delta}_2 \tilde{z} &= \mu \tilde{\Delta}_0 \tilde{z} \\
\end{align*}
\]

are common regular eigenvalues of (\Delta).

For Q2EP and model updating we can show that

regular eigenvalues of (2EP) = eigenvalues of (\(\tilde{\Delta}\)) = regular eigenvalues of (\Delta).
Conclusions

J-D works for nonsingular two-parameter eigenvalue problems.

The harmonic approach can be generalized to the 2EP.

Singular 2EP: work in progress ...