Numerical methods for two-parameter eigenvalue problems

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Partly joined work with: M. Hochstenbach (Eindhoven), T. Košir and A. Muhič (Ljubljana)
• Two-parameter eigenvalue problem (2EP)

• Jacobi–Davidson type methods for 2EP

• Harmonic Rayleigh–Ritz for 2EP

• Singular 2EP
Two-parameter eigenvalue problem

- **Two-parameter eigenvalue problem:**

  \[
  A_1 x = \lambda B_1 x + \mu C_1 x \\
  A_2 y = \lambda B_2 y + \mu C_2 y,
  \]

  where \(A_i, B_i, C_i\) are \(n \times n\) matrices, \(\lambda, \mu \in \mathbb{C}\), \(x, y \in \mathbb{C}^n\).

- **Eigenvalue:** a pair \((\lambda, \mu)\) that satisfies (2EP) for nonzero \(x\) and \(y\).

- **Eigenvector:** the tensor product \(x \otimes y\).

- There are \(n^2\) eigenvalues.

- **Goal:** eigenvalues \((\lambda, \mu)\) close to a target \((\sigma, \tau)\) and eigenvectors \(x \otimes y\).
Separation of variables: $\Delta u + \nu u = 0$ on $\Omega$, $u|_{\partial\Omega} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \implies$ two S–L equations ($\nu = \lambda + \mu$)

\[
\begin{align*}
x'' + \lambda x &= 0, & x(0) &= x(a) = 0, \\
y'' + \mu y &= 0, & y(0) &= y(b) = 0.
\end{align*}
\]
Separation of variables: \( \Delta u + \nu u = 0 \) on \( \Omega \), \( u|_{\partial \Omega} = 0 \)

Rectangle: \( \Omega = [0, a] \times [0, b] \implies \) two S–L equations (\( \nu = \lambda + \mu \))

\[
\begin{align*}
x'' + \lambda x &= 0, & x(0) &= x(a) = 0, \\
y'' + \mu y &= 0, & y(0) &= y(b) = 0.
\end{align*}
\]

Circle: \( \Omega = \{x^2 + y^2 \leq a^2\} \), polar coordinates \( \implies \) a triangular situation

\[
\begin{align*}
\Phi'' + \lambda \Phi &= 0, & \Phi(0) &= \Phi(2\pi) = 0, \\
r^{-1}(rR)' + (\nu - \lambda r^{-2})R &= 0, & R(0) &< \infty, R(a) = 0.
\end{align*}
\]
Separation of variables: $\Delta u + \nu u = 0$ on $\Omega$, $u|_{\partial \Omega} = 0$

Rectangle: $\Omega = [0, a] \times [0, b] \implies$ two S–L equations ($\nu = \lambda + \mu$)

$$
x'' + \lambda x = 0, \quad x(0) = x(a) = 0,$$
$$y'' + \mu y = 0, \quad y(0) = y(b) = 0.
$$

Circle: $\Omega = \{x^2 + y^2 \leq a^2\}$, polar coordinates $\implies$ a triangular situation

$$
\Phi'' + \lambda \Phi = 0, \quad \Phi(0) = \Phi(2\pi) = 0,$$
$$r^{-1}(rR')' + (\nu - \lambda r^{-2})R = 0, \quad R(0) < \infty, R(a) = 0.
$$

Ellipse: $\Omega = \{(x/a)^2 + (y/b)^2 \leq 1\}$, elliptic coordinates ($c$ focus)

$\implies$ modified Mathieu’s and Mathieu’s DE ($4\lambda = c^2\nu$)

$$
\nu_1'' + (2\lambda \cosh(2y_1) - \mu)\nu_1 = 0, \quad \nu_1(0) = \nu_1(d) = 0,$$
$$\nu_2'' - (2\lambda \cos(2y_1) - \mu)\nu_2 = 0, \quad \nu_2(0) = \nu_2(\pi/2) = 0.
$$
Tensor product approach

\[ A_1x = \lambda B_1x + \mu C_1x \]
\[ A_2y = \lambda B_2y + \mu C_2y \] (2EP)

- On \( \mathbb{C}^n \otimes \mathbb{C}^n \) we define \( n^2 \times n^2 \) matrices

\[ \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \]
\[ \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \]
\[ \Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2. \]

- 2EP is equivalent to a coupled GEP

\[ \Delta_1z = \lambda \Delta_0z \]
\[ \Delta_2z = \mu \Delta_0z, \] (\( \Delta \))

where \( z = x \otimes y \).

- 2EP is nonsingular \( \iff \) \( \Delta_0 \) is nonsingular

- \( \Delta_0^{-1} \Delta_1 \) and \( \Delta_0^{-1} \Delta_2 \) commute.
Right definite problem

\[(2EP) \quad \begin{align*}
A_1 x &= \lambda B_1 x + \mu C_1 x \\
A_2 y &= \lambda B_2 y + \mu C_2 y \\
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2 \\
\Delta_1 z &= \lambda \Delta_0 z \\
\Delta_2 z &= \mu \Delta_0 z
\end{align*} \quad (\Delta)\]

2EP is right definite when \(A_i, B_i, C_i\) are Hermitian and \(\Delta_0\) is positive definite.

If 2EP is right definite then

- eigenpairs are real

- there exist \(n^2\) linearly independent eigenvectors

- eigenvectors of distinct eigenvalues are \(\Delta_0\)-orthogonal, i.e., \((x_1 \otimes y_1)^T \Delta_0 (x_2 \otimes y_2) = 0\)
Numerical methods - $\Delta$ matrices

\[(2\text{EP}) \quad A_1 x = \lambda B_1 x + \mu C_1 x \quad \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \]
\[A_2 y = \lambda B_2 y + \mu C_2 y \quad \Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \]
\[\Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2 \quad \Delta_1 z = \lambda \Delta_0 z \quad \Delta_2 z = \mu \Delta_0 z \quad (\Delta)\]

Hochstenbach, Košir, P. (2005):

1. $Q^* \Delta_0 Z = R$ and $Q^* \Delta_1 Z = S$, where $Q, Z$ are unitary, $R, S$ upper triangular, and the multiple values of $\lambda_i := s_{ii}/r_{ii}$ are clustered along the diagonal of $R^{-1} S$.

\[R = \begin{bmatrix}
R_{11} & R_{12} & \cdots & R_{1p} \\
0 & R_{22} & \cdots & R_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_{pp}
\end{bmatrix}, \quad S = \begin{bmatrix}
S_{11} & S_{12} & \cdots & S_{1p} \\
0 & S_{22} & \cdots & S_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & S_{pp}
\end{bmatrix} \]

2. Compute diagonal blocks $T_{11}, \ldots, T_{pp}$ of $T = Q^* \Delta_2 Z$.
3. Compute the eigenvalues $\mu_{i1}, \ldots, \mu_{im_j}$ of the GEP $T_{ii} w = \mu R_{ii} w$ for $i = 1, \ldots, p$.
4. Reindex $(\lambda_1, \mu_{11}), \ldots, (\lambda_1, \mu_{1m_1}); \ldots; (\lambda_p, \mu_{p1}), \ldots, (\lambda_p, \mu_{pm_p})$ into $(\lambda_1, \mu_1), \ldots, (\lambda_{n^2}, \mu_{n^2})$.
5. For each eigenvalue $(\lambda_j, \mu_j)$ compute its eigenvector $x_j \otimes y_j$.

Time complexity: $\mathcal{O}(n^6)$
Algorithms that work with matrices $A_i, B_i, C_i$

(2EP) \[ \begin{align*}
A_1x &= \lambda B_1x + \mu C_1x \\
A_2y &= \lambda B_2y + \mu C_2y
\end{align*} \]

\[ \begin{align*}
\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2
\end{align*} \]

\[ \begin{align*}
\Delta_1z &= \lambda \Delta_0z \\
\Delta_2z &= \mu \Delta_0z \quad (\Delta)
\end{align*} \]

- **Gradient method**: Blum, Curtis, Geltner (1978), Browne, Sleeman (1982)
- **Newton’s method for eigenvalues**: Bohte (1980)
- **Generalized Rayleigh Quotient Iteration**: Ji, Jiang, Lee (1992)
Subspace methods (Arnoldi, Lanczos, J-D, ...) compute eigenpairs from low dimensional subspaces. The main ingredients are:

- **Extraction**: We compute an approximation to an eigenpair from a given search subspace (Rayleigh-Ritz, harmonic Rayleigh-Ritz, ...).
- **Expansion**: After each step we expand the subspace by a new direction.

**Jacobi–Davidson method:**

- a new subspace direction is orthogonal or oblique to the last chosen Ritz vector,
- approximate solutions of certain correction equations are used for expansion.

J-D method can be efficiently applied to two-parameter eigenvalue problems. This is not clear for subspace methods based on Krylov subspaces.
J-D for the right definite case

Extraction: Ritz–Galerkin conditions: search spaces = test spaces: \( u \in \mathcal{U}_k, \ v \in \mathcal{V}_k \)

\[
(A_1 - \sigma B_1 - \tau C_1)u \perp \mathcal{U}_k \\
(A_2 - \sigma B_2 - \tau C_2)v \perp \mathcal{V}_k
\]

\( \Rightarrow \) projected right definite two-parameter eigenvalue problem

\[
\mathcal{U}_k^T A_1 \mathcal{U}_k c = \sigma \mathcal{U}_k^T B_1 \mathcal{U}_k c + \tau \mathcal{U}_k^T C_1 \mathcal{U}_k c \\
\mathcal{V}_k^T A_2 \mathcal{V}_k d = \sigma \mathcal{V}_k^T B_2 \mathcal{V}_k d + \tau \mathcal{V}_k^T C_2 \mathcal{V}_k d
\]

Ritz value: \((\sigma, \tau)\), Ritz vectors: \(u = \mathcal{U}_k c, \ v = \mathcal{V}_k d\), where \(c, d \in \mathbb{R}^k\)

Expansion: Correction equation for new directions \(s, t\):

\[
(I - uu^T)(A_1 - \sigma B_1 - \tau C_1)(I - uu^T)s = -(A_1 - \sigma B_1 - \tau C_1)u \\
(I - vv^T)(A_2 - \sigma B_2 - \tau C_2)(I - vv^T)t = -(A_2 - \sigma B_2 - \tau C_2)v.
\]

Works well for exterior eigenvalues.
Two-sided J-D for a general problem

Petrov–Galerkin conditions: search spaces $u_i \in U_{ik}$, test spaces $v_i \in V_{ik}$

$$(A_1 - \sigma B_1 - \tau C_1)u_1 \perp V_{1k},$$

$$(A_2 - \sigma B_2 - \tau C_2)u_2 \perp V_{2k},$$

$\Rightarrow$ projected two-parameter eigenvalue problem

$$V_{1k}^* A_1 U_{1k} c_1 = \sigma V_{1k}^* B_1 U_{1k} c_1 + \tau V_{1k}^* C_1 U_{1k} c_1$$

$$V_{2k}^* A_2 U_{2k} c_2 = \sigma V_{2k}^* B_2 U_{2k} c_2 + \tau V_{2k}^* C_2 U_{2k} c_2,$$

where $u_i = U_{ik} c_i \neq 0$ for $i = 1, 2$ and $\sigma, \tau \in \mathbb{C}$.

Petrov value: $(\sigma, \tau)$, Petrov vectors: $u_i = U_{ik} c_i$, $v_i = V_{ik} d_i$, where $c_i, d_i \in \mathbb{C}^k$

Usually performs better than the one-sided method.

Works well for exterior eigenvalues.
Harmonic Rayleigh–Ritz for 2EP

GEP: \[ Ax = \lambda Bx \]
subspace is \( U_k \), target is \( \tau \)

Rayleigh–Ritz: \[ Au - \theta Bu \perp U_k \]

Spectral transformation: \[ (A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x \]

Harmonic Rayleigh–Ritz: \[ Au - \theta Bu \perp (A - \tau B)U_k \]

2EP:
\[ A_1x = \lambda B_1x + \mu C_1x \]
\[ A_2y = \lambda B_2y + \mu C_2y \]
subspace is \( U_k \otimes V_k \), target is \( (\sigma, \tau) \)

Rayleigh–Ritz:
\[ (A_1 - \theta B_1 - \eta C_1) u \perp U_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp V_k \]
Harmonic Rayleigh–Ritz for 2EP

GEP: \[ Ax = \lambda Bx \]
subspace is \( \mathcal{U}_k \), target is \( \tau \)

Rayleigh–Ritz: \[ Au - \theta Bu \perp \mathcal{U}_k \]

Spectral transformation: \[ (A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x \]

Harmonic Rayleigh–Ritz: \[ Au - \theta Bu \perp (A - \tau B)\mathcal{U}_k \]

2EP:
\[ A_1x = \lambda B_1x + \mu C_1x \]
\[ A_2y = \lambda B_2y + \mu C_2y \]
subspace is \( \mathcal{U}_k \otimes \mathcal{V}_k \), target is \( (\sigma, \tau) \)

Rayleigh–Ritz:
\[ (A_1 - \theta B_1 - \eta C_1) u \perp \mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp \mathcal{V}_k \]

Spectral transformation: \[ ? \ ? \ ? \]
Harmonic Rayleigh–Ritz for 2EP

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subspace is \( \mathcal{U}_k \), target is \( \tau \)

Rayleigh–Ritz: \( Au - \theta Bu \perp \mathcal{U}_k \)

Spectral transformation: \( (A - \tau B)^{-1}Bx = (\lambda - \tau)^{-1}x \)

Harmonic Rayleigh–Ritz: \( Au - \theta Bu \perp (A - \tau B)\mathcal{U}_k \)

2EP:

\[ A_1x = \lambda B_1x + \mu C_1x \]
\[ A_2y = \lambda B_2y + \mu C_2y \]

subspace is \( \mathcal{U}_k \otimes \mathcal{V}_k \), target is \( (\sigma, \tau) \)

Rayleigh–Ritz:

\[ (A_1 - \theta B_1 - \eta C_1) u \perp \mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp \mathcal{V}_k \]

Spectral transformation: ???

Harmonic Rayleigh–Ritz:

\[ (A_1 - \theta B_1 - \eta C_1) u \perp (A_1 - \sigma B_1 - \tau C_1)\mathcal{U}_k \]
\[ (A_2 - \theta B_2 - \eta C_2) v \perp (A_2 - \sigma B_2 - \tau C_2)\mathcal{V}_k \]
1. \( s = u_1 \) and \( t = v_1 \) (starting vectors), \( U_0 = V_0 = [ ] \) 

   for \( k = 1, 2, ... \)

2. \((U_{k-1}, s) \rightarrow U_k, (V_{k-1}, t) \rightarrow V_k\)

3. Extract appropriate harmonic Ritz pair \(((\xi_1, \xi_2), c \otimes d)\)

4. Take \( u = U_k c, v = V_k d \) and compute tensor Rayleigh quotient

\[
\theta = \frac{(u \otimes v)^* \Delta_1 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* A_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* A_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}
\]

\[
\eta = \frac{(u \otimes v)^* \Delta_2 (u \otimes v)}{(u \otimes v)^* \Delta_0 (u \otimes v)} = \frac{(u^* B_1 u)(v^* A_2 v) - (u^* A_1 u)(v^* B_2 v)}{(u^* B_1 u)(v^* C_2 v) - (u^* C_1 u)(v^* B_2 v)}
\]

5. \[ r_1 = (A_1 - \theta B_1 - \eta C_1)u \]
   \[ r_2 = (A_2 - \theta B_2 - \eta C_2)v \]

6. Stop if \( (||r_1||^2 + ||r_2||^2)^{1/2} \leq \varepsilon \)

7. Solve (approximately) an \( s \perp u, t \perp v \) from corr. equation(s)

Hochstenbach, P. (2008) ETNA
Numerical example, \( n = 1000 \)

Goal: 50 eigenvalues closest to the origin using at most 2500 outer iterations.

<table>
<thead>
<tr>
<th>GMRES</th>
<th>Rayleigh-Ritz</th>
<th>Harmonic Rayleigh-Ritz</th>
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<tbody>
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<td>64</td>
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Convergence graphs for the Rayleigh-Ritz (left) and the harmonic R-R extraction (right) for the first 40 outer iterations (8 GMRES steps for the correction eq.).
Model updating (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find the smallest perturbation of matrix $A$ by a linear combination of matrices $B$ and $C$, such that $A - \lambda B - \mu C$ has the prescribed eigenvalues $\sigma_1$ and $\sigma_2$. 
Model updating (Cottin 2001, Cottin and Reetz 2006): finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find the smallest perturbation of matrix $A$ by a linear combination of matrices $B$ and $C$, such that $A - \lambda B - \mu C$ has the prescribed eigenvalues $\sigma_1$ and $\sigma_2$.

The problem can be expressed as a two-parameter eigenvalue problem

\[
\begin{align*}
(A - \sigma_1 I)x &= \lambda Bx + \mu Cx, \\
(A - \sigma_2 I)y &= \lambda By + \mu Cy.
\end{align*}
\]

\[
\det(B \otimes C - C \otimes B) = 0 \implies \text{singular 2EP}
\]
Quadratic 2EP as a singular 2EP

\[(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda \mu E_1 + \mu^2 F_1)x = 0\]
\[(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda \mu E_2 + \mu^2 F_2)y = 0,\]

where \(A_i, B_i, \ldots, F_i\) are \(n \times n\) matrices, \((\lambda, \mu)\) is an eigenvalue, and \(x \otimes y\) is the corresponding eigenvector. In the generic case the problem has \(4n^2\) eigenvalues.
Quadratic 2EP as a singular 2EP

\[
(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda \mu E_1 + \mu^2 F_1)x = 0
\]

\[
(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda \mu E_2 + \mu^2 F_2)y = 0,
\]

where \( A_i, B_i, \ldots, F_i \) are \( n \times n \) matrices, \((\lambda, \mu)\) is an eigenvalue, and \( x \otimes y \) is the corresponding eigenvector. In the generic case the problem has \( 4n^2 \) eigenvalues.

We can write Q2EP as a two-parameter eigenvalue problem, one option is

\[
\begin{pmatrix}
A_1 & B_1 & C_1 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
+ \lambda
\begin{pmatrix}
0 & D_1 & \frac{1}{2}E_1 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \mu
\begin{pmatrix}
0 & \frac{1}{2}E_1 & F_1 \\
0 & 0 & 0 \\
-I & 0 & 0
\end{pmatrix}
\begin{bmatrix}
x \\
\lambda x \\
\mu x
\end{bmatrix}
= 0
\]

\[
\begin{pmatrix}
A_2 & B_2 & C_2 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
+ \lambda
\begin{pmatrix}
0 & D_2 & \frac{1}{2}E_2 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \mu
\begin{pmatrix}
0 & \frac{1}{2}E_2 & F_2 \\
0 & 0 & 0 \\
-I & 0 & 0
\end{pmatrix}
\begin{bmatrix}
y \\
\lambda y \\
\mu y
\end{bmatrix}
= 0,
\]

where matrices are of size \( 3n \times 3n \). Singular 2EP
Numerical method for singular 2EP \((\det(\Delta_0) = 0)\)

\[
\begin{align*}
(2\text{EP}) & \quad A_1x = \lambda B_1x + \mu C_1x \\
& \quad A_2y = \lambda B_2y + \mu C_2y \\
\Delta_0 & = B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 & = A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 & = B_1 \otimes A_2 - A_1 \otimes B_2 \\
\Delta_1z & = \lambda \Delta_0z \\
\Delta_2z & = \mu \Delta_0z
\end{align*}
\]

For singular 2EP, there are no general results linking the eigenv. of \((2\text{EP})\) and \((\Delta)\).
Numerical method for singular 2EP \((\det(\Delta_0) = 0)\)

\[(2\text{EP})\]
\[
A_1 x = \lambda B_1 x + \mu C_1 x \\
A_2 y = \lambda B_2 y + \mu C_2 y
\]

\[
\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\
\Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2 \\
\Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2
\]

\[
\Delta_1 z = \lambda \Delta_0 z \\
\Delta_2 z = \mu \Delta_0 z \tag{\Delta}
\]

For singular 2EP, there are no general results linking the eigenv. of (2EP) and (\(\Delta\)).

Numerical method: we extract the common regular part of matrix pencils (\(\Delta\)). Thus we obtain matrices \(\tilde{\Delta}_0, \tilde{\Delta}_1,\) and \(\tilde{\Delta}_2\), such that \(\tilde{\Delta}_0\) is nonsingular and eigenvalues of

\[
\tilde{\Delta}_1 \tilde{z} = \lambda \tilde{\Delta}_0 \tilde{z} \\
\tilde{\Delta}_2 \tilde{z} = \mu \tilde{\Delta}_0 \tilde{z} \tag{\tilde{\Delta}}
\]

are common regular eigenvalues of (\(\Delta\)).

For Q2EP and model updating we can show that

regular eigenvalues of (2EP) = eigenvalues of (\(\tilde{\Delta}\)) = regular eigenvalues of (\(\Delta\)).
Conclusions

J-D works for nonsingular two-parameter eigenvalue problems.

The harmonic approach can be generalized to the 2EP.

Singular 2EP: work in progress ...