Optimal arrangements of n points on a sphere and in a circle

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Introduction

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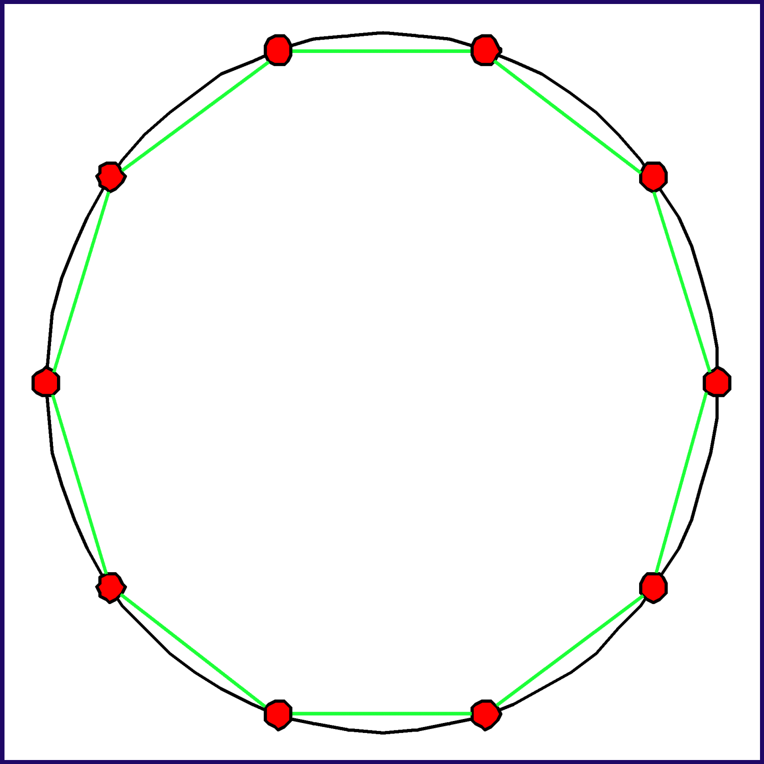

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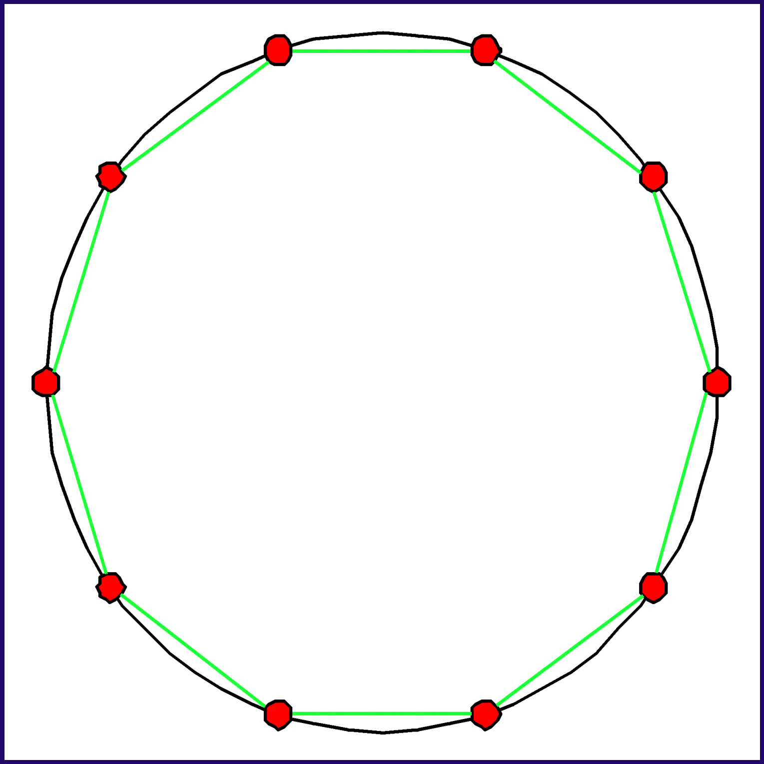

But, how to distribute n points uniformly on a sphere?

Notation

We denote arrangement of n points with unit vectors

$$\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_n,$$

where

$$\mathbf{r}_i = (x_i, y_i, z_i)$$

is the position of the i-th point on the unit sphere S.

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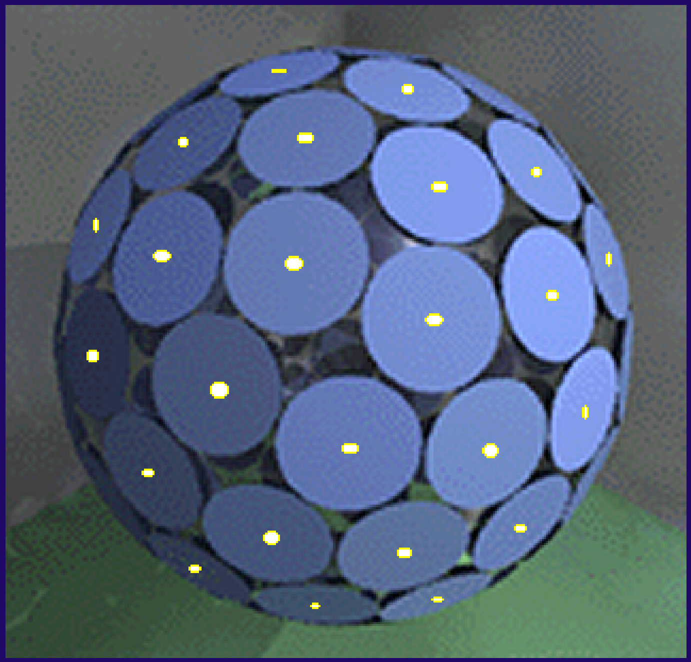

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A solution of the Tammes problem has the following properties:

1. Each point r_i is equally distant to its three nearest neighbours.

2. There exists d such that $\min_{i \neq j} |\boldsymbol{r}_i - \boldsymbol{r}_j| = d$ for all i.

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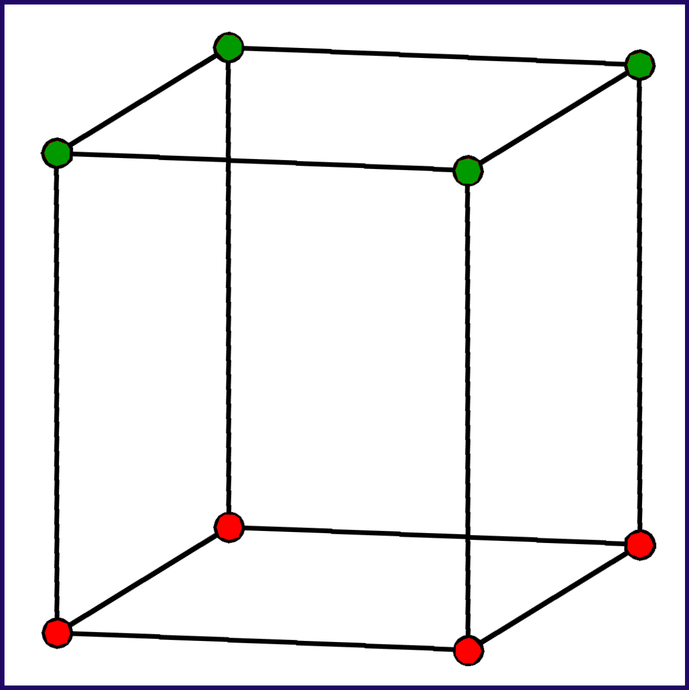

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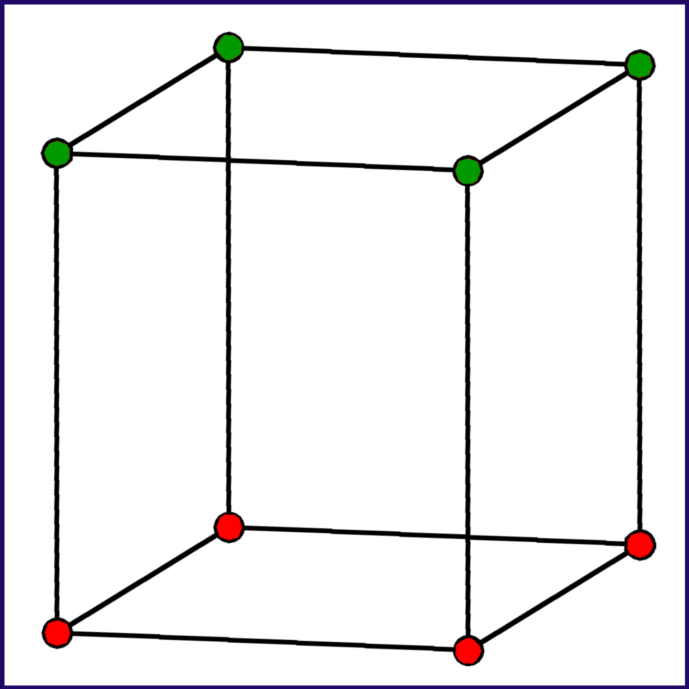

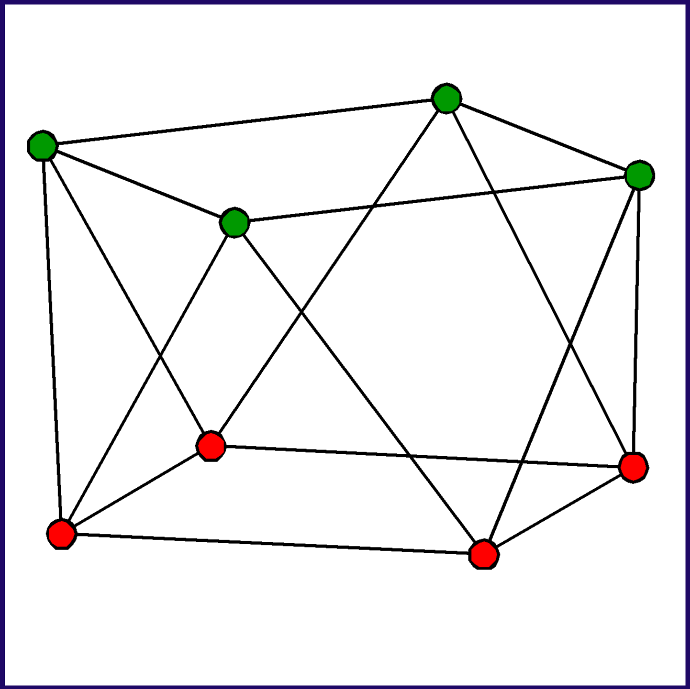

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c) Use small random perturbations and search for the arrangement with greater minimum distance. When one is found, return to Step a).

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Find the minimum Coulomb potential

$$\sum_{i<j} \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

Also known as:

- *the electron problem*,
- *the Coulomb potential problem*.

The covering problem

The maximum distance from any point on the sphere to the nearest point of the set has to be as small as possible.

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Energy function

For an arrangement $\boldsymbol{R}$ and a given $\alpha \in \mathbb{R}$ we define α-energy

$$E(\boldsymbol{R}, \alpha) := \begin{cases} \displaystyle\sum_{1 \le i < j \le n} |\boldsymbol{r}_i - \boldsymbol{r}_j|^{\alpha} & \text{for } \alpha \neq 0 \\ \displaystyle\sum_{1 \le i < j \le n} \log \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} & \text{for } \alpha = 0. \end{cases}$$

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$\alpha = 0$: maximal product of distances between all pairs.

Numerical algorithms

Local optimization method on the energy function:

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Our computer program for Thompson's problem uses gradient method with the random walk method to escape from a local minimum. Two methods combine together in a kind of a simulated annealing method.

Known Tammes and Thompson exact solutions

Tammes : 2,3,4,5,6,7,8,9,10,11,12,24.

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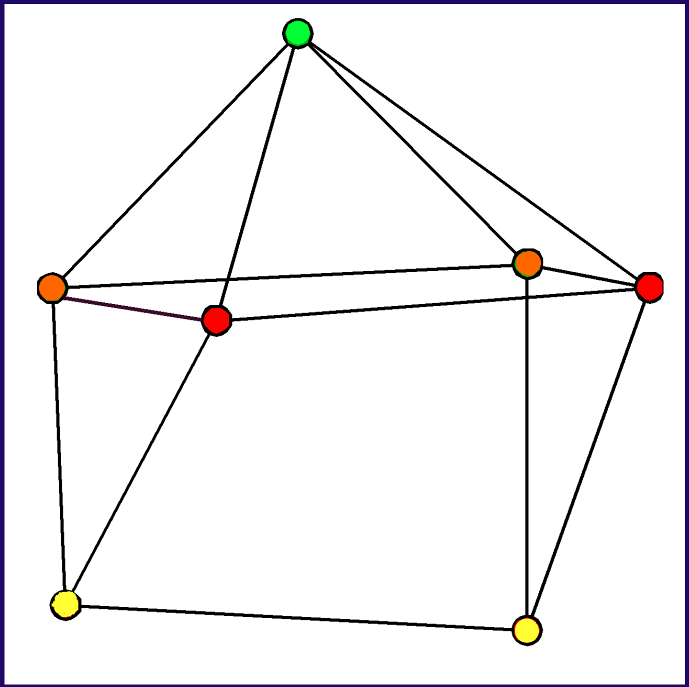
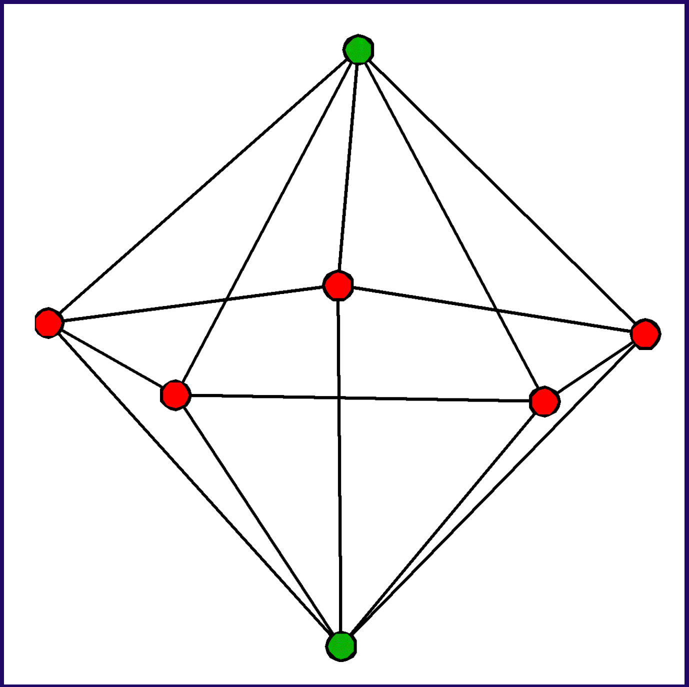
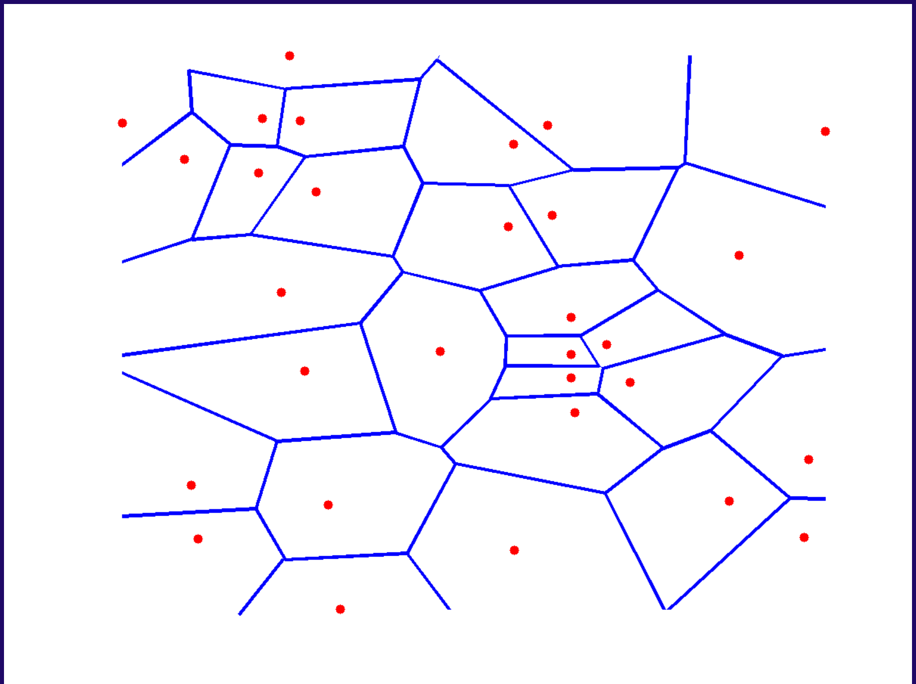




Tammes and Thompson configuration for $n = 7$.

Voronoi polyhedra

To each arrangement $\mathbf{R}$ we may associate a *Voronoi polyhedron*.



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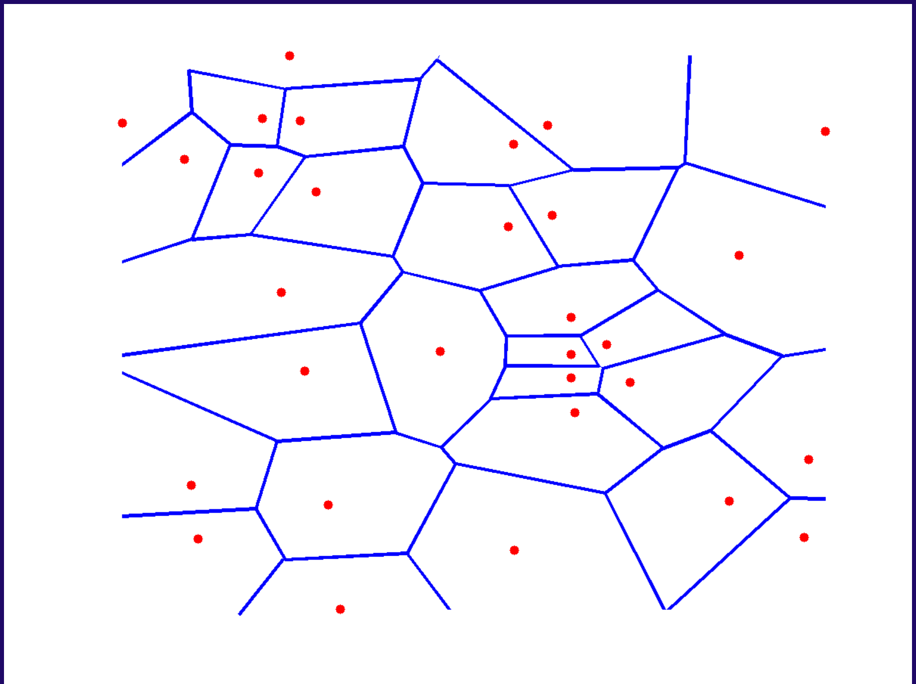
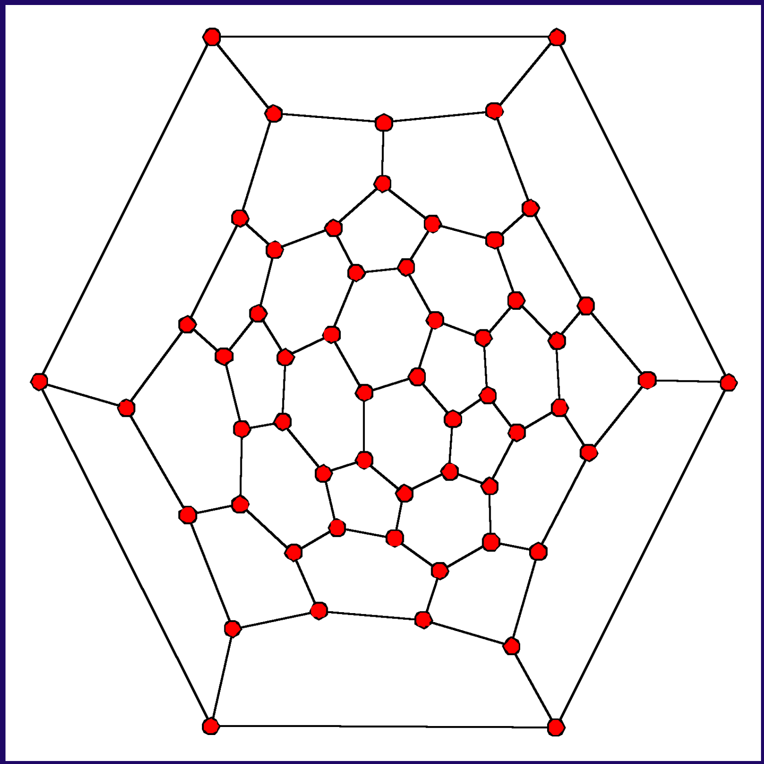
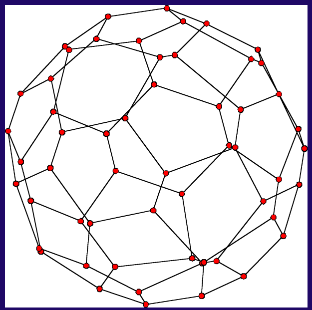


For most n the associated Voronoi polyhedron is a fullerene.



The Schlegel diagram of Voronoi polyhedron of Thompson's arrangement for $n = 30$.



Uniform arrangements in a circle

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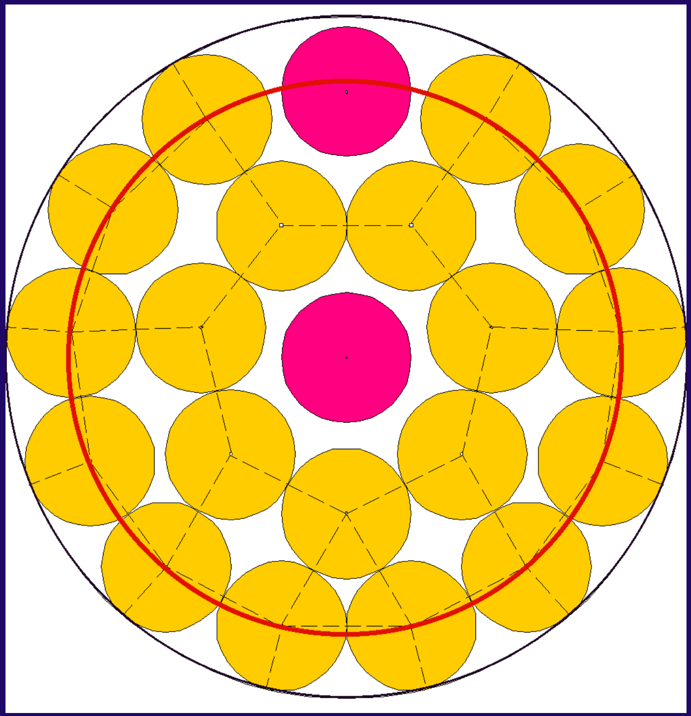
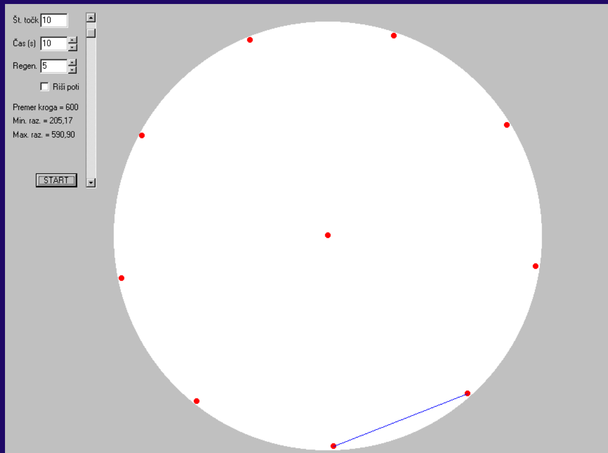
This problem is equivalent to the problem of *packing of circles in a circle*.



The centers of circles represent the best arrangement of points in the circle going through the centers of the boundary circles.



by Peter Pogačar