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Electrostatics of two charged conducting spheres

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We prove that two charged conducting spheres will almost always attract each other at close approach, even when they have like charges. The one exception is when the two spheres have the same charge ratio that they would obtain by being brought into contact. In this case, they repel, and we derive an analytical expression for the force at contact, for any size ratio, generalizing a force formula for equal spheres obtained by Kelvin in 1853. We also give the electrostatic energy of two arbitrarily charged spheres, and its analytical forms at large and small separations. Expressions are derived for the surface charge densities of the two spheres. Attraction occurs between two positively charged spheres because of mutual polarization: one of the spheres obtains a negatively charged region (neighbouring the other sphere).

Keywords: electrostatics; charged spheres; Kelvin force

1. Introduction

Like charges repel each other, opposite charges attract. But is it true that two conductors, each positively charged, always repel? We shall show that for conducting spheres, this is not so, and indeed that if two spheres with like charges are brought close enough, they will attract. Surprisingly, this holds for all sphere sizes and all charge magnitudes, except one. The one exception (when the spheres have charges in the ratio that would make them an equipotential surface on contact) is related to a problem solved, for equal spheres, by Kelvin (Thomson 1853), namely that of the force between two charged spheres in contact. If the spheres both have radius a and charge Q_a, the repulsive force acting between them is

$$F_0 = \frac{Q_a^2}{(2a)^2}\frac{4\ln 2 - 1}{6(\ln 2)^2}. \tag{1.1}$$

This force is smaller than that between two charges Q_a separated by distance $2a$, namely $Q_a^2/(2a)^2$, by the factor given in (1.1) (which is approximately 0.6149) because of redistribution of charge on the spheres.

We shall generalize (1.1) to spheres of radii a and b, give general formulae for the electrostatic energy and force between two spheres, and also consider an earlier problem of Kelvin's (Thomson 1845), namely the interaction between a charged and an earthed sphere.

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Section 6 makes reference to earlier works that found attraction between like-charged conductors, from 1836 to the present, both experimental and theoretical. Among what is new in this paper is a proof of the theorem that attraction occurs in all cases, except when the spheres have the same charge ratio that they would obtain by being brought into contact.

The potential energy of a system of N conductors with charges Q_i and potentials V_i is (Maxwell 1891, §84; Jackson 1975, §1.11)

$$W = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i. \tag{1.2}$$

The capacitance coefficients C_{ij} of a system of conductors are defined by the equation (Maxwell 1891, §87; Jackson 1975, §1.11)

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j \quad (i = 1, 2, \ldots, N). \tag{1.3}$$

Thus, if we know the capacitance coefficients, we can calculate the electrostatic energy for specified charges Q_i or potentials V_i on the conductors. This energy will be a function of the relative positions of the conductors, and by differentiations with respect to the coordinates, we can find the forces acting on the conductors.

For two spherical conductors, of radii a and b and distance between centres c (figure 1), the capacitance coefficients are known (Maxwell 1891, §173; Russell 1909; Jeffery 1912; Smythe 1950, §5.08),

$$\left.\begin{aligned} C_{aa} &= ab\sinh U \sum_{n=0}^{\infty} [a\sinh nU + b\sinh(n+1)U]^{-1}, \\ C_{bb} &= ab\sinh U \sum_{n=0}^{\infty} [b\sinh nU + a\sinh(n+1)U]^{-1} \\ \text{and} \qquad C_{ab} &= -\frac{ab}{c}\sinh U \sum_{n=1}^{\infty} [\sinh nU]^{-1}. \end{aligned}\right\} \tag{1.4}$$

The dimensionless parameter U is related to a, b and c by

$$\cosh U = \frac{c^2 - a^2 - b^2}{2ab}. \tag{1.5}$$

The electrostatics of two conductors is determined by the specified conditions. As the simplest example, consider Kelvin's (Thomson 1845) problem in which one conductor (say the sphere of radius a) is charged, and the other conductor (the sphere of radius b) is earthed. (A related problem is that of two spheres held at a constant potential difference, considered by Warren & Cuthrell (1975) and Lekner (2012). The force between the spheres is always attractive in that case.) Kelvin expressed the force between spheres with equal radii as an infinite sum, with recurrence relations between the successive terms. Here, we make use of the electrostatic equations (1.2) and (1.3) for a two-conductor system,

$$W = \tfrac{1}{2}Q_a V_a + \tfrac{1}{2}Q_b V_b, \quad Q_a = C_{aa}V_a + C_{ab}V_b \quad \text{and} \quad Q_b = C_{ab}V_a + C_{bb}V_b. \tag{1.6}$$

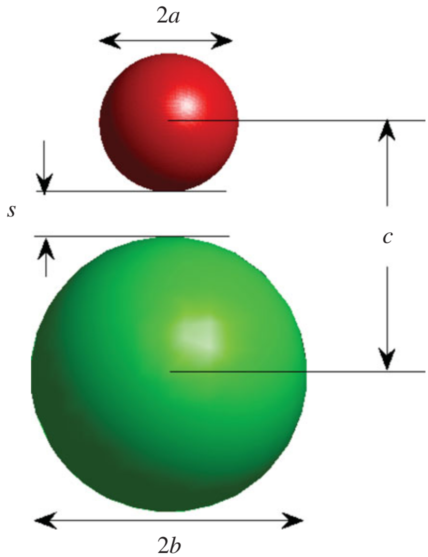

Figure 1. The system of two conducting spheres under consideration. Their radii are a and b, the distance between their centres is c, and their separation is $s = c - a - b$. (Online version in colour.)

These reduce, when $V_b = 0$ and Q_a or V_a are given, to

$$W = \tfrac{1}{2} Q_a V_a, \quad Q_a = C_{aa} V_a \quad \text{and} \quad Q_b = C_{ab} V_a, \tag{1.7}$$

from which we obtain

$$V_a = \frac{Q_a}{C_{aa}}, \quad W = \frac{Q_a^2}{2C_{aa}} \quad \text{and} \quad Q_b = \frac{C_{ab}}{C_{aa}} Q_a. \tag{1.8}$$

Since C_{aa} is positive and C_{ab} is negative, W is positive and Q_a and Q_b have opposite sign: the potential of the sphere of radius b is made zero by the flow of charge from Earth. Figure 2 shows W divided by the energy required to charge an isolated sphere of radius a to charge Q_a, namely $Q_a^2/2a$. From (1.8), $W/(Q_a^2/2a) = a/C_{aa}$. Since C_{aa} decreases monotonically as the distance c between the sphere centres increases, W is an increasing function of the separation and the force between the charged and earthed spheres is always attractive.

Anticipating the results of §3, C_{aa} grows logarithmically in close approach, and so the force between the spheres grows without bound as their separation $s = c - a - b$ tends to zero,

$$W = \frac{Q_a^2}{2C_{aa}} \to \frac{a+b}{2ab} Q_a^2 \frac{1}{[\frac{1}{2}\ln(2ab/(a+b)s) - \psi(b/(a+b))]} \tag{1.9}$$

and

$$F = -\partial_s W \to -\frac{(a+b)Q_a^2}{4ab} \frac{1}{s} \frac{1}{[\frac{1}{2}\ln(2ab/(a+b)s) - \psi(b/(a+b))]^2}. \tag{1.10}$$

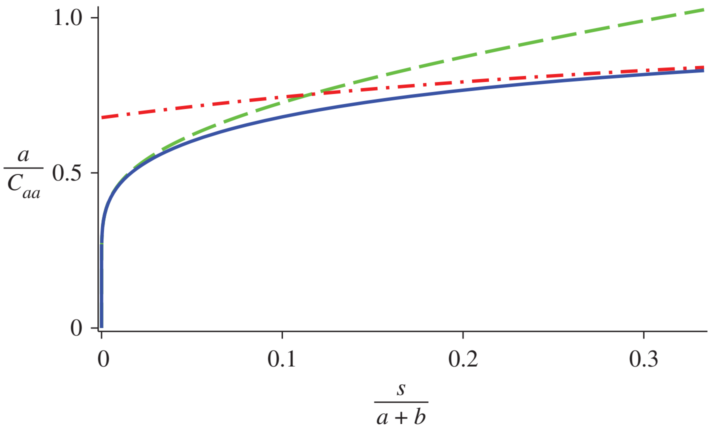

Figure 2. The potential energy of a sphere of radius a with charge Q_a and an *earthed* sphere of radius b, with $b=2a$. The plot shows W divided by the energy of an isolated sphere, $W/(Q_a^2/2a)=a/C_{aa}$, as a function of the separation s divided by $a+b$. Also shown are the small and large separation expressions, denoted with dashed and dot-dashed curves, respectively. For these, C_{aa} is approximated by the terms given in (3.2) and (2.1), respectively. (Online version in colour.)

Here, and throughout the paper, ψ represents the logarithmic derivative of the gamma function. As we shall see in §2, at large separations, $C_{aa}=a+a^2b/c^2+O(c^{-4})$, and this leads to the attractive force $-Q_a^2b/c^3+O(c^{-5})$.

We turn now to the main problem under consideration in this paper, the electrostatic interaction of two spheres with specified charges Q_a and Q_b. From the second and third equations in (1.6), we find that the potentials are

$$V_a=\frac{Q_aC_{bb}-Q_bC_{ab}}{C_{aa}C_{bb}-C_{ab}^2} \quad \text{and} \quad V_b=\frac{Q_bC_{aa}-Q_aC_{ab}}{C_{aa}C_{bb}-C_{ab}^2}. \tag{1.11}$$

Thus, the electrostatic energy of the two-sphere system is

$$W=\frac{Q_a^2C_{bb}-2Q_aQ_bC_{ab}+Q_b^2C_{aa}}{2(C_{aa}C_{bb}-C_{ab}^2)}. \tag{1.12}$$

Once W is known, the force between the spheres is given by $F=-\partial_c W$ (or by $-\partial_s W$). Where the energy increases with c, the force F will be attractive (and negative). A local decrease of W with c makes the force repulsive. In application of the expressions (1.4) for the capacitance coefficients, it is useful to know the analytical forms when the spheres are far apart and when they are close together. These cases will be considered in §§2 and 3.

2. Electrostatics of two widely separated spherical conductors

When the distance c between the centres of the two spheres is large compared with the sum of their radii, $c\gg a+b$, the dimensionless parameter U defined in (1.5) is large, and the sums defining the capacitance coefficients converge rapidly. The numerics at large separations are thus no problem. To obtain analytical results, we use the fact that the quantities $\sinh nU/\sinh U$, which appear in the sums, are polynomials in $\cosh U = (c^2 - a^2 - b^2)/2ab$, in fact Chebyshev polynomials of the second kind. One can thus generate expansions of the capacitance coefficients and of the electrostatic energy in inverse powers of c. We shall give only the first few terms,

$$\left.\begin{aligned} C_{aa} &= a + \frac{a^2 b}{c^2} + \frac{a^2 b^2 (a+b)}{c^4} + \frac{a^2 b^2 (a^3 + a^2 b + 2ab^2 + b^3)}{c^6} + O(c^{-8}),\\ C_{bb} &= b + \frac{ab^2}{c^2} + \frac{a^2 b^2 (a+b)}{c^4} + \frac{a^2 b^2 (a^3 + 2a^2 b + ab^2 + b^3)}{c^6} + O(c^{-8})\\ \text{and}\quad -C_{ab} &= \frac{ab}{c} + \frac{a^2 b^2}{c^3} + \frac{a^2 b^2 (a^2 + ab + b^2)}{c^5} + \frac{a^2 b^2 (a^2 + ab + b^2)^2}{c^7} + O(c^{-9}). \end{aligned}\right\} \tag{2.1}$$

The electrostatic energy of spheres with charges Q_a and Q_b follows from (1.12),

$$\begin{aligned} W &= \frac{Q_a^2}{2a} + \frac{Q_b^2}{2b} + \frac{Q_a Q_b}{c} - \frac{Q_a^2 b^3 + Q_b^2 a^3}{2c^4} - \frac{Q_a^2 b^5 + Q_b^2 a^5}{2c^6}\\ &\quad + \frac{2Q_a Q_b a^3 b^3}{c^7} - \frac{Q_a^2 b^7 + Q_b^2 a^7}{2c^8} + \frac{3Q_a Q_b a^3 b^3 (a^2 + b^2)}{c^9} + O(c^{-10}). \end{aligned} \tag{2.2}$$

(Remarkably, Maxwell calculated the energy to terms of order c^{-22} (Maxwell 1891, §146). He used expansions of the potential in spherical harmonics, not the capacitance coefficient sums (1.4); I have checked all the terms to the order shown in his Treatise, and all are correct.)

We see from (2.2) that negative terms appear in the electrostatic energy, even if the charges Q_a and Q_b on the two spheres have the same sign. Considering the terms in order, the first two are the self-energies of the two charged spheres, followed by the Coulomb energy $Q_a Q_b/c$. The next term is always negative and comes from the mutual polarization of the spheres: the polarizability of sphere a is a^3, and (to lowest order in c^{-1}) the field acting on it is $E_b = Q_b/c^2$, so the dipole moment of sphere a is $p_a = Q_b a^3/c^2$. The interaction energy of this dipole with the field that created it is $-\frac{1}{2}p_a E_b = -Q_b^2 a^3/2c^4$. Note that the terms containing c^{-6} and c^{-8} are also negative and monotonic in c, contributing with the dipolar term to attraction between the spheres, whatever the signs of Q_a and Q_b may be.

Figure 3 shows the energy of spheres with radii in the ratio $b/a = 2$, and charges in the ratio $Q_b/Q_a = 1/2$, as a function of the separation s between nearest points of the spheres. From the graph, we see that there is an attractive force between the spheres, which have charges of the same sign, for $s/(a+b)$ less than about 0.26, and that the force increases without limit as the sphere separation tends to zero. In the figure, we have used only the terms shown in (2.2), to demonstrate the accuracy of the expansion to order c^{-9}. When terms to c^{-22} are included, the agreement with the exact energy is much better, of course.

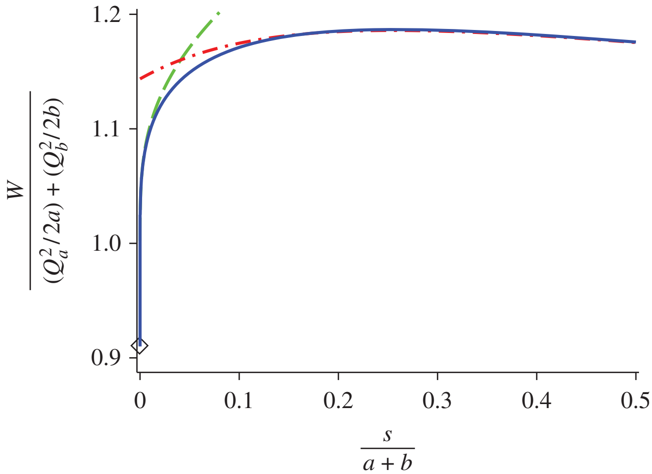

Figure 3. Mutual energy of two spheres as a function of their separation, drawn for $b=2a$, $Q_b=\frac{1}{2}Q_a$. For these parameters, the energy ratio at contact is $1/\ln 3 \approx 0.91$ (indicated by the diamond). The dashed curve is the short-range expression (3.3), and the dot-dashed curve is the long-range expansion to order c^{-9}, as given in (2.2). The energy ratio plotted is W divided by the sum of their self-energies $Q_a^2/2a + Q_b^2/2b$, which is the energy at infinite separation. (Online version in colour.)

3. Capacitance coefficients and electrostatic energy in close approach

As the separation $s=c-a-b$ between the spheres tends to zero, the parameter U also tends to zero,

$$U = \text{arc cosh}\left(\frac{c^2-a^2-b^2}{2ab}\right) = \left[\frac{2(a+b)s}{ab}\right]^{\frac{1}{2}} + O(s^{3/2}), \tag{3.1}$$

and the sums defining the capacitance coefficients converge more and more slowly. Since close approach is physically the most interesting configuration, expansions at small s have been developed for this case (Russell 1909; Lekner 2011*a*). The results to leading order in s are

$$\left.\begin{aligned} C_{aa} &= \frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] - \psi\left(\frac{b}{a+b}\right) + O(s)\right\}, \\ C_{bb} &= \frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] - \psi\left(\frac{a}{a+b}\right) + O(s)\right\} \\ \text{and}\quad C_{ab} &= -\frac{ab}{a+b}\left\{\frac{1}{2}\ln\left[\frac{2ab}{(a+b)s}\right] + \gamma + O(s)\right\}. \end{aligned}\right\} \tag{3.2}$$

As before, $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ and $\gamma = -\psi(1) = 0.5772\ldots$ is Euler's constant.

The logarithmic terms in (3.2) grow without bound as the sphere separation s tends to zero. The energy expression (1.12) then contains these logarithmic terms in both the numerator and the denominator. The $[\ln s]^2$ terms cancel in the denominator $C_{aa}C_{bb} - C_{ab}^2$, but terms linear in $\ln s$ remain,

$$W = -\frac{a+b}{2ab}\frac{\{(Q_a+Q_b)^2\ln[2ab/(a+b)s] + 4Q_aQ_b\gamma - 2Q_a^2\psi(a/(a+b)) - 2Q_b^2\psi(b/(a+b))\}}{\{[2\gamma+\psi(a/(a+b))+\psi(b/(a+b))]\ln[2ab/(a+b)s] + 2\gamma^2 - 2\psi(a/(a+b))\psi(b/(a+b))\}} + O(s). \tag{3.3}$$

A surprise comes when we differentiate with respect to the separation distance s to find the force: the leading term is always attractive, unless the charges Q_a and Q_b are in a special ratio. We find the force $F = -\partial_s W$ to be

$$F = -\frac{a+b}{abs}\frac{\{Q_a[\gamma+\psi(a/(a+b))] - Q_b[\gamma+\psi(b/(a+b))]\}^2}{\{[2\gamma+\psi(a/(a+b))+\psi(b/(a+b))]\ln[2ab/(a+b)s] + 2\gamma^2 - 2\psi(a/(a+b))\psi(b/(a+b))\}^2} + O(1). \tag{3.4}$$

When $a = b$, the force between the spheres simplifies to

$$F = -\frac{\{Q_a - Q_b\}^2}{2as\{\ln[\frac{4a}{s}] + 2\gamma\}^2} + O(1).$$

Thus, the logarithmic terms in the energy lead to an attractive force at short range, which (in theory) increases without limit as the separation s of the spheres tends to zero. In practice, the spheres will have some roughness, and an electrical short will allow flow of charge between the spheres at a separation of the order of a nanometre or greater. Here, the charges on the spheres will equilibrate to the precise ratio that annihilates the leading term in the force as given in (3.4), as we shall see in §4. Here, we note that in the limit $s \to 0$, the electrostatic energy given in (3.3) tends to

$$W_0 = \frac{(Q_a+Q_b)^2}{2(a+b)}\frac{-1}{\beta(1-\beta)\,[2\gamma+\psi(\beta)+\psi(1-\beta)]}, \quad \beta = \frac{b}{a+b}. \tag{3.5}$$

This energy is positive, the quantity in square brackets being negative. The factor multiplying $(Q_a+Q_b)^2/2(a+b)$ varies between unity when $\beta \to 0$ or 1 (one sphere much larger than the other) and $1/\ln 2 \approx 1.44$ when $\beta = 1/2$ (spheres of equal size). These special cases follow from (Davis 1972, eqns 6.3.7 and 6.3.3)

$$\lim_{z\to 0}[z\psi(z)] = -1, \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2. \tag{3.6}$$

4. The repulsive force between two spheres that are or have been in contact

The force between two equal charged spheres in contact was first considered by Kelvin. He derived the force given in formula (1.1) by the method of images, which leads to a double series (Thomson 1853, eqn (k)). Adding columns gives divergent series, but adding by horizontal lines leads to the equivalent of equation (1.1). The force equation (1.1) has been rederived by less suspect methods than the summing of conditionally convergent series (which Riemann showed can be summed to any desired value): see Smith & Barakat (1975) and O'Meara & Saville (1981). The force between *unequal* touching spheres was also calculated numerically by O'Meara & Saville (1981); here we shall derive an analytical expression for this force.

Maxwell (1891, §175) showed that when two spheres of radii a and b are in contact and at potential $\tilde{V}$ (here and henceforth we denote by '~' those quantities associated with spheres *in contact*), the charge on the sphere of radius a is

$$\tilde{Q}_a = \frac{a^2 b \tilde{V}}{(a+b)^2} \sum_{n=1}^{\infty} \frac{1}{n(n - a/(a+b))} = -\frac{ab\tilde{V}}{a+b}\left[\gamma + \psi\left(\frac{b}{a+b}\right)\right]. \tag{4.1}$$

The second equality in (4.1) follows from the relation (Davis 1972, eqn 6.3.16)

$$\psi(1-z) = -\gamma - z\sum_{n=1}^{\infty}\frac{1}{n(n-z)} \quad (z \neq 1, 2, \ldots). \tag{4.2}$$

Similarly, the charge on sphere b is

$$\tilde{Q}_b = -\frac{ab\tilde{V}}{a+b}\left[\gamma + \psi\left(\frac{a}{a+b}\right)\right]. \tag{4.3}$$

The ratio $\tilde{Q}_b/\tilde{Q}_a$ is approximately equal to $(b/a)^2(\pi^2/6)^{(a-b)/(a+b)}$; this expression is correct when $a = b$, and also when one of a or b is much greater than the other. The maximum error of slightly less than 2.4 per cent occurs when one radius is about four times the other.

We note that the relative magnitudes of $\tilde{Q}_a$ and $\tilde{Q}_b$ are precisely such as to make the leading attractive force term (3.4) zero. Thus, we shall need to include terms of order U^2 or equivalently of order s (which were omitted from (3.2)) in order to calculate the force. However, the capacitance coefficients listed in (3.2) are sufficient to determine the contact energy,

$$\tilde{W}_0 = \frac{-ab\tilde{V}^2}{2(a+b)}[2\gamma + \psi(\beta) + \psi(1-\beta)], \quad \beta = \frac{b}{a+b}. \tag{4.4}$$

To compare this energy with W_0 given in equation (3.5), we need to relate the potential $\tilde{V}$ of the two spheres in contact to the total charge $Q = Q_a + Q_b = \tilde{Q}_a + \tilde{Q}_b$. The capacitance of two spheres in contact is (Russell 1909; Moussiaux & Ronveaux 1979; Lekner 2011*a*)

$$\tilde{C} = -\frac{ab}{a+b}[2\gamma + \psi(\beta) + \psi(1-\beta)]. \tag{4.5}$$

Figure 4 shows $\tilde{C}/(a+b) = -\beta(1-\beta)[2\gamma + \psi(\beta) + \psi(1-\beta)]$ as a function of β. As noted in connection with the close-approach energy W_0 of equation (3.5), this varies between unity when $\beta \to 0$ or $\beta \to 1$ (when one sphere is much larger than the other) and $\ln 2$ when $\beta = 1/2$.

The voltage $\tilde{V}$ when Q is the total charge on the two-sphere system is $\tilde{V} = Q/\tilde{C}$. Thus,

$$\tilde{W}_0 = \frac{1}{2}Q\tilde{V} = \frac{1}{2}\tilde{C}\tilde{V}^2 = \frac{1}{2}\frac{Q^2}{\tilde{C}}. \tag{4.6}$$

Equation (3.5) can also be written in terms of $\tilde{C}$, and we find the remarkable result $W_0 = \tilde{W}_0$. That is, the energy of the two-sphere system *just prior* to electrical contact (and the resulting sharing of charge) is equal to the energy *after* the charges Q_a and Q_b have equilibrated to $\tilde{Q}_a$ and $\tilde{Q}_b$. The two very different charge distributions (before and after electrical contact) have exactly the same energy.

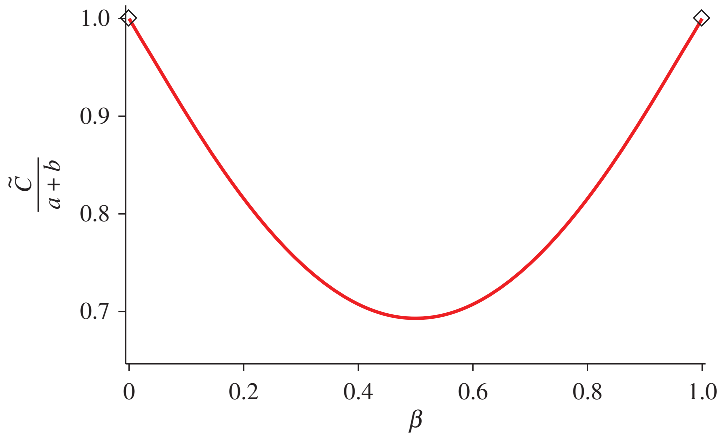

Figure 4. The capacitance of two spheres of radii a and b in contact. The plot shows $\tilde{C}/(a+b)$ as a function of $\beta = b/(a+b)$. The diamonds show the limiting value of unity when one sphere is much larger than the other. (Online version in colour.)

In reality, as discussed in §3, electrical contact will be made before $s = 0$, and the energy $W(s)$ will differ from $W_0 = \tilde{W}_0 = Q^2/2\tilde{C}$ by a term of order $\{\ln[2ab/(a+b)s]\}^{-1}$. This term goes to zero with s, but its derivative is proportional to $1/s$. Thus the approach of $W(s)$ to W_0 is vertical, as seen in figure 3. At some small value of s, an electrical short will cause the energy to drop from $W(s)$ to W_0. After that the charges on the two spheres will be $\tilde{Q}_a$ and $\tilde{Q}_b$, and the spheres will repel each other, as we shall now see.

To find the force acting between the two spheres, we set $Q_a = \tilde{Q}_a$ and $Q_b = \tilde{Q}_b$ into the energy formula (1.12). With these charge values, the close-approach logarithmic terms cancel, and we need the terms of order s that were omitted from (3.2). The results to order s, derived in appendix A, are most conveniently expressed in terms of the parameters U defined in equations (1.5) or (3.1) and $\beta = b/(a+b)$,

$$\left.\begin{aligned}
C_{aa} &= \frac{ab}{a+b}\left\{\ln\frac{2}{U} - \psi(\beta) + \frac{U^2}{6(a+b)^3}\left[(a^3+b^3)\left(\ln\frac{2}{U} - \psi(\beta)\right)\right.\right.\\
&\qquad \left.\left. -\ ab(a-b)\psi'(\beta) + \frac{1}{12}(a+b)(a^2-4ab+b^2)\right]\right\},\\
C_{bb} &= \frac{ab}{a+b}\left\{\ln\frac{2}{U} - \psi(1-\beta) + \frac{U^2}{6(a+b)^3}\left[(a^3+b^3)\left(\ln\frac{2}{U} - \psi(1-\beta)\right)\right.\right.\\
&\qquad \left.\left. +\ ab(a-b)\psi'(1-\beta) + \frac{1}{12}(a+b)(a^2-4ab+b^2)\right]\right\}\\
\text{and}\quad C_{ab} &= -\frac{ab}{a+b}\left\{\ln\frac{2}{U} + \gamma + \frac{U^2}{6(a+b)^2}\left[(a^2-ab+b^2)\left(\ln\frac{2}{U} + \gamma\right)\right.\right.\\
&\qquad \left.\left. +\ \frac{1}{12}(a+b)^2\right]\right\}.
\end{aligned}\right\} \tag{4.7}$$

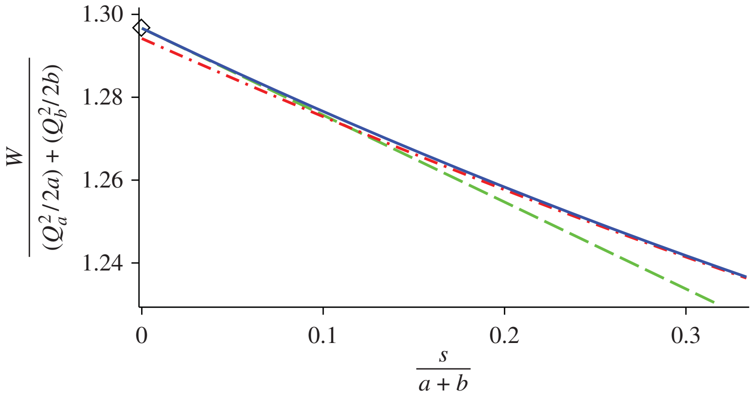

Figure 5. The electrostatic energy of two spheres with charges $\tilde{Q}_a$ and $\tilde{Q}_b$, as a function of the sphere separation. The plot shows $\tilde{W}$ (full curve) and $\tilde{W}_0 - \tilde{F}_0 s$ (dashed line) divided by $\tilde{Q}_a^2/2a + \tilde{Q}_b^2/2b$, and also the large separation expression (dotted-dashed curve), divided by the same energy. The plot is for $b = 2a$. The diamond is at the contact value of the energy ratio, in this case, $36\ln(3)/(\pi^2 - 2\pi\sqrt{3}\ln(3) + 27\ln(3)^2)$. (Online version in colour.)

We shall denote by $\tilde{W}$ the energy of the two-sphere system when $Q_a = \tilde{Q}_a$ and $Q_b = \tilde{Q}_b$. The result to order U^2 (or equivalently to order s) is

$$\tilde{W} = \tilde{W}_0 + U^2 Q^2 \frac{ab(a+b) + (a^3+b^3)[2\gamma + \psi(\beta) + \psi(1-\beta)] + ab(a-b)[\psi'(\beta) - \psi'(1-\beta)]}{12ab(a+b)^2[2\gamma + \psi(\beta) + \psi(1-\beta)]}. \tag{4.8}$$

The force between the two spheres is given by

$$-\tilde{F} = \partial_c \tilde{W} = \frac{\mathrm{d}U}{\mathrm{d}c}\partial_U \tilde{W} = \frac{c}{ab\sinh U}\partial_U \tilde{W}$$
$$= \frac{[a^2 + b^2 + 2ab\cosh U]^{\frac{1}{2}}}{ab\sinh U}\partial_U \tilde{W} = \left[\frac{a+b}{ab}U^{-1} + O(U)\right]\partial_U \tilde{W}. \tag{4.9}$$

Thus, the force at close approach is independent of the sphere separation (since $\tilde{W} = \tilde{W}_0 + O(U^2)$, $\partial_U \tilde{W} = O(U)$) and is equal to $-2(a+b)/(ab)$ times the coefficient of U^2 in (4.8). Figure 5 shows the energy $\tilde{W}$ and its constant force approximation $\tilde{W}_0 - \tilde{F}_0 s$ as a function of the sphere separation.

We wish to compare the repulsive contact force $\tilde{F}_0$ with the force $\tilde{Q}_a \tilde{Q}_b/(a+b)^2$ between point charges $\tilde{Q}_a$ and $\tilde{Q}_b$ separated by distance $a+b$, so we set

$$\tilde{F}_0 = \frac{\tilde{Q}_a \tilde{Q}_b}{(a+b)^2} f_0. \tag{4.10}$$

From (4.8) and (4.9), we deduce, in terms of the parameter $\beta = b/(a+b)$, that

$$f_0 = \frac{\beta(1-\beta)\{(2\beta-1)[\psi'(\beta) - \psi'(1-\beta)] - 1\} - (1 - 3\beta + 3\beta^2)[2\gamma + \psi(\beta) + \psi(1-\beta)]}{6\beta^2(1-\beta)^2[\gamma + \psi(\beta)][\gamma + \psi(1-\beta)]}. \tag{4.11}$$

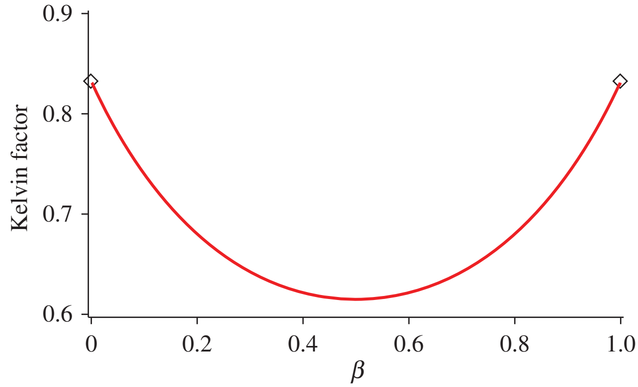

Figure 6. Generalization of the Kelvin factor: the plot shows $f_0(\beta)$ of equation (4.11), as a function of $\beta = b/(a+b)$. The diamonds are at the limiting value (4.13), approached when one sphere is much larger than the other. The Kelvin value (4.12) is at $\beta = 1/2$. (Online version in colour.)

When $\beta = 1/2\,(a = b)$, this reproduces the Kelvin (Thomson 1853) value,

$$f_0(\beta = \tfrac{1}{2}) = \frac{4\ln 2 - 1}{6(\ln 2)^2} = 0.61490\ldots \tag{4.12}$$

If one sphere is much larger than the other ($\beta \to 0$ or $\beta \to 1$), we find

$$f_0(\beta \to 0 \text{ or } 1) = \frac{6\zeta(3) + 1}{\pi^2} = 0.83208\ldots. \tag{4.13}$$

Figure 6 shows the variation of the factor f_0 over the full range of β. The analytical formula (4.11) agrees with the numerical values of F_4 given in table 3 of O'Meara & Saville (1981), given that their force between the spheres is expressed as $(Q^2/a^2)F_4$.

5. Experimental consequences

Some interesting implications follow from the theory given above. Consider, for example, a colloidal suspension of spheres that are conducting or have been coated with a conducting surface, and which have been charged. According to the results of §3, each pair of spheres will attract each other at short range (unless the charges happen to be equal to the charges that the spheres would attain in contact). This electrostatic attraction is reinforced by the attractive van der Waals interaction (Hamaker 1937). As they come into contact, they will share their charge, and then repel each other with the electrostatic force derived in §4.

The simplest case to consider is that of spheres all of radius a, but with unequal charges Q_i. Thermal motion in the suspension, aided by the van der Waals interaction and the short-ranged attraction between spheres of different charge, will lead to pair contact and sharing of charge between pairs. Eventually, each sphere will have a charge close to the average $\langle Q_i \rangle$. The electrostatic and van der Waals forces thus act to equalize the charges on the spheres, and consequently to electrostatic repulsions between them.

At short range, the attractive van der Waals force between two spheres of radius a separated by distance s is approximately $-(Aa/12)\, s^{-2}$, where A is the Hamaker constant (Hamaker 1937; Hunter 1989, ch. 4). The repulsive electrostatic Kelvin force is nearly independent of the separation, at short range. It was given in equation (1.1), in Gaussian units. (In SI units, both the energy and the force expressions derived above are to be divided by the factor $4\pi\varepsilon_0 \approx 111\ pF/m$. So far, we have taken the space between the conductors to be vacuum; in a medium of relative dielectric constant ε, the energy and force expressions of this paper are to be divided by ε or $4\pi\varepsilon\varepsilon_0$ in Gaussian or SI units, respectively. In electrolytes, there is the additional effect of screening of the electrostatic interactions; we do not consider screening here.) The van der Waals force (in theory) increases without bound as the spheres approach each other, and so will eventually dominate over the electrostatic repulsion, leading to coagulation, unless polymerically stabilized (Hunter 1989, ch. 8).

Let us express the charges on the spheres in terms of the electronic charge e, $Q_i = N_i e$. Since $e^2/4\pi\varepsilon_0 \approx 1.44\,\text{eV nm}$, $Q_i^2/4\pi\varepsilon_0 a^2 \approx 1.44\,\text{eV nm}\ N_i^2(\text{nm}/a)^2$. If we take, for example, the parameter values of Meyer *et al.* (2006), namely $a = 30\,\text{nm}$, $n = 1000$, $A = 1.5\,\text{eV}$, the Kelvin electrostatic repulsive force will be about $246\,\text{eV nm}^{-1}$ ($1\text{eV nm}^{-1} \approx 0.16\,\text{nN}$). The attractive van der Waals force at close range, for the above parameter values, is approximately $-3.75\,\text{eV nm}^{-1}\ (\text{nm}/s)^2$, and will dominate for $s < 0.12\,\text{nm}$. Polymeric stabilization could prevent the van der Waals forces from gaining dominance at short range.

After the charge exchange has taken place, the spheres will have repulsive pairwise electrostatic interactions. By varying the temperature, or the amount of original charging, or both, it may be possible to observe a liquid to solid phase transition in the colloid.

Similar considerations apply to electrostatic interactions between charged droplets, of importance in cloud formation, spray atomization, fuel injection and ink-jet printing (see Khachatourian & Wistrom (2001); Su (2006); and references therein). In those examples, the droplets are usually better modelled as spheres with a complex dielectric constant.

In flames, the particles are charged, as evidenced by the ionic wind produced by an applied electric field (Lawton & Weinberg 1969; Weinberg 1973; Gaydon & Wolfhard 1978; Saito *et al.* 1997). We may expect the same kind of interaction between the charged particles as we derived for conducting spheres, with the major difference that the ionized particles in a flame are probably highly variable in shape. Thus, torques as well as forces will be important in their interactions. It is likely that electrostatic forces do play an important role in flames, but in a much more complex way than described for colloidal suspensions of conducting or conductively coated spheres and for charged droplets.

6. Summary and discussion

We have generalized Kelvin's (Thomson 1853) formula for the repulsive force between two conducting spheres in contact. For arbitrary charges, and arbitrary

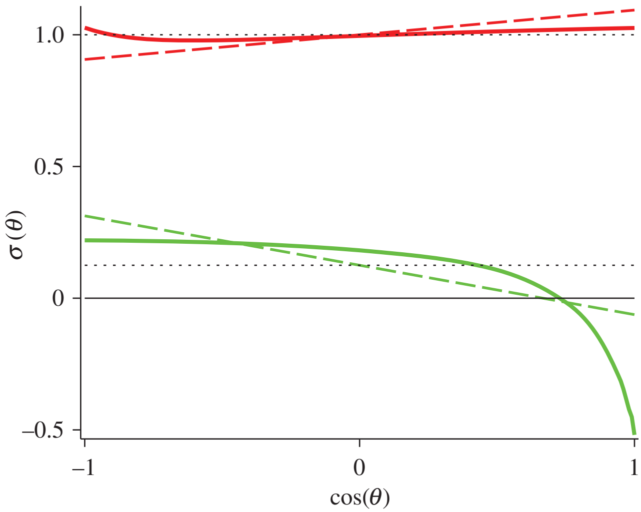

Figure 7. Surface charge densities on the two spheres with $b = 2a$, $Q_b = Q_a/2$, for which the mutual electrostatic energy was plotted in figure 3. The chosen centre-to-centre separation $c = 4a$ $(s = a)$ corresponds to the point $s/(a + b) = 1/3$ in figure 3. The upper and lower surface charge density curves are for sphere a and sphere b, plotted versus the cosines of the polar angle $\cos(A)$ or $\cos(B)$, respectively. The thick curves denote the exact densities, the dashed lines give the dipolar approximations and the dotted lines the undisturbed charge densities, all in units of $Q_a/4\pi a^2$. For each sphere, all three curves have the same (algebraic) area beneath them, since this area is proportional to Q_a and Q_b, respectively. (Online version in colour.)

separations, Maxwell's large-separation expansion contains attractive terms, even for like charges, owing to the mutual polarization of the two spheres. We proved that, at short range, the force between two charged spheres is always attractive, irrespective of the relative sign of their charges, unless their charge ratio is that attained by bringing the spheres into contact (in which case, they repel with a force given by our generalization of the Kelvin formula).

The reader may exclaim 'surely this would have been found experimentally!'. Well, possibly it may have been, for discs. Snow Harris (1836) carried out experiments on charged discs with a torsion balance. We quote directly from §17, Experiment E of his paper: '... the law of force, which at first was $1/d^2$, became at a certain point irregular as the distance decreased, and after being as $1/d$ became in some cases again irregular, until at last the repulsion vanished altogether, and was superseded by attraction'. The capacitance coefficients of a pair of discs of finite thickness are not known, but it seems likely that mutual polarization of the discs, or of any pair of conductors, will lead to attraction between them at short range. (Attraction between like-charged colloidal spheres has also been observed, by Grier (2000), but only when confined either by charged glass walls, or by neighbouring spheres. Wistrom & Khachatourian (1999) measured forces between spheres held at constant potential, rather than between those having fixed charges, as considered here.)

On the theoretical side, Davis (1964) gave series expressions for the coefficients of Q_a^2, Q_aQ_b and Q_b^2 in the force between two spheres, and evaluated them numerically for special values of a, b and s. Simpson (1978) found by numeric computation of the recurrence relations derived by Kelvin (Thomson 1853) that the force between like-charged dust grains (assumed to be spherical) becomes attractive at short range. The charges on the grains were taken to be proportional to their radii. Soules (1990) found attraction between equally charged spheres of different radii. Khachatourian & Wistrom (2000, 2001) calculated attractive forces at short range, as did Su (2006), Bichoutskaia *et al.* (2010) and Kolikov *et al.* (2012).

Thus, the phenomenon of attraction between bodies carrying like charges was known. Not proved was that there will almost always be an attractive force between conducting spheres carrying like charges (the one exception being spheres that have been in contact and have shared their charge), and that this force increases without limit as the spheres approach each other.

The physical explanation lies in charge redistribution on the two spheres owing to their mutual polarization: attraction arises because as the spheres approach each other, a negative charge density appears around the pole of one of the spheres, as calculated and discussed in appendix B. We then have a configuration in which the nearby north and south poles of the two spheres have opposite charge, and the attraction of these near charges dominates over the repulsion of the overall like charges. Figure 7 shows the surface charge distributions on the two spheres for which the energy was displayed in figure 3. We see that the negative charge around the north pole of sphere b is substantial, even at a separation for which the force is still repulsive.

The author is grateful to Pablo Etchegoin, Eric Le Ru, Ben Ruck and the reviewers for helpful comments, and especially for the suggestion of an Editorial Board Member to include details of the surface charge distributions on the two spheres.

Appendix A. Capacitance coefficients in close approach

We wish to derive the close-approach forms of the capacitance coefficients C_{aa}, C_{bb} and C_{ab} given in (4.6). As in Lekner (2011*a*), we shall start with the exact integral equivalents of the defining formulae (1.4), namely,

$$\left.\begin{aligned}
\frac{C_{aa}}{ab} &= \frac{1}{2b} + \frac{\sinh U}{cU}\ln\left(\frac{a+b\mathrm{e}^{U}+c}{a+b\mathrm{e}^{U}-c}\right)\\
&\quad + 2(a+b\cosh U)\sinh U\int_0^\infty \mathrm{d}y\,\frac{\sin Uy(\mathrm{e}^{2\pi y}-1)^{-1}}{(a+b\cosh U)^2-c^2\cos^2 Uy},\\
\frac{C_{bb}}{ab} &= \frac{1}{2a} + \frac{\sinh U}{cU}\ln\left(\frac{a\mathrm{e}^{U}+b+c}{a\mathrm{e}^{U}+b-c}\right)\\
&\quad + 2(a\cosh U+b)\sinh U\int_0^\infty \mathrm{d}y\,\frac{\sin Uy(\mathrm{e}^{2\pi y}-1)^{-1}}{(a\cosh U+b)^2-c^2\cos^2 Uy}\\
\text{and}\quad \frac{-C_{ab}}{ab} &= \frac{1}{2c} + \frac{\sinh U}{cU}\ln\left(\frac{\mathrm{e}^{U}+1}{\mathrm{e}^{U}-1}\right)\\
&\quad + \frac{2}{c}\cosh U\sinh U\int_0^\infty \mathrm{d}y\,\frac{\sin Uy(\mathrm{e}^{2\pi y}-1)^{-1}}{\cosh^2 U-\cos^2 Uy}.
\end{aligned}\right\}\qquad (A\,1)$$

Since

$$\cosh U = 1 + \frac{(a+b)s}{ab} + \frac{s^2}{2ab}, \tag{A 2}$$

the parameter U goes to zero with s (compare to (3.1)),

$$s = \frac{ab}{2(a+b)} U^2 \left[1 + \frac{a^2 - ab + b^2}{12(a+b)^2} U^2 + O(U^4) \right]. \tag{A 3}$$

In expanding the equations (A 1) in powers of U, we encounter the known integral (Davis 1972, eqn 6.3.21)

$$I(z) = \int_0^\infty \mathrm{d}y \frac{y}{(y^2+z^2)(\mathrm{e}^{2\pi y}-1)} = \tfrac{1}{2}\left[\ln z - \frac{1}{2z} - \psi(z) \right] \quad (\mathrm{Re}(z) > 0), \tag{A 4}$$

and also the three integrals

$$J_n(z) = \int_0^\infty \mathrm{d}y \frac{y^n}{(y^2+z^2)^2(\mathrm{e}^{2\pi y}-1)}, \quad n = 1, 3, 5 \quad (\mathrm{Re}(z) > 0). \tag{A 5}$$

$J_1(z)$ can be evaluated from $\mathrm{d}I/\mathrm{d}z$,

$$J_1(z) = \frac{1}{4z}\left[\psi'(z) - \frac{1}{z} - \frac{1}{2z^2} \right]. \tag{A 6}$$

To evaluate $J_3(z)$, we make use of the identity

$$\frac{y^2}{(y^2+z^2)^2} = \frac{1}{y^2+z^2} - \frac{z^2}{(y^2+z^2)^2}, \tag{A 7}$$

from which it follows that

$$J_3(z) = I(z) - z^2 J_1(z). \tag{A 8}$$

Similarly,

$$J_5(z) = \int_0^\infty \mathrm{d}y \frac{y^3}{(y^2+z^2)(\mathrm{e}^{2\pi y}-1)} - z^2 \int_0^\infty \mathrm{d}y \frac{y^3}{(y^2+z^2)^2(\mathrm{e}^{2\pi y}-1)}. \tag{A 9}$$

In the first integral, we set

$$\frac{y^3}{y^2+z^2} = y - \frac{yz^2}{y^2+z^2}, \tag{A 10}$$

and expand in series to evaluate

$$\int_0^\infty \mathrm{d}y \frac{y}{\mathrm{e}^{2\pi y}-1} = \sum_{n=1}^\infty \int_0^\infty \mathrm{d}y\, y\, \mathrm{e}^{-2\pi n y} = \sum_1^\infty \frac{1}{(2\pi n)^2} = \frac{1}{4\pi^2}\frac{\pi^2}{6} = \frac{1}{24}. \tag{A 11}$$

Thus,

$$J_5(z) = \frac{1}{24} - z^2 I(z) - z^2 J_3(z). \tag{A 12}$$

Now all the required integrals are evaluated in terms of $\ln z$, $\psi(z)$ and $\psi'(z)$. It remains to substitute

$$c = \sqrt{a^2 + b^2 + 2ab\cosh U} \tag{A 13}$$

into (A1) and expand in powers of U. Use of the integrals $I(z)$ and $J_n(z)$ then leads to the results (4.6).

Appendix B. Surface charge distributions on the two spheres

To calculate the surface charge densities σ_a and σ_b on the two spheres, we shall use the solution of Laplace's equation in bispherical coordinates (u, v), first formulated by Jeffery (1912). Let the sphere centres lie on the z-axis, with the sphere of radius a above that of radius b, as in figure 1. Then, the sphere surfaces are $u = u_a$ and $u = -u_b$ in bispherical coordinates, where (Lekner 2011*b*)

$$\sinh u_a = \frac{\ell}{a} \quad \text{and} \quad \sinh u_b = \frac{\ell}{b}, \quad \ell = \frac{ab}{c}\sinh U, \tag{B 1}$$

with $U = u_a + u_b$ related to a, b and the distance between the sphere centres c by equation (1.5), namely $\cosh U = (c^2 - a^2 - b^2)/2ab$. The bispherical scale length ℓ can be expressed directly in terms of a, b and c,

$$\ell = \frac{1}{2c}\left[(c - a - b)(c + a - b)(c - a + b)(c + a + b)\right]^{\frac{1}{2}}. \tag{B 2}$$

Jeffery (1912) showed that Laplace's equation is solved by

$$V(u, v) = (2\cosh u - 2\cos v)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[A_n \mathrm{e}^{(n+1/2)u} + B_n \mathrm{e}^{-(n+1/2)u}\right] P_n(\cos v). \tag{B 3}$$

Let the electrostatic potentials on the two spheres be $V(u_a, v) = V_a$ and $V(-u_b, v) = V_b$. The identity (Morse & Feshbach 1953, p. 129)

$$(2\cosh u - 2\cos v)^{\frac{1}{2}} \sum_{n=0}^{\infty} \mathrm{e}^{-(n+1/2)|u|} P_n(\cos v) = 1 \tag{B 4}$$

allows us to remove the $(2\cosh u_a - 2\cos v)^{1/2}$ factor on equating V_a to $V(u_a, v)$, and similarly for V_b. The orthogonality of the Legendre polynomials $P_n(\cos v)$ then gives us the expansion coefficients A_n and B_n,

$$A_n = \frac{\mathrm{e}^{(2n+1)u_b} V_a - V_b}{\mathrm{e}^{(2n+1)U} - 1} \quad \text{and} \quad B_n = \frac{\mathrm{e}^{(2n+1)u_a} V_b - V_a}{\mathrm{e}^{(2n+1)U} - 1}. \tag{B 5}$$

In the problem under consideration, the total charges Q_a and Q_b are fixed. The surface charge density on sphere a is given by (Jeffery 1912, §6)

$$4\pi\sigma_a = \frac{\cosh u_a - \cos v}{\ell} \partial_u V\Big|_{u=u_a}. \tag{B 6}$$

On using Jeffery's integrals,

$$\left.\begin{aligned}\int_{-1}^{1}\mathrm{d}(\cos v)\frac{P_n(\cos v)}{(2\cosh u-2\cos v)^{\frac{1}{2}}}&=\frac{\mathrm{e}^{-(n+1/2)|u|}}{n+\frac{1}{2}}\\ \text{and}\qquad \int_{-1}^{1}\mathrm{d}(\cos v)\frac{P_n(\cos v)}{(2\cosh u-2\cos v)^{\frac{3}{2}}}&=\frac{\mathrm{e}^{-(n+1/2)|u|}}{\sinh|u|},\end{aligned}\right\}\tag{B 7}$$

we find that the total charge Q_a is

$$Q_a=\ell\int_{-1}^{1}\mathrm{d}(\cos v)\frac{\partial_u V|_{u=u_a}}{2\cosh u_a-2\cos v}=2\ell\sum_{n=0}^{\infty}A_n.\tag{B 8}$$

The total charge on sphere b is $Q_b=2\ell\sum_{n=0}^{\infty}B_n$, by a parallel calculation. As in Lekner (2011*b*), we define the sums

$$S(U,u)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{(2n+1)u}}{\mathrm{e}^{(2n+1)U}-1}\quad\text{and}\quad S(U,0)\equiv S_0(U).\tag{B 9}$$

Then, with the shorthand notation $S(U,u_a)=S_a$, $S(U,u_b)=S_b$, we have

$$\frac{Q_a}{2\ell}=\sum_{n=0}^{\infty}A_n=V_aS_b-V_bS_0\quad\text{and}\quad\frac{Q_b}{2\ell}=\sum_{n=0}^{\infty}B_n=V_bS_a-V_aS_0.\tag{B 10}$$

Solving for the sphere potentials, we find

$$V_a=\frac{Q_aS_a+Q_bS_0}{2\ell(S_aS_b-S_0^2)}\quad\text{and}\quad V_b=\frac{Q_bS_b+Q_aS_0}{2\ell(S_aS_b-S_0^2)}.\tag{B 11}$$

We note in passing that the electrostatic energy of the two-sphere system is

$$W=\tfrac{1}{2}Q_aV_a+\tfrac{1}{2}Q_bV_b=\frac{Q_a^2S_a+2Q_aQ_bS_0+Q_b^2S_b}{4\ell(S_aS_b-S_0^2)}.\tag{B 12}$$

Comparison with equation (1.12), which expresses the energy in terms of the capacitance coefficients C_{aa}, C_{bb} and C_{ab}, gives us

$$C_{aa}=2\ell S_b,\quad C_{bb}=2\ell S_a\quad\text{and}\quad C_{ab}=-2\ell S_0.\tag{B 13}$$

This implies, on comparing (1.4) with the sums derived from (B 9), the (apparently new) identities between sums

$$\sum_{n=0}^{\infty}\frac{c}{a\sinh nU+b\sinh(n+1)U}=2\sum_{n=0}^{\infty}\frac{\mathrm{e}^{(2n+1)u_b}}{\mathrm{e}^{(2n+1)U}-1},\tag{B 14}$$

$$\sum_{n=0}^{\infty}\frac{c}{b\sinh nU+a\sinh(n+1)U}=2\sum_{n=0}^{\infty}\frac{\mathrm{e}^{(2n+1)u_a}}{\mathrm{e}^{(2n+1)U}-1}\tag{B 15}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\sinh nU}=2\sum_{n=0}^{\infty}\frac{1}{\mathrm{e}^{(2n+1)U}-1}.\tag{B 16}$$

We return to the problem at hand, namely the calculation of surface charge densities. Equation (B 6) gives us the charge density on sphere a in terms of $\cos v$. Let A and B be the polar angles on spheres a and b, with $A=0=B$ at their north poles, $A=\pi=B$ at their south poles. Then, geometry gives us $\cos v$ on the two spheres,

$$\cos v_a = \frac{\cosh u_a \cos A + 1}{\cosh u_a + \cos A} \quad \text{and} \quad \cos v_b = \frac{1-\cosh u_b \cos B}{\cosh u_b - \cos B}, \tag{B 17}$$

where

$$\cosh u_a = \frac{c^2+a^2-b^2}{2ac} \quad \text{and} \quad \cosh u_b = \frac{c^2-a^2+b^2}{2bc}. \tag{B 18}$$

To calculate the surface charge density, we shall need the more general sums

$$\left.\begin{aligned} S(U,u,v) &= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{(2n+1)u}}{\mathrm{e}^{(2n+1)U}-1} P_n(\cos v) \\ \text{and} \qquad T(U,u,v) &= \sum_{n=0}^{\infty} \frac{(2n+1)\mathrm{e}^{(2n+1)u}}{\mathrm{e}^{(2n+1)U}-1} P_n(\cos v) = \partial_u S(U,u,v). \end{aligned}\right\} \tag{B 19}$$

Since U is fixed at u_a+u_b, we denote $S(U,u,v_a)$ by $S^a(u)$, etc. Then, from (B 6), we find that the charge density on sphere a is

$$\begin{aligned} 4\pi\sigma_a &= (2\cosh u_a - 2\cos v_a)^{1/2} \frac{V_a}{2a} [S^a(u_b+u_a/2) - S^a(-u_a/2)] \\ &\quad + (2\cosh u_a - 2\cos v_a)^{3/2} \frac{1}{4\ell} \{V_a[T^a(u_b+u_a/2) + T^a(-u_a/2)] \\ &\quad - 2V_b T^a(u_a/2)\}. \end{aligned} \tag{B 20}$$

Similarly, on sphere b, the surface charge density is given by

$$\begin{aligned} 4\pi\sigma_b &= (2\cosh u_b - 2\cos v_b)^{1/2} \frac{V_b}{2b} [S^b(u_a+u_b/2) - S^b(-u_b/2)] \\ &\quad + (2\cosh u_b - 2\cos v_b)^{3/2} \frac{1}{4\ell} \{V_b[T^b(u_a+u_b/2) + T^b(-u_b/2)] \\ &\quad - 2V_a T^b(u_b/2)\}. \end{aligned} \tag{B 21}$$

At the north pole of sphere a and at the south pole of sphere b we have $\cos(\nu)=1$, and $P_n(\cos v)=1$. There, the charge densities have the following expansions in reciprocal powers of c, the centre-to-centre distance between the spheres:

$$\left.\begin{aligned} \sigma_a(N) &= \frac{Q_a}{4\pi a^2} + \frac{3Q_b}{4\pi c^2} - \frac{5aQ_b}{4\pi c^3} + O(c^{-4}) \\ \text{and} \qquad \sigma_b(S) &= \frac{Q_b}{4\pi b^2} + \frac{3Q_a}{4\pi c^2} - \frac{5bQ_a}{4\pi c^3} + O(c^{-4}). \end{aligned}\right\} \tag{B 22}$$

At the south pole of sphere a, and at the north pole of sphere b, $\cos v=-1$ and $P_n(\cos v)=(-1)^n$. The charge densities at these neighbouring pole sites have the expansions

$$\left.\begin{aligned}\sigma_a(S) &= \frac{Q_a}{4\pi a^2} - \frac{3Q_b}{4\pi c^2} - \frac{5aQ_b}{4\pi c^3} + O(c^{-4})\\ \text{and}\qquad \sigma_b(N) &= \frac{Q_b}{4\pi b^2} - \frac{3Q_a}{4\pi c^2} - \frac{5bQ_a}{4\pi c^3} + O(c^{-4}).\end{aligned}\right\} \tag{B 23}$$

Thus, as expected, the charge densities are initially diminished (from the large-separation values $Q_a/4\pi a^2$ and $Q_b/4\pi b^2$) at the mutually closest points of the two spheres if the total charges Q_a and Q_b have the same sign. As two positively charged spheres approach closer, one gets a negative charge density at the pole closest to the other sphere, and then the other can get an increased positive charge density at its neighbouring pole, as seen in figure 7. Suppose that $b > a$, and $Q_a > Q_b$ (with both Q_a and Q_b positive), as is the case in our example of $b = 2a$, $Q_b = Q_a/2$ illustrated in figures 3 and 7. Then, the larger effect of surface charge diminishment will be at the north pole of sphere b, and according to the second equation of (B 23), we can expect a negative surface charge $\sigma_b(N)$ when $c \lesssim b(3Q_a/Q_b)^{1/2}$. For the case illustrated, the right-hand side of the inequality is $2a\sqrt{6} \approx 4.9a$. In fact, we find numerically that $\sigma_b(N)$ becomes negative at a larger separation, namely for $c \lesssim 6.62a$. The term $-3Q_a/4\pi c^2$ in $\sigma_b(N)$ of (B 23) is just the dipolar correction: a conducting sphere of radius b is polarized by a *uniform* external field E_0 along the negative z-direction to have a surface charge density $-3/4\pi E_0 \cos B$, corresponding to a dipole at the centre of the sphere. When the separation c is large compared with $a + b$, the field originating from sphere a is Q_a/c^2, so the polarization surface charge density at the north pole of sphere b is $-3Q_a/4\pi c^2$, in agreement with the leading correction term in (B 23).

As the spheres come together, the field of one at the other is highly non-uniform, and the dipolar approximation fails. The full solutions, (B 20) and (B 21), give much greater charge redistribution, as illustrated in figure 7.

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