Subcube isoperimetry and power of coalitions

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The notion of isoperimetry

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Spectral methods

For a d-regular G with the second eigenvalue λ_2 of its adjacency matrix,

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}$$

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A set $S \subset \{0,1\}^n$ is good if $|S| = 1$ or there are $C_m \simeq Q_m$, $C_{m+1} \simeq Q_{m+1}$, $2^m < |S| \leq 2^{m+1}$ s.t. $V(C_m) \subset S \subseteq V(C_{m+1})$ and $S \setminus V(C_m)$ is good.

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A useful estimate [Chung, Füredi, Graham, Seymour]

$$\Phi_E(Q_n, k) \geq k(n - \log_2 k)$$

with equality for $k = 2^d$ attained by a d-dimensional subcube.

Subcube isoperimetric problem in the hypercube

Let $f_n(k,d) = \max_{S \subset V}\{\#_d(S);\ |S| = k\}$ where $\#_d(S)$ denotes the number of (induced) subcubes of dimension d in $Q_n[S]$. (inner subcubes)

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Let $g_n(k,d) = \min_{S \subset V}\{\sigma_d(S);\ |S| = k\}$ where $\sigma_d(S)$ denotes the number of (induced) Q_d's with a vertex in S and a vertex in $\overline{S}$. (border subcubes)

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Corollary

$$g_n(k,d) = \binom{n}{d} 2^{n-d} - f_n(k,d) - f_n(2^n - k, d)$$

for every $n \geq 1$, $0 < k < 2^n$, $d \geq 0$.

Labeling of the hypercube

For a bijection $c : \{0,1\}^n \to [0, 2^n - 1]$, a set $S \subseteq \{0,1\}^n$, and $d \geq 1$ let

$$\delta_c(S) = |S| \max_{u \in S} c(u) - \sum_{u \in S} c(u) \qquad \text{(the maximal deviation of } c \text{ on } S\text{)},$$

$$\Delta_n^d(c) = \sum_{Q_d \simeq C \subseteq Q_n} \delta_c(V(C)) \qquad \text{(the total maximal deviation of } c \text{ on } Q_d\text{'s)}.$$

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Integer coding scenario

1. encode (uniformly) chosen $0 \leq I < 2^n$ by $u = c^{-1}(I) \in \{0,1\}^n$,
2. (at most) d coordinates D are chosen uniformly in random,
3. an adversary with knowledge of c may flip any bit from D in u => u',
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Problem: Find coding c that minimizes expected error $I' - I$.

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Theorem

The binary coding c minimizes $\Delta_n^d(c)$ for every $d \geq 1$.

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   The hyperedge isoperimetry & relevance approach requires an order on $V(\mathcal{H})$ whose initial segments minimize relevance in border hyperedges.

4) Question: Which (classes of) hypergraphs have such an order?

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Smallest total influence [Hart]

$$\min_{f : p_1(f) = k/2^n} I_f = \frac{\Phi_E(Q_n, k)}{2^{n-1}} \quad \text{where } p_1(f) = \Pr_x[f(x) = 1] \ \ (\text{bias})$$

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Theorem [Kahn, Kalai, Linial]

For every $f : \{0,1\}^n \to \{0,1\}$ with $p_1(f) = \frac{1}{2}$ there exists $i \in [n]$ with

$$I_f(i) \geq \frac{c \log n}{n} \qquad \text{where } c \text{ is an absolute constant.}$$

Harmonic analysis of Boolean functions

A Boolean function: $f : \{-1, 1\}^n \to \{-1, 1\}$.

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Influence in Fourier coefficients

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Influence for monotonous functions

$$I_f(S) = \sum_{\substack{T \subseteq S \\ |T| \text{ odd}}} \widehat{f}(T), \quad I_f^d = \sum_{\substack{S \subseteq [n] \\ |S| \text{ odd}}} \widehat{f}(S) \binom{n - |S|}{n - d}$$

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Representation of f by formula φ_1 from k

$$\varphi_n : \begin{cases} \top & b_n = 0 \\ x_n & b_n = 1 \end{cases}, \qquad \varphi_i : \begin{cases} x_i \vee (\varphi_{i+1}) & b_i = 0 \\ x_i \wedge (\varphi_{i+1}) & b_i = 1 \end{cases}$$

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Fourier transform from formula

$$\mathbf{p}_{x_i} = x_i \qquad \mathbf{p}_{\varphi \wedge \psi} = \mathbf{p}_\varphi \mathbf{p}_\psi$$

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Harmonic analysis of good functions

$f : \{-1,1\}^n \to \{-1,1\}$ s.t. $f^{-1}(1)$ is a good set of size $k = \sum_{i=1}^{n} b_i 2^{n-i} < 2^n$

Representation of f by formula φ_1 from k

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Fourier coefficients from k

$$\widehat{f}(S) = \left(\frac{c_j}{2} - \frac{1}{2^n} - \sum_{l=j+1}^{n} \frac{c_l}{2^l} \right) \prod_{i \in S} c_i \quad \text{where } j = \max(S),\ c_i = 1 - 2b_i \in \{-1, 1\}$$

$$I_f = \sum_{i \in [n]} \widehat{f}(\{i\}) = 1 - \frac{1}{2^n} - \frac{1}{2^n} \sum_{i=1}^{n} c_i - \sum_{i<j} \frac{c_i c_j}{2^j}$$

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5) Connections between (minimal) representations of Boolean functions and influence of coalitions.

The story of Dido

Dido - Queen of Carthage / Vergilius: Aeneid

P.-N. Guérin: Aeneas tells Dido the misfortunes of the Trojan city.