Girth of a group

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Bovec, Slovenija, L-L Meeting

September 22, 2012

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We (G. Exoo, RJ) propose a new measure of the complexity of a group related to the widely studied **Cage Problem**.

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We (G. Exoo, RJ) propose a new measure of the complexity of a group related to the widely studied **Cage Problem**.

Fact 1.: Many of the best known graphs for the Cage Problem are Cayley graphs.

Fact 2: Several people started to consider the restriction of the cage problem to vertex-transitive or Cayley graphs:

For given degree k and girth g, find the smallest vertex-transitive (Cayley) graph of degree k and girth g.

Smallest cubic vertex-transitive and Cayley graphs

g	$rec(k,g)$	$n_{cay}(3,g)$	$n_{vt}(3,g)$
3	4	4	4
4	6	6	6
5	10	50	10
6	14	14	14
7	24	30	26
8	30	42	30
9	58	60	60
10	70	96	80
11	112	192	192
12	126	162	126
13	202	272	272
14	258	406	406
15	384	864	620
16	512	1008	1008

Primož et al.

The Girth of a Group

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- one needs to decide whether we require connected Cayley graphs

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- unpublished paper by Saul Schleimer – different definition; involutions make the girth 2
- many people have been thinking along these terms (Biggs, . . .)

Girth of Nilpotent Groups

Theorem (Conder, Exoo, RJ)

If Γ is a nilpotent group of nilpotency class n, then the girth g of Γ (of degree at least 3) is bounded from above as follows:

$$
\begin{array}{ll}
g \leq 4, & \text{if } n = 1, \\
g \leq 8, & \text{if } n = 2, \\
g \leq (n+1)^2, & \text{if } n \geq 3.
\end{array}
$$

Girth of Solvable Groups

Theorem (Conder, Exoo, RJ)

If Γ is a solvable group with derived series of length n, then the girth g of Γ (of degree at least 3) is bounded from above as follows:

$$\begin{aligned} g &\leq 4, && \text{if } n = 1, \\ g &\leq 14 \cdot 4^{n-2}, && \text{if } n \geq 2. \end{aligned}$$

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Specifically, if $Cay(\Gamma, X)$ is a Cayley graph of a solvable group Γ of derived length n and of degree at least 3, $|X| \ge 3$. Then

$g \le 44$,	if $n = 3$ and X contains at least three inv's,
$g \le 48$,	if $n = 3$, and X contains at least two distinct non-inv's,
$g \le 50$,	if $n = 3$, and X consists of one inv and one non-inv,
$g \le 148$,	if $n = 4$ and and X contains at least three inv's,
$g \le 168$,	if $n = 4$, and X contains at least two distinct non-inv's.

General Upper Bounds on the Girth of a Group

- for given girth g and degree k, the order of G cannot exceed the Moore bound:

$$\begin{array}{ll} 1+k\frac{(k-1)^{(g-1)/2}-1}{k-2}, & g \text{ odd} \\ 2\frac{(k-1)^{g/2}-1}{k-2}, & g \text{ even} \end{array}$$

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the maximum order of an element in $\mathbb{S}_n$ is proportional to $e^{\sqrt{n \log n}}$

Theorem (Exoo, RJ, Širáň)

Let G be a k-regular graph of girth g whose edge set can be partitioned into a family $\mathcal{F}$ of k_1 1-factors, F_i, $1 \leq i \leq k_1$, and k_2 oriented 2-factors F_i, $k_1 + 1 \leq i \leq k_1 + k_2$ (where $k_1 + 2k_2 = k$). If $\Gamma_{\mathcal{F}}$ is the finite permutation group acting on the set $V(G)$ generated by the set

$$X = \{\delta_{F_i} \mid 1 \leq i \leq k_1\} \cup \{\sigma_{F_i} \mid k_1 + 1 \leq i \leq k_1 + k_2\} \cup \{\sigma_{F_i}^{-1} \mid k_1 + 1 \leq i \leq k_1 + k_2\},$$

then the Cayley graph $Cay(\Gamma_{\mathcal{F}}, X)$ is k-regular of girth at least g.

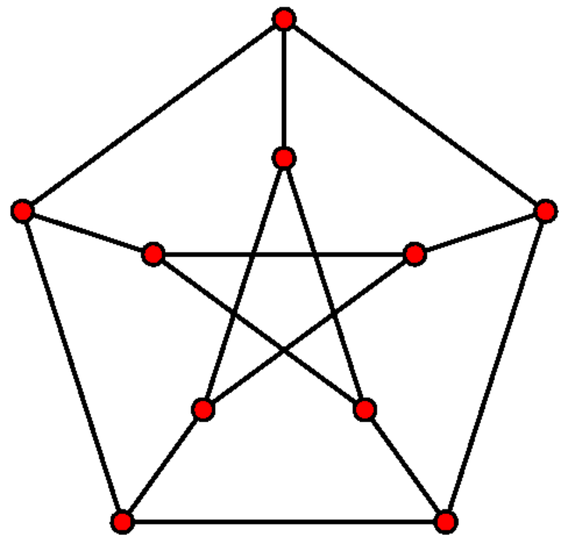

Figure: Smallest $(3,5)$-graph

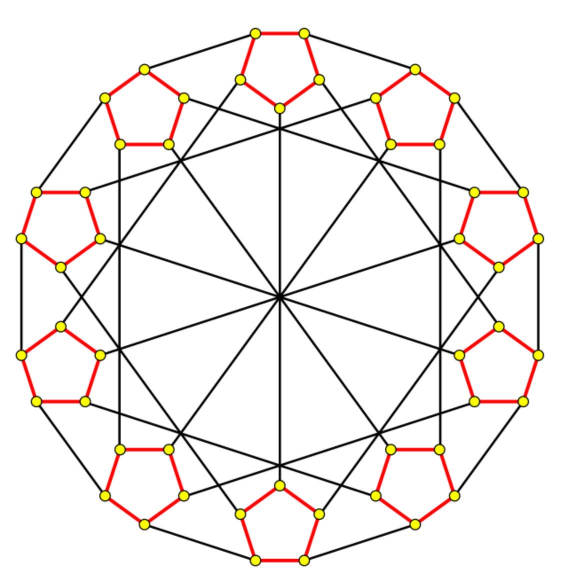

Figure: Smallest Cayley (3, 5)-graph

Theorem (Exoo, RJ, Širáň)

Let $Cay(\Gamma, X)$ be a k-regular graph of girth g. Suppose that Γ has a permutation representation $\gamma \to \sigma_\gamma$, $\gamma \in \Gamma$, on a set V, satisfying the property that no non-reversing product of the permutations σ_x, $x \in X$, of length smaller that g fixes a vertex $v \in V$, and for every $v \in V$, the images $\sigma_x(v)$ are all different. Then the graph G_Γ with vertex set V and edge set $E = \{\{v, \sigma_x(v)\} \mid v \in V, x \in X\}$ is k-regular of girth g.

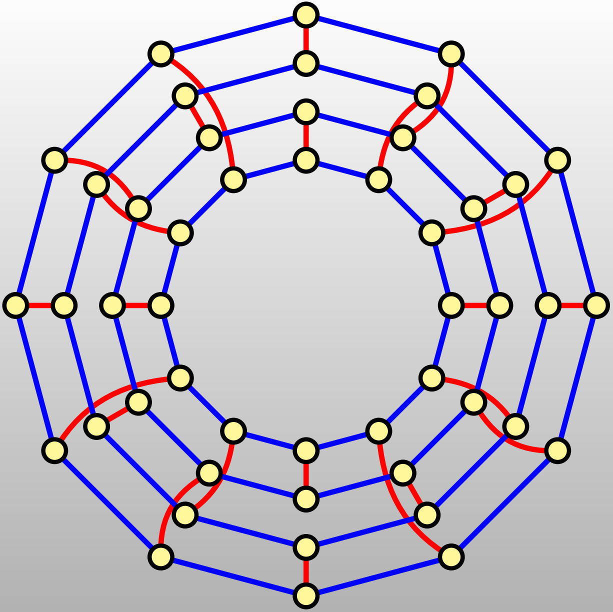

Figure: A Cayley Graph of the group $(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}_3$

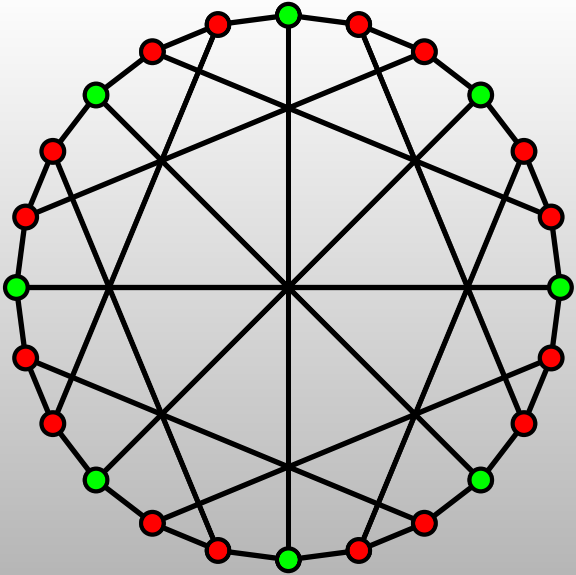

Figure: The McGee Graph

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Theorem (Biggs; Exoo, RJ)

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- the Cayley graph of $\Gamma = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ has girth at least $2r$

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- Γ appears to be either $\mathbb{S}_n$ or $\mathbb{A}_n$
- the girth of these graphs appears quite a bit larger than $2r$, in fact, it appears to be close to n, the order of $T_{k,r}$
- this seems to suggest the existence of Cayley graphs for $\mathbb{S}_n$ of girth at least n

Girth of $\mathbb{S}_n$

n	Type 1 girth	Type 2 girth
05	09	
06	10	
07	14	12
08	20	15
09	22	20
10	24	27
11	30	30
12	32	36
13	36	40
14	40	45
15	44	50
16	48	52
17		54
18		58

Thank You