Rainbow-free colorings in PG(n, q)

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Overview

1. A survey on upper chromatic number of projective planes (results of *G. Bacsó, T. Héger, T. Szőnyi* and *Zs. Tuza*).
2. New constructions of rainbow-free colorings in projective spaces and some bounds on the balanced chromatic numbers of spaces (joint work with *G. Araujo-Pardo* and *A. Montejano*).

Hypergraph coloring

A *C-hypergraph* $\mathcal{H} = (X, \mathcal{C})$ has an underlying vertex set X and a set system $\mathcal{C}$ over X. A *vertex coloring* of $\mathcal{H}$ is a mapping ϕ from X to a set of colors $\{1, 2, \ldots, k\}$.
A *strict rainbow-free k-coloring* is a mapping $\phi : X \to \{1, \ldots, k\}$ that uses each of the k colors on at least one vertex such that each $\mathcal{C}$-*edge* $C \in \mathcal{C}$ has at least two vertices with a *common* color.
The *upper chromatic number* of $\mathcal{H}$, denoted by $\bar{\chi}(\mathcal{H})$, is the largest k admitting a strict rainbow-free k-coloring.

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If $X_i = \phi^{-1}(i)$, then a different but equivalent view is a *color partition* $X_1 \cup \cdots \cup X_k = X$ with k nonempty classes. A coloring is called *balanced,* if

$$-1 \leq |X_i| - |X_j| \leq 1$$

holds for all $i, j \in \{1, 2, \ldots, k\}$.

Coloring of projective spaces

Let Π be an n-dimensional projective space and $0 < d < n$ be an integer. Then Π may be considered as a hypergraph, whose vertices and hyperedges are the points and the d-dimensional subspaces of the space, respectively.

Upper chromatic number of finite planes

Theorem (Bacsó, Tuza (2007))

1. *As $q \to \infty$, any projective plane Π_q of order q satisfies*

$$\bar{\chi}(\Pi_q) \leq q^2 - q - \sqrt{q}/2 + o(\sqrt{q}).$$

2. *If q is a square, then the Galois plane of order q satisfies*

$$\bar{\chi}(\mathrm{PG}(2,q)) \geq q^2 - q - 2\sqrt{q}.$$

The decrement

The result usually formulated in a complementary form, because both the number of points and the upper chromatic number of Π_q are around q^2.

Definition

The decrement of Π_q *is the quantity*
$\mathrm{dec}(\Pi_q) := q^2 + q + 1 - \bar{\chi}(\Pi_q)$.

Double blocking sets

Definition

In the plane Π_q, $B \subset \Pi$ is a double blocking set if every line intersects B in at least two points.
Let τ_2 denotes the size of a smallest double blocking set in Π_q.

The estimation of the double blocking number is a challenging problem and it has a large literature. Lower bounds are much more often considered, mostly in $\mathrm{PG}(2, q)$. However, due to the lack of constructions, we have only weak upper bounds in general.

Coloring and double blocking sets

If B is a double blocking set in Π_q, coloring the points of B with one color and all points outside B with mutually distinct colors, one gets a rainbow-free coloring with $q^2 + q + 1 - |B| + 1$ colors. We call such a coloring a *trivial coloring*. To achieve the best possible out of this idea, one should take B a smallest double blocking set. We have obtained

Proposition

$$\bar{\chi}(\Pi_q)) \geq q^2 + q + 1 - \tau_2 + 1, \quad (1)$$

equivalently,

$$\mathrm{dec}(\Pi_q) \leq \tau_2 - 1.$$

Best known result

Theorem (Bacsó, Héger, Szőnyi (2012))

Let $q = p^h$, p prime. Let $\tau_2 = 2(q+1) + c$ denote the size of the smallest double blocking set in $\mathrm{PG}(2,q)$. Suppose that one of the following two conditions holds:

1. $206 \le c \le c_0 q - 13$, *where* $0 < c_0 < 2/3$, $q \ge q(c_0) = 2(c_0 + 2)/(2/3 - c_0) - 1$, *and* $p \ge p(c_0) = 50c_0 + 24$.
2. $q > 256$ *is a square.*

Then $\mathrm{dec}(\mathrm{PG}(2,q)) = \tau_2 - 1$, and equality is reached if and only if the only color class having more than one point is a smallest double blocking set.

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For arbitrary finite projective planes this result may be false or hopeless to prove.

Balanced colorings

The difference of the sizes of any two color classes is at most 1.
Π_q a projective plane of order q, $v = q^2 + q + 1$.

Proposition

Each balanced rainbow-free coloring of Π_q consists of at most $v/3$ color classes.

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Theorem (G. Araujo-Pardo, Gy. K., A. Montejano (2012))

If 3 divides v then each cyclic plane of order q has a balanced rainbow-free coloring with $v/3$ color classes.

The cyclic model

Example in the case $q = 3$.
The plane of order 3 have $3^2 + 3 + 1 = 13$ points and 13 lines. Take the vertices of a regular 13-gon $P_1 P_2 \ldots P_{13}$. The chords obtained by joining distinct vertices of the polygon have 6 $(= 3(3+1)/2)$ different lengths. Choose 4 $(= 3+1)$ vertices of the regular 13–gon so that all the chords obtained by joining pairs of these points have different lengths. Four vertices define $4 \times 3/2 = 6$ chords. For example the vertices P_1, P_2, P_5 and P_7 form a good subpolygon. Let us denote this quadrangle by Λ_0.

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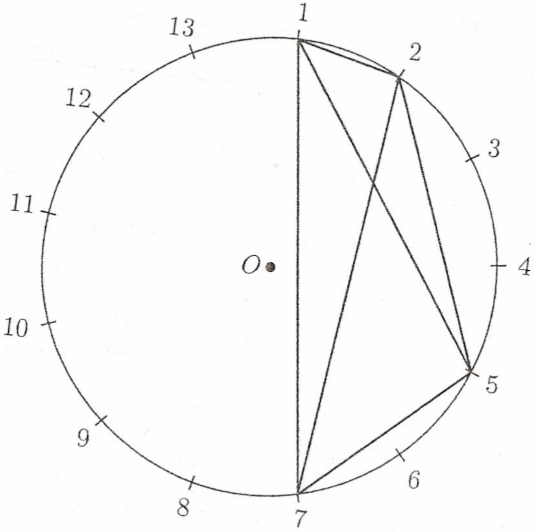
Cyvlic model



We can construct a projective plane of order q, if we are able to choose $q+1$ vertices of the regular (q^2+q+1)-gon in such a way that no two chords spanned by the choosen vertices have the same length.

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One can find easily such sets of vertices if q is a prime power (algebraic method, points of $\mathrm{PG}(2,q) \leftrightarrow$ elements of the cyclic group $\mathrm{GF}^\star(q^3)/\mathrm{GF}^\star(q)$.

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Proof. Let the color classses be the sets $\{P_i, P_{i+v/3}, P_{i+2v/3}\}$ Each line contains exactly one pair of points of the form $(P_j, P_{j+v/3})$ and these two points are in the same class, hence the coloring is rainbow-free.

Balanced colorings

The same construction works if v has a "small" divisor.

Theorem

Suppose that v is not a prime. If $1 < s$ is the smallest nontrivial divisor of v, then each cyclic plane of order q has a balanced rainbow-free coloring with v/s color classes.

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Corollary

If v is not a prime then each cyclic plane of order q has a balanced rainbow-free coloring with more than $q+1$ color classes.

Proof. q does not divide v, hence if s is the smallest nontrivial divisor of v, then $s < q$. This implies $v/s > q+1$.

Balanced colorings

Proposition

In the cyclic model if we divide the circle into q arcs of equal length (the difference between any two lengths is at most 1) and the union of k consecutive arcs contains $k+2$ elements of the difference set, then there exists a balanced rainbow-free coloring with v/m color classes (each arc is a color class).

Theorem

If q is large enough ($q >\approx 15$) then each cyclic plane of order q has a balanced rainbow-free coloring with $\lceil 5q/4 \rceil$ color classes.

Balanced colorings in $\mathrm{PG}(3, q)$

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Theorem

If $\mathrm{PG}(2, q)$ *has a balanced rainbow-free coloring with* k *color classes then in* $\mathrm{PG}(3, q)$ *there also exists a balanced rainbow-free coloring of the points with respect to the lines with the same number of color classes.*

Balanced colorings in $\mathrm{PG}(3, q)$

Theorem

The size of the color classes in any balanced rainbow-free coloring of the points with respect to the lines in $\mathrm{PG}(3, q)$ *is at least* $2q + 2$.

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Theorem

If $q \equiv 1 \pmod 3$ then in $\mathrm{PG}(3, q)$ there is a balanced rainbow-free coloring of the points with respect to the lines with $(q^2 + q + 1)/3$ color classes.
One of the classes has size $3q + 1$, all other classes have size $3q$.

Balanced colorings in higher dimensions

Theorem

Suppose that $k+1|n+1$. *Then* $q^{n+1}/q^{k+1} = A$ *is an integer. In* $\mathrm{PG}(n,q)$ *there is a balanced rainbow-free coloring of the points with respect to the k-dimensional subspaces with* $(q^n + q^{n-1} + \ldots + q + 1) - A$ *color classes.*

Theorem

Suppose that $k+1|n+1$. Then $q^{n+1}/q^{k+1} = A$ is an integer. In $\mathrm{PG}(n,q)$ there is a balanced rainbow-free coloring of the points with respect to the k-dimensional subspaces with $(q^n + q^{n-1} + \ldots + q + 1) - A$ color classes.

Theorem

If $\mathrm{PG}(n,q)$ has a balanced rainbow-free coloring of the points with respect to the lines with k color classes then in $\mathrm{PG}(n+1,q)$ there also exists a balanced rainbow-free coloring of the points with respect to the lines with the same number of color classes.

THANKS FOR YOUR ATTENTION!