Realizations of the Game Domination Number

Boštjan Brešar, Sandi Klavžar, Gašper Košmrlj, Doug Rall

Faculty of Mathematics and Physics

University of Ljubljana, Slovenia

20th September 2012

Domination Game

- game on a finite graph
- two players, Dominator (male) and Staller (female)
- legal move: set of dominated vertices enlarges by at least one
- Dominator-start game (γ_g) and Staller-start game (γ'_g)

Basic problems

1. For a given graph G find $\gamma_g(G)$ and/or $\gamma'_g(G)$.
2. For a given pair $(k, l) \in \mathbb{N} \times \mathbb{N}$ find a graph G for which $\gamma_g(G) = k$ and $\gamma'_g(G) = l$.

Examples:

- $\gamma_g(P) = 5$, $\gamma'_g(P) = 4$, P Petersen graph
- $\gamma_g(P_n) = \lceil \frac{n}{2} \rceil$
- K_n realizes $(1, 1)$ for any $n \geq 1$
- $K_{m,n}$ realizes $(3, 2)$ for any $m, n \geq 1$

What pairs can be realizable?

For a vertex subset S of a graph G, $G|S$ denotes the partially dominated graph in which vertices from S are already dominated.

Continuation Principle: [Kinnersley, West, Zemani] For any graph G and $B \subseteq A \subseteq V(G)$ it follows that $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.

Corollary: $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.
Proof:

$$u \in V(G), \qquad \gamma_g(G) \leq \gamma'_g(G|N[u]) + 1 \overset{CP}{\leq} \gamma'_g(G) + 1$$

$$\gamma'_g(G) = \max_{x \in V(G)} \gamma_g(G|N[x]) + 1 \overset{CP}{\leq} \gamma_g(G) + 1$$

Realizations and lexicographic product

Problem: For every $r \geq 1$ and pair $p \in \{(k, k+1), (k, k), (k+1, k)\}$ find a family of r-connected graphs $\{G_k; k \geq 1\}$ that realizes pair p.

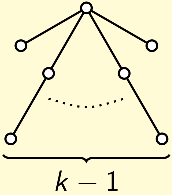
Theorem: For every $n \geq 1$, $\gamma_g(G \circ K_n) = \gamma_g(G)$ and $\gamma_g'(G \circ K_n) = \gamma_g'(G)$.

Recall that $\kappa(G \circ H) = \kappa(G)|V(H)|$ (if G not complete).

Graph is *prime* if it is **not** constructed as in the above theorem.

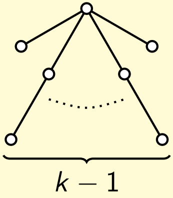
Pair $(k, k+1)$

1-connected prime: T_k

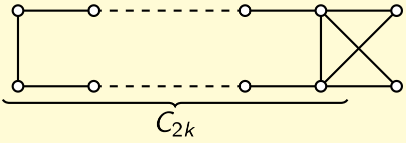


Pair $(k, k+1)$

1-connected prime: T_k



2-connected prime: G_k



Pair (k, k)

1-connected prime: P_{2k} [K, W, Z]

Pair (k, k)

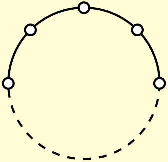
1-connected prime: P_{2k} [K, W, Z]

2-connected prime: C'_k

Pair (k, k)

1-connected prime: P_{2k} [K, W, Z]

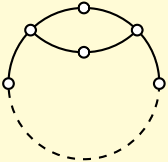
2-connected prime: C'_k



Pair (k, k)

1-connected prime: P_{2k} [K, W, Z]

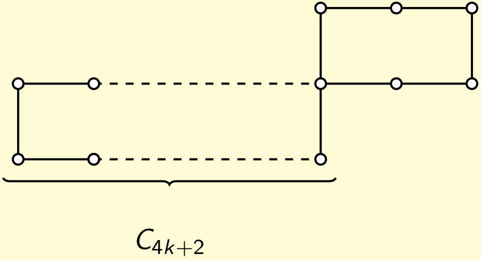
2-connected prime: C'_k



C'_k

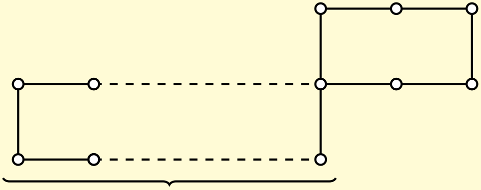
Pair $(2k+1, 2k)$

1-connected prime: CC_k



Pair $(2k+1, 2k)$

1-connected prime: CC_k



C_{4k+2}

2-connected prime: C_{4k+2} [K, W, Z]

Pair $(2k+2, 2k+1)$

$(2,1)$ not realizable

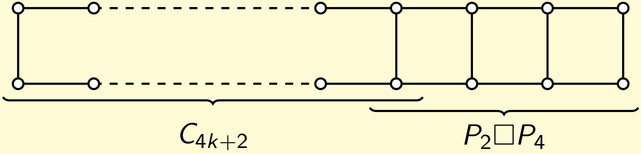
1-connected prime: a very complicated one [Z]

Pair $(2k + 2, 2k + 1)$

$(2, 1)$ not realizable

1-connected prime: a very complicated one [Z]

2-connected prime: BL_k



Extremal realizations

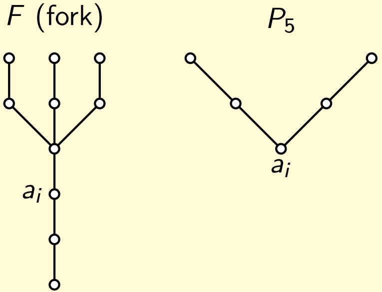
3/5-conjecture: If G is an isolate-free graph of order n, then $\gamma_g(G) \leq 3n/5$.

Is the bound tight?

Extremal realizations

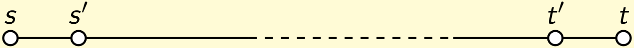
3/5-conjecture: If G is an isolate-free graph of order n, then $\gamma_g(G) \leq 3n/5$.

Is the bound tight? yes



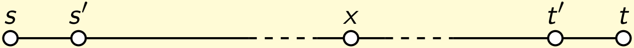
Construction using paths and forks

Path of order $k \geq 6$:



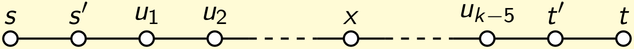
Construction using paths and forks

Choose one of the middle vertices:



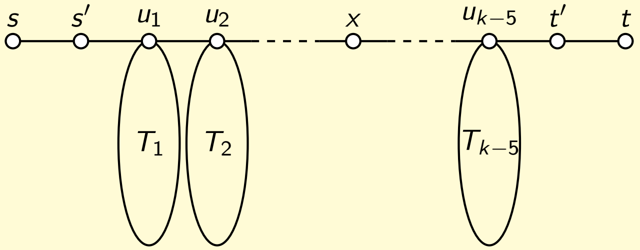
Construction using paths and forks

Label the rest of the vertices:



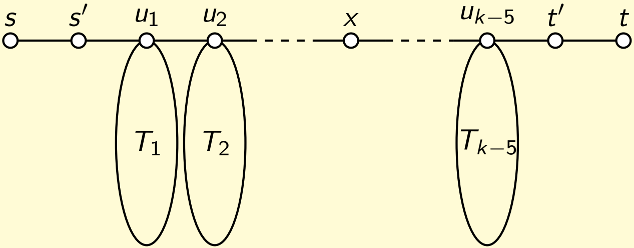
Construction using paths and forks

Identify u_i with $a_i \in V(T_i)$, where $T_i \in \{P_5, F\}$ for $i = 1, 2, \ldots, k-5$:



Notation: $T_k^{\ell}[T_1, T_2, \ldots, T_{k-5}]$ where $\ell = d(s,x) + 1$

Construction using paths and forks

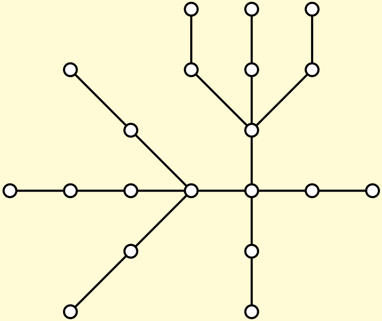


Notation: $T_k^\ell[T_1, T_2, \ldots, T_{k-5}]$ where $\ell = d(s,x)+1$

Theorem: $T_k^\ell[T_1, T_2, \ldots, T_{k-5}]$ is a 3/5-tree.

Example

$T^3_7[P_5, F]$:



Attachable trees

T a tree and $x \in V(T)$, then *attachable tree* is a pair (T, x) provided:

1. x is an optimal-start vertex for Dominator in Game 1 on T
2. $\gamma_g(T|x) = \gamma_g(T)$
3. $\gamma_g'(T) = \gamma_g(T)$

Theorem: $(T_k^{\ell}[T_1, \ldots, T_{k-5}], x)$ is attachable.

Another construction

G graph with $V(G) = \{v_1, \ldots, v_n\}$ and H_i, $1 \leq i \leq n$, be a connected graph of order $m_i \geq 2$, and $x_i \in V(H_i)$. We denote by $G[H_1[x_1], H_2[x_2], \ldots, H_n[x_n]]$ the graph of order $\sum_{i=1}^{n} m_i$ formed by identifying x_i and v_i for $1 \leq i \leq n$.

Theorem: (T_1, x_1) and (T_2, x_2) attachable 3/5-trees, then $K_2[T_1[x_1], T_2[x_2]]$ is a 3/5-tree.

General case

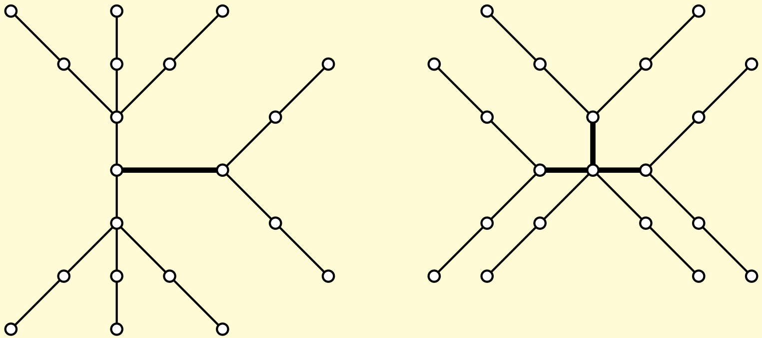
Attachable tree (T, x) is *special* if for any optimal first move of Staller in Game 2 that is different from x, Dominator can optimally reply with a move on x.

Theorem: If G connected graph of order n, (T_i, x_i) special attachable 3/5-trees for $i = 1, \ldots, n$, then $G[T_1[x_1], \ldots, T_n[x_n]]$ is a 3/5-graph.

Examples

Special attachable 3/5-trees: P_5, F , $T_7^3[P_5, P_5]$, $T_7^4[P_5, P_5]$
(Question: Are all T_k^ℓ-like trees special?)

$K_2[T_7^4[P_5, P_5], P_5]$ and $K_{1,3}[P_5, P_5, P_5, P_5]$:



Thank you for your attention!

Questions?