Distance-regular Cayley graphs of abelian groups

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1. Definition
2. Distance-regular cayley graphs over cyclic groups
3. Distance-regular cayley graphs over dihedral groups
4. “Minimal” distance-regular Cayley graphs on abelian groups

A connected finite graph is **distance-regular** if the cardinality of the intersection of two spheres depends only on their radiuses and the distance between their centres.

Distance-regular graphs

A connected graph Γ with diameter D is **distance-regular**, whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in V(\Gamma)$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\{z \in V(\Gamma) : \partial(x, z) = i \text{ and } \partial(y, z) = j\}|$$

is independent of x and y.

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The numbers p_{ij}^{h} are called the **intersection numbers** of Γ.

Antipodal distance-regular graphs

Let Γ be a distance-regular graph with diameter D. Then Γ is **antipodal**, if the relation "being at distance 0 or D" is an equivalence relation on the vertex set of Γ.

Antipodal distance-regular graphs

Assume Γ is an antipodal distance-regular graph with diameter D.

Define graph $\overline{\Gamma}$ as follows: the vertex set of $\overline{\Gamma}$ are the equivalence classes of the above equivalence relation, and two vertices (equivalence classes) are adjacent in $\overline{\Gamma}$ if and only if there is an edge in Γ connecting them.

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$\overline{\Gamma}$ is called the **antipodal quotient** of Γ. It is distance-regular, non-antipodal, and its diameter is $\lfloor \frac{D}{2} \rfloor$.

Bipartite distance-regular graphs

Assume Γ is a bipartite distance-regular graph with diameter D. A connected component of its second distance graph, denoted by $\frac{1}{2}\Gamma$, is again distance-regular, non-bipartite, and its diameter is $\lfloor \frac{D}{2} \rfloor$.

Primitive distance-regular graphs

If distance-regular graph Γ is non-antipodal and non-bipartite, then it is called **primitive**.

Primitive distance-regular graphs

Assume Γ is non-primitive distance regular graph. Then the following hold:

- If Γ is antipodal and non-bipartite, then $\overline{\Gamma}$ is primitive.
- If Γ is bipartite and non-antipodal, then $\frac{1}{2}\Gamma$ is primitive.
- If Γ is antipodal and bipartite with odd diameter, then $\overline{\Gamma}$ and $\frac{1}{2}\Gamma$ are primitive.
- If Γ is antipodal and bipartite with even diameter, then $\overline{\Gamma}$ is bipartite and $\frac{1}{2}\Gamma$ is antipodal. Moreover, $\frac{1}{2}\overline{\Gamma}$ and $\overline{\frac{1}{2}\Gamma}$ are isomorphic and primitive.

Cayley graphs

Let G be a finite group with identity 1, and let S be an inverse-closed subset of $G \setminus \{1\}$. A Cayley graph $\mathrm{Cay}(G; S)$ has elements of G as its vertices, the edge-set is given by $\{\{g, gs\} : g \in G, s \in S\}$.

Distance-regular circulants

Miklavič and Potočnik (2003):

Γ is (nontrivial) distance-regular Cayley graph of cyclic group if and only if Γ is a Paley graph on p vertices, where p is a prime congruent to 1 modulo 4.

Idea of the proof

Cyclic groups of non-prime order are examples of B-groups.

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If Γ is a Cayley distance-regular graph over a B-group G, then G is either antipodal, or bipartite, or a complete graph.

If Γ is distance-regular circulant, then also $\overline{\Gamma}$ and $\frac{1}{2}\Gamma$ are circulants.

Idea of the proof

Therefore, if Γ is a distance-regular circulant, then one of the following holds:

- Γ is a complete graph.
- Γ is bipartite or antipodal with diameter $D \in \{2, 3\}$.
- Γ is bipartite and antipodal with diameter $D \in \{4, 6\}$.
- Γ is primitive of prime order.
- Γ is bipartite (or antipodal) with diameter $D \geq 4$, and $\overline{\Gamma}$ (or $\frac{1}{2}\Gamma$) is distance-regular circulant of prime order.

Distance-regular dihedrants

Miklavič and Potočnik (2007):

Every (nontrivial) distance-regular Cayley graph of dihedral group is bipartite, non-antipodal graph of diameter 3, and it arises from certain difference set either in a cyclic or dihedral group.

What if a group is not a B-group?

Group $\mathbb{Z}_m \times \mathbb{Z}_n$ is a B-group if and only if $m \neq n$.

What if a group is not a B-group?

Muzychuk (2005):

Classification of strongly regular Cayley graphs of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$, where p is a prime.

"Minimal" drcg on abelian groups

Assume now that G is abelian. Let S be an inverse-closed subset of $G \setminus \{1\}$ which generates G. Assume that there exists $s \in S$, such that $S \setminus \{s, s^{-1}\}$ does not generate G.

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Problem

Clasify distance-regular Cayley graphs $\mathrm{Cay}(G; S)$, where G and S are as above.

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From now on assume that G, S and s are as on the previous slide. As cycles are clearly of this kind, we will assume that the valency $k = |S|$ of $\mathrm{Cay}(G; S)$ is at least 3.

Let $H = \langle S \setminus \{s, s^{-1}\}\rangle$ and let $o(s)$ denote the order of s.

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Theorem

With the above notation we have $[G : H] \leq 4$.

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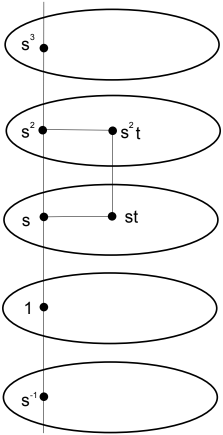

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Theorem

With the above notation, assume that $o(s) = [G : H]$. Then the following hold.

- If $[G : H] \in \{4, 2\}$, then $\mathrm{Cay}(G; S)$ is a Hamming graph $H(d, 2)$, that is, a hypercube.
- If $[G : H] = 3$, then $\mathrm{Cay}(G; S)$ is a Hamming graph $H(d, 3)$.

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Theorem

With the above notation, assume that $[G : H] = 3$ and $o(s) \geq 6$. Then $\mathrm{Cay}(G : S)$ is $K_{3,3}$.

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Theorem

With the above notation, assume that $[G : H] = 2$ and $o(s) \geq 6$. Then $\mathrm{Cay}(G : S)$ is $K_{6,6} - 6K_2$ or $K_{2,2,2}$.

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Theorem

With the above notation, assume that $[G : H] = 2$ and $o(s) = 4$. Then $\mathrm{Cay}(G : S)$ is either the antipodal quotient of $H(d, 2)$, or a Hamming graph graph $H(d, 4)$, or a Doobs graph $D(n, m)$ with $n \geq 1$.

THANK YOU!