On graphs whose complement and square are isomorphic

Martin Milanič, Anders Sune Pedersen, Daniel Pellicer

University of Primorska, Koper, Slovenia

University of Southern Denmark

UNAM, Mexico

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Main definitions

Graph complement

G: a simple graph

The complement of G is the graph $\overline{G}$ defined as:

- $V(\overline{G}) = V(G)$
- $E(\overline{G}) = \{uv \,:\, u,v \in V(G) \wedge u \neq v \wedge uv \notin E(G)\}$

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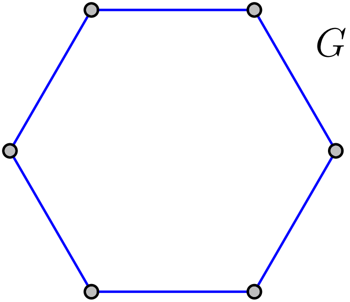

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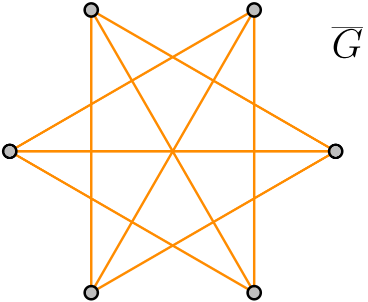

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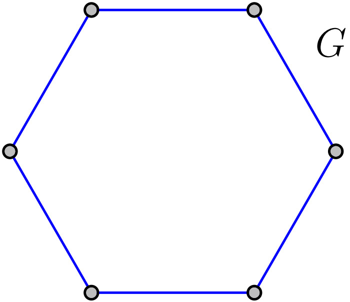

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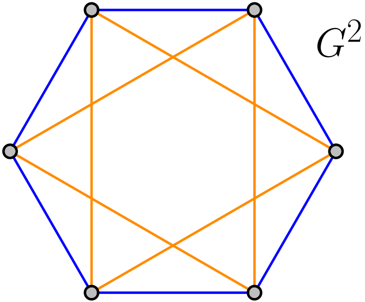

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Recall that two graphs G and H are isomorphic if there exists an isomorphism from G to H, that is, a bijection $\varphi : V(G) \to V(H)$ that preserves adjacencies and non-adjacencies.

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- $\psi(G) = G$: self-complementary graphs

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Equivalently:

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Examples

Square-complementary graphs

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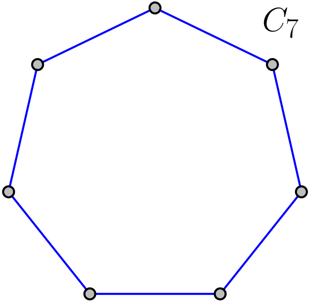

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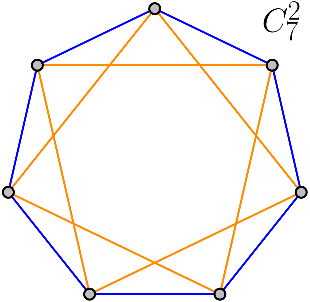

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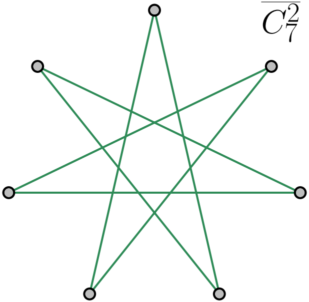
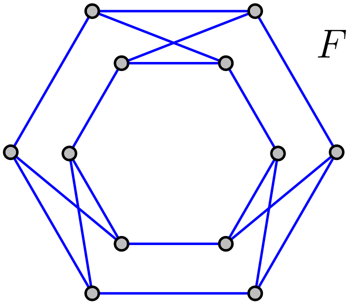


Square-complementary graphs

Another (bipartite) example: the Franklin graph



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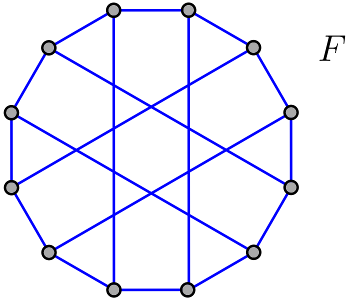

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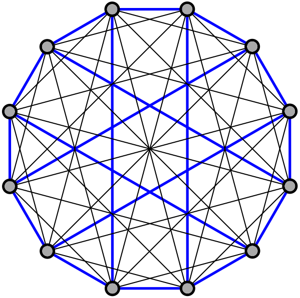

F^2

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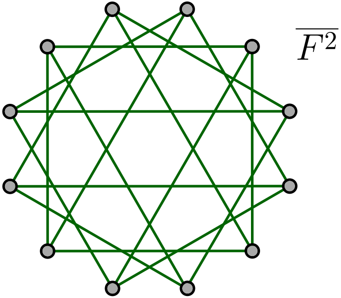

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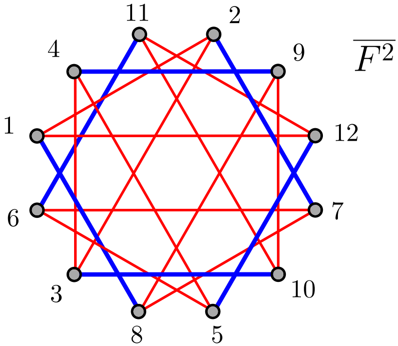

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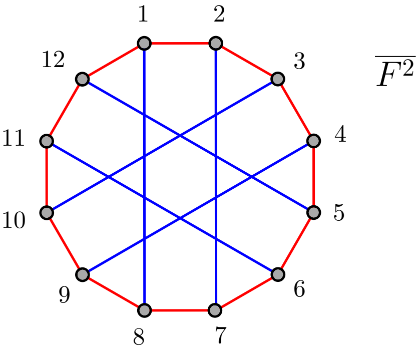
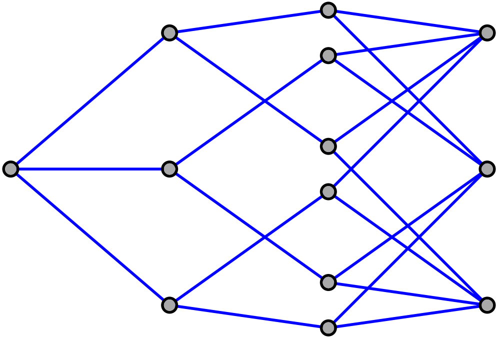


Square-complementary graphs

Another bipartite example, on 13 vertices:



Constructions

Infinite families of squco graphs

G: a graph, k: a positive integer

$G[k]$: the graph obtained by

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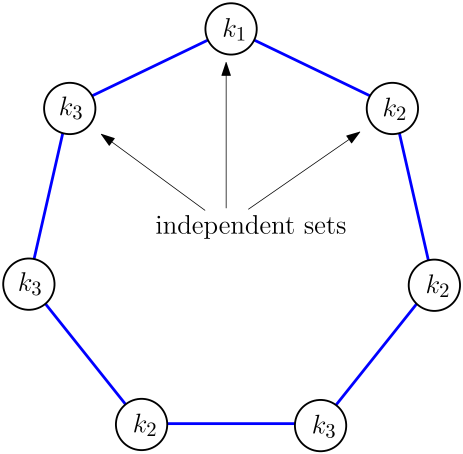
- For every nontrivial vertex transitive squco graph G and every $u \in V(G)$, the graph obtained by replacing u with a set of $k \geq 1$ non-adjacent vertices is squco.

Infinite families of squco graphs

Every graph in the following family of graphs arising from C_7 is a squco graph:



Properties of squco graphs

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- G is of radius at least 3.
- G is of diameter at most 4.
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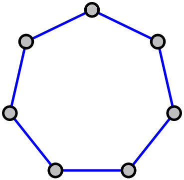
- G has no cut vertex.
- no proper spanning subgraph of G is a squco graph.

Characterizations in particular graph classes

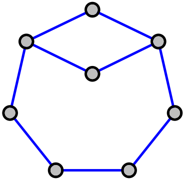
Subcubic squco graphs

The only squco graphs of maximum degree at most 3 are the following four graphs:

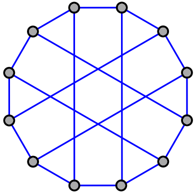
K_1



C_7



C_7^+



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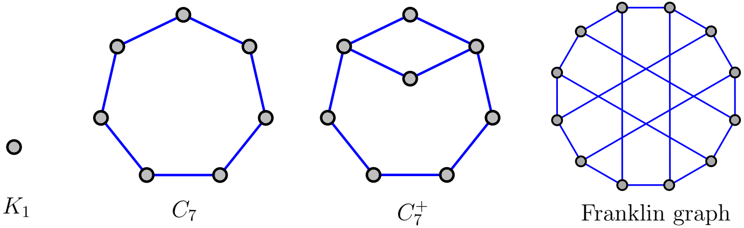

Corollary

There exists a squco graph on n vertices if and only if $n = 1$ or $n \geq 7$.

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A bipartite graph G is a squco graph if and only if it satisfies the following two conditions:

- *G is a connected bipartite self-complementary graph.*

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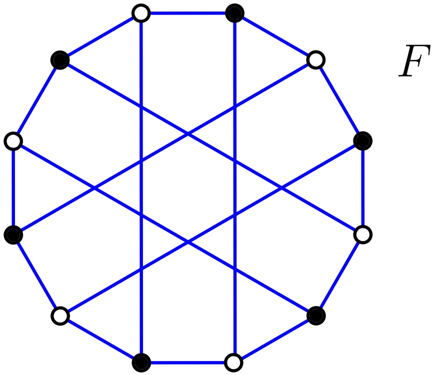
Theorem

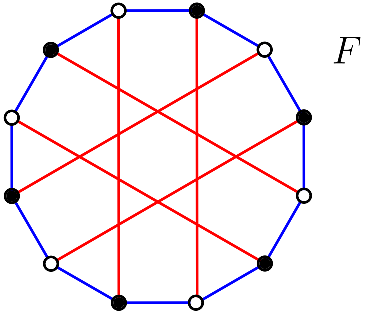
A bipartite graph G is a squco graph if and only if it satisfies the following two conditions:

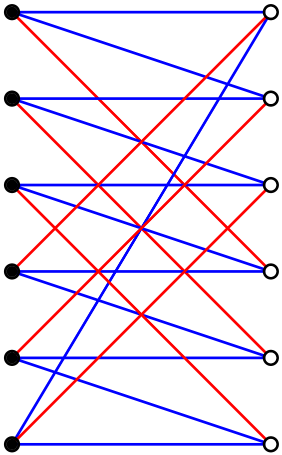
- *G is a connected bipartite self-complementary graph.*
- *Every two vertices in the same part have a common neighbor.*

Another look at the Franklin graph



Cyclic Haar graphs

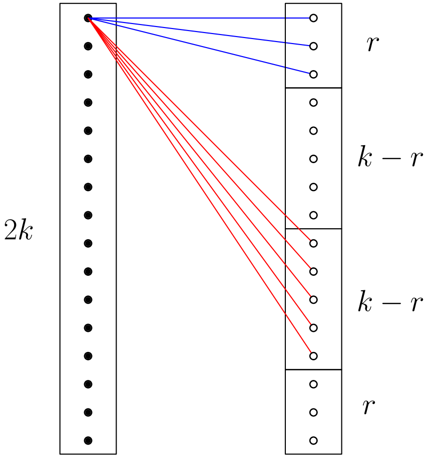




Cyclic Haar Graphs (Hladnik, Marušič, Pisanski, 2002)

Cyclic Haar graphs

The following cyclic Haar graphs are all squco:



$k \geq 3$

$2 \leq r \leq \frac{k+1}{2}$

$k = 3,\ r = 2$:
Franklin graph

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- Is there a nontrivial chordal squco graph?
- Is there a squco graph containing a triangle?
- Can squco graphs contain arbitrarily long induced paths?

Questions?

Thank you for your attention!

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