Formulas for various domination numbers of products of paths and cycles

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Ljubljana-Leoben 2012

Polygraphs

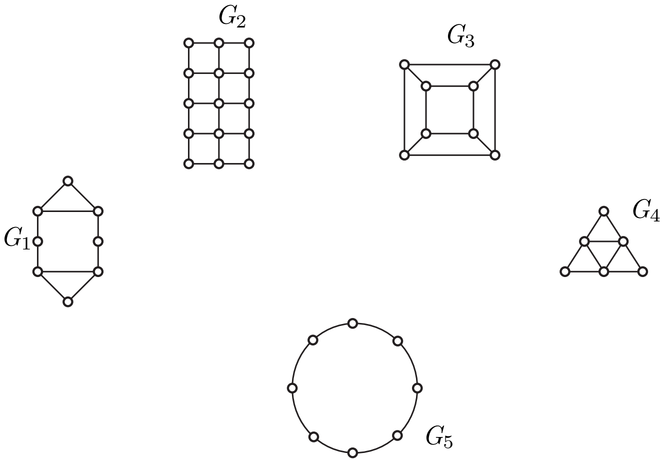
Definition. *A polygraph* $\Omega_n = \Omega_n(G_1, \ldots G_n; X_1, \ldots X_n)$ *over mutually disjoint monographs* $G_1, \ldots, G_n$ *has the vertex set*

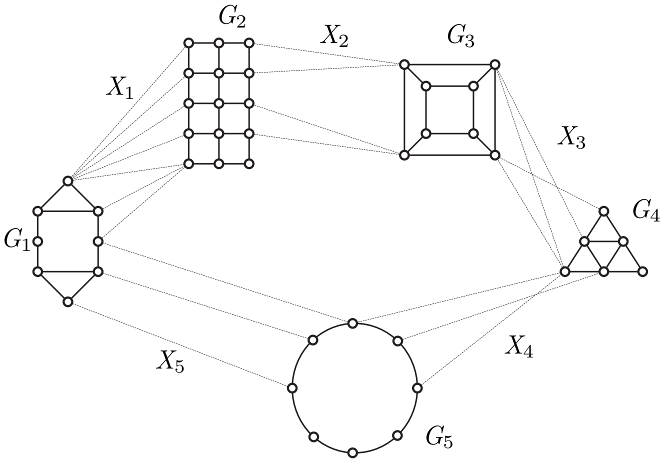
$$V(\Omega_n) = V(G_1) \cup \ldots \cup V(G_n),$$

and the edge set

$$E(\Omega_n) = E(G_1) \cup X_1 \cup \ldots \cup E(G_n) \cup X_n,$$

where $X_i \subseteq V(G_i) \times V(G_{i+1})$ *for* $i = 1, \ldots, n$ *and* $G_{n+1} \cong G_1$*.*


G_2
G_3
G_1
G_4
G_5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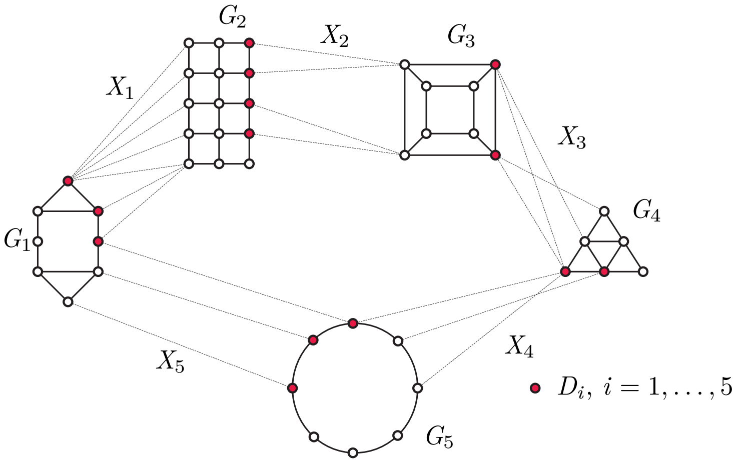
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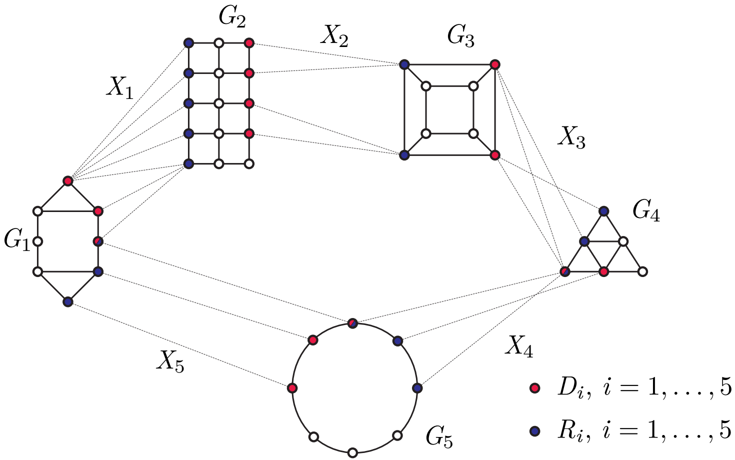
$D_i,\ i = 1, \ldots, 5$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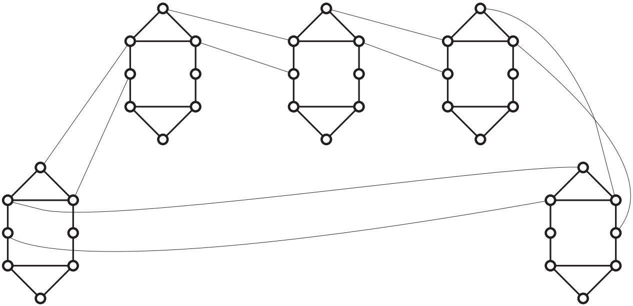
Rotagraphs and fasciagraphs



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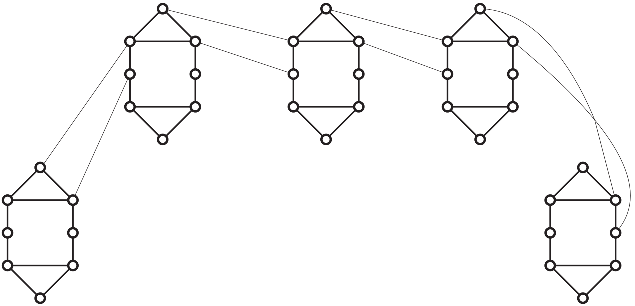

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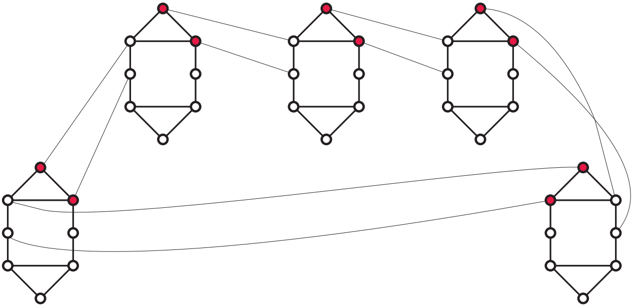

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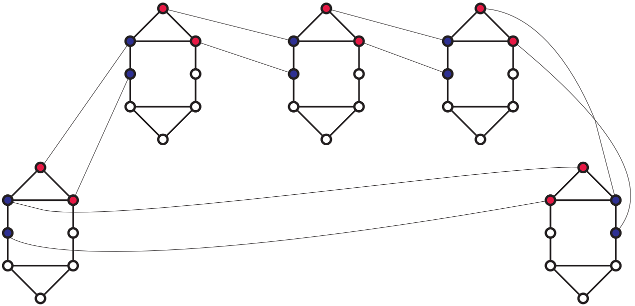
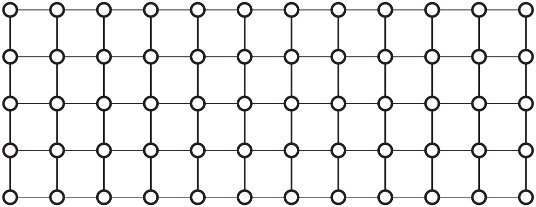
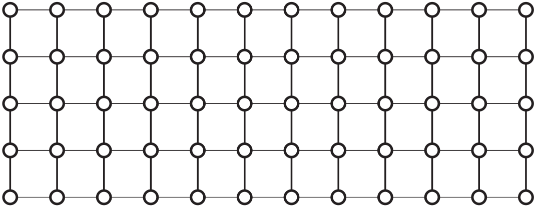




$P_{12} \square P_5$



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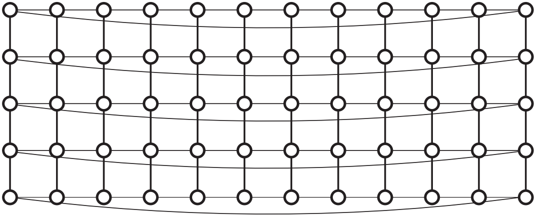

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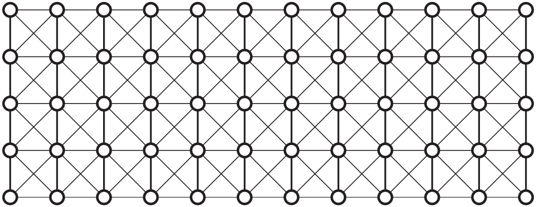
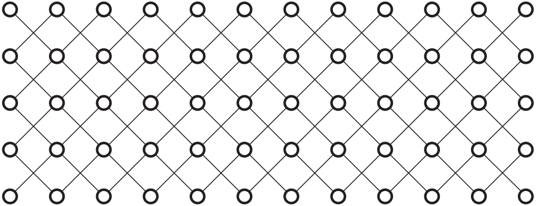


$P_{12} \boxtimes P_5$



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The domination number and its variations

Definition. *A set $D \subseteq V$ of a graph $G = (V, E)$ is a dominating set, if $N[D] = V$. The size of the smallest dominating set of a graph is the domination number, $\gamma(G)$.*

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Complexity results

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Still NP-complete for

- bipartite graphs (Chang et al., 1984), chordal graphs (Booth and Johnson, 1985),...
- other domination types.

Path algebra

Definition. *A semiring* $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ *is a set* P *on which two binary operations,* $\oplus$ *and* $\circ$ *are defined such that:*

1. $(P, \oplus)$ *is a commutative monoid with* $e^{\oplus}$ *as a unit;*
2. $(P, \circ)$ *is a monoid with* e° *as a unit;*
3. $\circ$ *is left– and right–distributive over* $\oplus$*;*
4. *for every* $x \in P$, $x \circ e^{\oplus} = e^{\oplus} = e^{\oplus} \circ x$

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Examples of path algebras:

$$\mathcal{P}_1 = (\{0,1\}, \max, \min, 0, 1)$$

$$\mathcal{P}_2 = (\mathbb{N}_0 \cup \{-\infty\}, \max, +, -\infty, 0)$$

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"tropical semiring"

Matrices with elements of a path algebra:
Let $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ be a path algebra and let $\mathcal{M}_n(\mathcal{P})$ be the set of all $n \times n$ matrices over P. Let $A, B \in \mathcal{M}_n(\mathcal{P})$ and define operations $\oplus$ and $\circ$ in the usual way:

$$\begin{aligned} (A \oplus B)_{ij} &= A_{ij} \oplus B_{ij}, \\ (A \circ B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \circ B_{kj}. \end{aligned}$$

Observation: $\mathcal{M}_n(\mathcal{P})$ equipped with above operations is a path algebra with the zero and the unit matrix as units of semiring.

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Example: $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

- The zero matrix:

$$\begin{bmatrix} \infty & \infty & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & \infty \end{bmatrix}$$

- The unit matrix:

$$\begin{bmatrix} 0 & \infty & \dots & \infty \\ \infty & 0 & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & 0 \end{bmatrix}$$

- $(A \oplus B)_{ij} = \min \{A_{ij}, B_{ij}\}$,
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Path algebras and digraphs

Let $\mathcal{P} = (P, \oplus, \circ, e^{\oplus}, e^{\circ})$ be a path algebra and let G be a labeled digraph, that is a digraph together with a labeling function ℓ which assigns to every arc of G an element of P: $\ell : E(G) \longrightarrow P$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and define the following matrix:

$$(A(G))_{ij} = \begin{cases} \ell(v_i, v_j); & \text{if } (v_i, v_j) \text{ is an arc of } G \\ e^{\oplus}; & \text{otherwise} \end{cases}$$

The labeling ℓ of G can be extended to walks: for a walk $Q = (v_{i_0}, v_{i_1})(v_{i_1}, v_{i_2}) \ldots (v_{i_{k-1}}, v_{i_k})$ of G let

$$\ell(Q) = \ell\left(v_{i_0}, v_{i_1}\right) \circ \ell\left(v_{i_1}, v_{i_2}\right) \circ \ldots \circ \ell\left(v_{i_{k-1}}, v_{i_k}\right).$$

Observation: Let S_{ij}^k be the set of all walks of order k from v_i to v_j in G. Then

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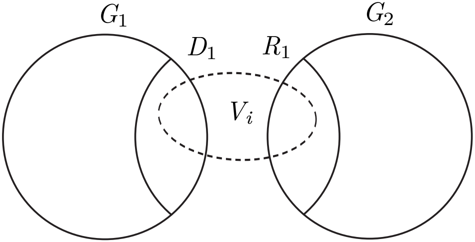
Algorithm, different graph invariants

$O(\log n)$ **algorithm (Klavžar and Žerovnik, 1996)**

Let $\omega_n(G;X)$ be a rotagraph and $\psi_n(G;X)$ a fasciagraph. Define a labeled digraph $\mathcal{G} = \mathcal{G}(G;X)$:

- $V(\mathcal{G}) \ldots$ subsets of $D \sqcup R$;
- $E(\mathcal{G}) \ldots$ between vertices that are not in "conflict".



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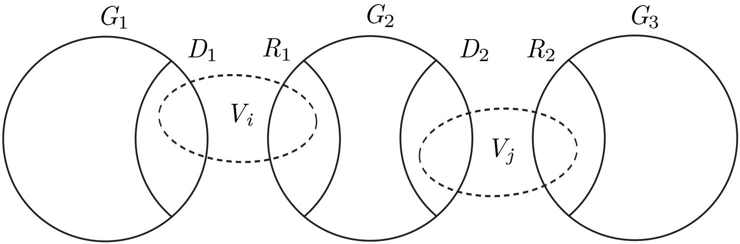
3. $\ell(V_i, V_j) = |V_i \cap R| + \gamma_{i,j}(G; X) + |D \cap V_j| - |V_i \cap R \cap D \cap V_j|$.



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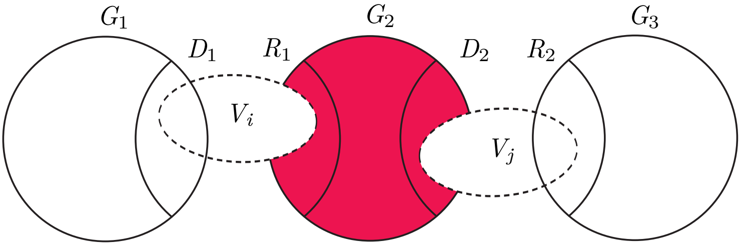

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Then

$$\gamma\left(\psi_n(G; X)\right) = \left(A\left(\mathcal{G}\right)^n\right)_{00}$$

and

$$\gamma\left(\omega_n(G; X)\right) = \min_i \left(A\left(\mathcal{G}\right)^n\right)_{ii}.$$

Generalization

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- Time complexity;
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- Other domination - type invariants;
- Implementation to get some closed expressions.

Calculating $A\left(\mathcal{G}\right)^n$ in $O(1)$

Let A^n denote $A\left(\mathcal{G}\right)^n$ and $A \in \mathcal{M}_n(\mathcal{P})$, where $\mathcal{P} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0)$.

Lemma (-, Žerovnik, 2012)

Let $N = |V(\mathcal{G}(G;X))|$, $K = |V(G)|$. Then there is an index q such that $A^q = A^p + C$ for some index $p < q$ and some constant matrix $C = [c]_{ij}$. Let $P = q - p$. Then for every $r \geq p$ and every $s \geq 0$ we have

$$A^{r+sP} = A^r + sC\,.$$

Fasciagraphs

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Assume that the j–th row of A^{n+P} and A^n differ for a constant, $a^{n+P}_{ji} = a^n_{ji} + C$ for all i. Then $\min_i a^{n+P}_{ji} = \min_i a^n_{ji} + C$.

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Rotagraphs

Lemma (-, Žerovnik, 2012)

Let $A^q = A^p + C$ and $P = q - p$. Then for every $t \in 0, 1, \ldots, P - 1$ there is a constant C_t such that for all $n \geq p$ with $t \equiv (n - p) \pmod{P}$ we have

$$\gamma(\psi_n(G; X)) - \gamma(\omega_n(G; X)) = C_t.$$

Results and remarks

Theorem (-, Žerovnik, 2012)

Domination numbers of fasciagraphs and rotagraphs can be computed in constant time, i.e. independently of the size of a monograph G.

k	$\gamma(P_n \Box C_k)$
3	$\left\lceil \frac{3n}{4} \right\rceil + 1$; **if** $n \equiv 0 \pmod 4$ $\left\lceil \frac{3n}{4} \right\rceil$; **otherwise**
4	n
5	**3**; **if** $n = 2$ **4**; **if** $n = 3$ $n + 2$; **otherwise**
6	$\left\lceil \frac{4n}{3} \right\rceil$; **if** $n \equiv 1 \pmod 3$ $\left\lceil \frac{4n}{3} \right\rceil + 1$; **otherwise**
7	$\left\lceil \frac{3n}{2} \right\rceil + 1$; **if** $n \equiv 1 \pmod 2$ $\left\lceil \frac{3n}{2} \right\rceil + 2$; **otherwise**

k	$\gamma(P_n \square C_k)$	
8	**4;**	**if** $n = 2$
	6;	**if** $n = 3$
	8;	**if** $n = 4$
	$\left\lfloor \frac{9n}{5} \right\rfloor + 1$;	**if** $n \equiv 5 \pmod{10}$
	$\left\lfloor \frac{9n}{5} \right\rfloor + 2$;	**otherwise**
9	**5;**	**if** $n = 2$
	7;	**if** $n = 3$
	10	**if** $n = 4$
	$2n + 2$;	**otherwise**
10	$2n + 2$;	**if** $n \leq 5$
	$2n + 3$;	**if** $6 \leq n \leq 9$
	$2n + 4$	**otherwise**
11	$\left\lfloor \frac{19n}{8} \right\rfloor + 1$;	**if** $n \in \{1, 2, 4, 6\}$ **or** $n \equiv 3 \pmod{8}$
	$\left\lfloor \frac{19n}{8} \right\rfloor + 2$;	**otherwise**

k	$\gamma(C_n \square P_k)$	$\gamma(C_n \square C_k)$
2	$\begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1; & \text{if } n \equiv 2 \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise} \end{cases}$	
3	$\left\lceil \frac{3n}{4} \right\rceil$	$\left\lceil \frac{3n}{4} \right\rceil$
4	$\begin{cases} n+1; & \text{if } n \in \{5, 9\} \\ n; & \text{otherwise} \end{cases}$	n
5	$\begin{cases} 4; & \text{if } n = 3 \\ \left\lceil \frac{6n}{5} \right\rceil + 1; & \text{if } n \equiv 3, 5, 9 \pmod{10} \\ \left\lceil \frac{6n}{5} \right\rceil; & \text{otherwise} \end{cases}$	$\begin{cases} n; & \text{if } n \equiv 0 \pmod 5 \\ n+2; & \text{if } n \equiv 3 \pmod 5 \\ n+1; & \text{otherwise} \end{cases}$
6	$\begin{cases} 9; & \text{if } n = 6 \\ \left\lceil \frac{10n}{7} \right\rceil + 1; & n \equiv 2, 6, 7, 9 \\ & 13 \pmod{14} \\ \left\lceil \frac{10n}{7} \right\rceil; & \text{otherwise} \end{cases}$	$\begin{cases} \left\lceil \frac{4n}{3} \right\rceil + 1; & n \equiv 2, 3, 8, 9 \pmod{18} \\ & 11, 14, 15, 17 \pmod{18} \\ \left\lceil \frac{4n}{3} \right\rceil; & \text{otherwise} \end{cases}$
7	$\begin{cases} 6; & \text{if } n = 3 \\ 16; & \text{if } n = 9 \\ 36; & \text{if } n = 21 \\ \left\lceil \frac{5n}{3} \right\rceil; & \text{otherwise} \end{cases}$	$\begin{cases} \left\lceil \frac{3n}{2} \right\rceil; & n \equiv 0, 5, 9 \pmod{14} \\ \left\lceil \frac{3n}{2} \right\rceil + 2; & n \equiv 2, 8, 12 \pmod{14} \\ \left\lceil \frac{3n}{2} \right\rceil + 1; & \text{otherwise} \end{cases}$