Sectional split extensions arising from lifts of groups

Rok Požar
University of Primorska

26^{th} Ljubljana-Leoben seminar, Bovec, Slovenia

September 21, 2012

Graph covers

Regular covering

A surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ and $p^{-1}(d)$ = orbits of a semiregular subgroup CT_p.

Regular covering

A surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ and $p^{-1}(d)$ = orbits of a semiregular subgroup CT_p.

Isomorphism of regular coverings

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \tilde{g}\ } & \tilde{X}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[\ g\ ]{} & X \end{array}$$

Graph covers

Regular covering

A surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ and $p^{-1}(d)$ = orbits of a semiregular subgroup CT_p.

Isomorphism of regular coverings

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[g]{} & X \end{array}$$

In particular, $g = \mathrm{id} \Rightarrow p$ and p' are **equivalent**.

Regular covering

A surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ and $p^{-1}(d)$ = orbits of a semiregular subgroup CT_p.

Isomorphism of regular coverings

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow[g]{} & X \end{array}$$

In particular, $g = \mathrm{id} \Rightarrow p$ and p' are **equivalent**.

Lifting automorphism along regular coverings

$\tilde{X} = \tilde{X}'$ and $p = p' \Rightarrow g$ **lifts along** p.

Stating Problem

Stating Problem

Setting

X a connected graph,

a group $G \leq \mathrm{Aut}(X)$,

a nonempty subset $\Omega \subseteq V_X$ invariant under the action of G.

Setting

X a connected graph,

a group $G \leq \mathrm{Aut}(X)$,

a nonempty subset $\Omega \subseteq V_X$ invariant under the action of G.

Problem

Find all connected regular covering projections $p\colon \tilde{X} \to X$ along which G lifts as a **sectional split extension** over Ω ($\exists$ a complement $\bar{G}$ with an invariant section over Ω).

Setting

X a connected graph,
a group $G \leq \mathrm{Aut}(X)$,
a nonempty subset $\Omega \subseteq V_X$ invariant under the action of G.

Problem

Find all connected regular covering projections $p\colon \tilde{X} \to X$
along which G lifts as a **sectional split extension** over Ω
($\exists$ a complement $\bar{G}$ with an invariant section over Ω).

Covers are given in terms of **voltage assignments** $\zeta\colon X \to \Gamma \cong \mathrm{CT}_p$.

Introducing a new vertex

Introducing a new vertex

Notation

$\widehat{X}(\Omega) = X + *$, where $*$ adjacent to Ω,

Notation

$$\widehat{X}(\Omega) = X + *, \text{ where } * \text{ adjacent to } \Omega,$$
$$\widehat{G} \leq \mathrm{Aut}(\widehat{X}(\Omega)) \text{ that fixes } * \text{ and acts on } X \text{ as } G,$$

Introducing a new vertex

Notation

$\widehat{X}(\Omega) = X + *$, where $*$ adjacent to Ω,

$\widehat{G} \leq \mathrm{Aut}(\widehat{X}(\Omega))$ that fixes $*$ and acts on X as G,

ζ on $X \Rightarrow$ an extension $\widehat{\zeta}$ on $\widehat{X}(\Omega)$ with

$$\widehat{\zeta}_x = \begin{cases} \zeta_x, & x \in D_X; \\ 1, & x \in D_{\widehat{X}(\Omega)} \setminus D_X, \end{cases}$$

Notation

$$\widehat{X}(\Omega) = X + *, \text{ where } * \text{ adjacent to } \Omega,$$

$$\widehat{G} \leq \mathrm{Aut}(\widehat{X}(\Omega)) \text{ that fixes } * \text{ and acts on } X \text{ as } G,$$

$$\zeta \text{ on } X \Rightarrow \text{an extension } \widehat{\zeta} \text{ on } \widehat{X}(\Omega) \text{ with}$$

$$\widehat{\zeta}_x = \begin{cases} \zeta_x, & x \in D_X; \\ 1, & x \in D_{\widehat{X}(\Omega)} \backslash D_X, \end{cases}$$

ζ on $\widehat{X}(\Omega)$ with the extra darts carrying the trivial voltage $\Rightarrow \bar{\zeta}$ a restriction to X.

Converting Problem

Converting Problem

Theorem 1 Let $p\colon \tilde{X} \to X$ be a regular covering projection of connected graphs. Then the following statements are equivalent

(1) G lifts along p as a sectional split extension over Ω

(2) p can be reconstructed by ζ on X such that $\widehat{G}$ lifts along $p_{\widehat{\zeta}}\colon \mathrm{Cov}(\widehat{\zeta}) \to \widehat{X}(\Omega)$.

Theorem 1 Let $p\colon \tilde{X} \to X$ be a regular covering projection of connected graphs. Then the following statements are equivalent

(1) G lifts along p as a sectional split extension over Ω

(2) p can be reconstructed by ζ on X such that $\widehat{G}$ lifts along $p_{\widehat{\zeta}}\colon \mathrm{Cov}(\widehat{\zeta}) \to \widehat{X}(\Omega)$.

Theorem 2 Let ζ be an assignment on $\widehat{X}(\Omega)$, that is trivial on the set of extra darts, and reconstructs p admitting the lift of $\widehat{G}$. If the restriction $\bar{\zeta}$ to X is connected, then G lifts along $p_{\bar{\zeta}}$ as a sectional split extension over Ω. Any connected regular cover of X along which G lifts as a sectional split extension over Ω arises in this way.

Reducing up to equivalence

Lemma 1 Let ζ and ζ' be two equivalent connected voltage assignments on $\widehat{X}(\Omega)$, which are trivial on the set of extra darts. Then their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ to X are also equivalent. Hence they are either both connected or both disconected.

Lemma 1 Let ζ and ζ' be two equivalent connected voltage assignments on $\widehat{X}(\Omega)$, which are trivial on the set of extra darts. Then their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ to X are also equivalent. Hence they are either both connected or both disconected.

Note: if ζ and ζ' are two nonequivalent connected assignments on $\widehat{X}(\Omega)$, it still might happen that their restrictions $\bar{\zeta}$ and $\bar{\zeta}'$ to X are both connected yet equivalent.

Example

All pairwise nonequivalent connected regular $\mathbb{Z}_p^k$ covers of K_4 along which $\mathbb{Z}_4$ lifts as a sectional split extension over $\Omega = K_4$

Example

All pairwise nonequivalent connected regular $\mathbb{Z}_p^k$ covers of K_4 along which $\mathbb{Z}_4$ lifts as a sectional split extension over $\Omega = K_4$

Line	Condition	Dim	Voltage array
1.	$p \equiv -1\,(4)$	1	$[1],[1],[1],[1],[0],[0]$
2.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
3.		3	$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$
4.	$p \equiv 1\,(4), \lambda_0^2 = -1$	1	$[1],[1],[1],[1],[0],[0]$
5.		1	$[1],[\lambda_0],[-1],[-\lambda_0],[0],[0]$
6.		1	$[1],[-\lambda_0],[-1],[\lambda_0],[0],[0]$
7.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-\lambda_0\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
8.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\\lambda_0\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\-\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
9.		2	$\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}\lambda_0\\-\lambda_0\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix},\begin{bmatrix}-\lambda_0\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}$
10.		3	$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\\lambda_0\\-\lambda_0\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\-\lambda_0\\\lambda_0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$
11.	$p = 2$	1	$[1],[1],[1],[1],[1],[1]$
12.		2	$\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}$

Thank you!