How vertex stabilizers grow?

Ljubljana-Leoben 2012

Pablo Spiga

Università degli studi di Milano
Dipartimento di Matematica Pura ed Applicata
pablo.spiga@unimib.it

Bovec, September 20–22, 2012
In this talk we are interested on the structure and on the size of vertex stabilizers in highly transitive graphs.

Graphs will be connected (not necessarily finite) and vertex stabilizers will be finite.

For a graph $\Gamma$ to be highly symmetric it is natural to require that $\text{Aut}(\Gamma)$ is transitive

1. on the vertices (vertex-transitive graphs): all the vertices look the same; or

2. on the edges (edge-transitive graphs): all the edges look the same; or

3. on the arcs (arc-transitive graphs): all the ordered pairs of adjacent edges look the same.
A natural measure for the degree of symmetry of a graph are the ratio
\[ \frac{|\text{Aut}(\Gamma)|}{|V\Gamma|} \] and \[ \frac{|\text{Aut}(\Gamma)|}{|E\Gamma|}, \]
so the size of the vertex stabilizers and of the edge stabilizers.

**Question:** What is the structure and what is the size of vertex stabilizers in a highly-transitive graph \( \Gamma \)?

This question is somehow too vague, so we consider three cases:

1. we fix some "graph-group" property;
2. we fix the "residue group", that is, the action of \( G_x \) on \( \Gamma_x \);
3. we fix the valency of \( \Gamma \);
4. we fix nothing (can we still say something about \( G_x \)?)

Pablo Spiga
Universitá degli studi di Milano

How vertex stabilizers grow?
Locally $s$-arc-transitive graphs

Given a graph $\Gamma$ and $s \geq 1$, an $s$-arc of $\Gamma$ is a sequence of vertices $\omega_0, \ldots, \omega_s$ of $\Gamma$ with $\omega_i$ adjacent to $\omega_{i+1}$, for every $i \in \{0, \ldots, s - 1\}$, and with $\omega_{i-1} \neq \omega_{i+1}$, for every $i \in \{1, \ldots, s - 1\}$.

Given a subgroup $G$ of the automorphism group of $\Gamma$, we say that $\Gamma$ is locally $s$-arc-transitive with respect to $G$ if, for every vertex $\omega$ of $\Gamma$, the vertex stabilizer $G_\omega$ is transitive on the $s$-arcs with initial vertex $\omega$. 
locally $s$-arc-transitive implies locally $(s - 1)$-arc-transitive.

(For $s \geq 1$.) The group $G$ acts transitively on the edges of $\Gamma$. So $G$ has at most two orbits on the vertices of $\Gamma$.

(For $s \geq 2$.) The vertex stabilizer $G_\omega$ acts 2-transitively on the neighbourhood $\Gamma_\omega$.

The bigger the $s$ the more symmetric the graph $\Gamma$ is.
How vertex stabilizers grow?
Examples Tutte 8-cage: $s = 5$.

Graphs with large $s$ are (for example) the incident graphs associated to some geometrical structures: like generalized $n$-gons. The classical examples arise from the Tit’s buildings associated to the Ree groups $^2F_4(2^n)$...plus some sporadic examples due to Marston Conder. (Typically graphs with large girth and small diameter may have large $s$.)

A generalized $n$-gon is an incidence structure consisting of a set of points $P$ and of a set of lines $L$, where in the bipartite incidence graph with vertex set $P \cup L$ has girth $2n$ and diameter $n$. 
Theorem (Richard Weiss)

*If G is transitive on the vertices of Γ of valency > 2 and G (and so Γ) is finite, then s ≤ 7.*

This is one of the first remarkable applications of the classification of the finite 2-transitive groups...

which in turn is one of the first important applications of the Classification of the Finite Simple Groups.
15 years and hundreds of pages later

**Theorem (Trofimov, Weiss)**

*If $G$ is transitive on the vertices of $\Gamma$ of valency $> 2$, $s \geq 2$ and $G$ (and so $\Gamma$) is finite, then $G_\omega^{[8]} = 1$.***

An automorphism that fixes pointwise the vertices in a ball of radius 8, must fix every vertex.

In particular, the size of $G_\omega$ is bounded by a function of the valency of $\Gamma$. 

*Pablo Spiga  Università degli studi di Milano
How vertex stabilizers grow?
Use this set of hypothesis:

1. \( \Gamma \) is locally \( s \)-arc-transitive with respect to \( G \);
2. for every vertex \( \omega \) of \( \Gamma \) the vertex stabilizer \( G_\omega \) is finite, and
3. the valency of every vertex of \( \Gamma \) is at least three.

**Theorem (, Bernd Stellmacher, John Van Bon)**

*If \( \Gamma \) and \( G \) are as above, then \( s \leq 9 \).*

You cannot be transitive on paths of length 10 starting at a vertex \( \omega \), for every \( \omega \).

Is \( G_\omega \) bounded above by a function of the valencies of \( \Gamma \)? Yes, if \( s \geq 6 \). No clue if \( 2 \leq s \leq 5 \).
Something more is true.

**Theorem (, Bernd Stellmacher, John Van Bon)**

Suppose that \( \Gamma \) and \( G \) as above. Then either \( s \leq 5 \), or \( G \) is a group with a weak \((B, N)\)-pair of rank 2 with respect to \( G_\alpha \) and \( G_\beta \), where \( \alpha \) and \( \beta \) are two adjacent vertices of \( \Gamma \).

Are all possible combinations of \( G_\alpha \) and \( G_\beta \) compatible? Can you construct a new generalized 8-gon.

Is there some connection between locally \( s \)-transitive graphs and generalized \( n \)-gons.

**Theorem (Feit-Higman)**

A thick generalized \( n \)-gon exists only for \( n = 2, 3, 4, 6, 8 \).
Fix the residue group: $G_{\alpha}^\Gamma$: the permutation group induced by $G_\alpha$ on $\Gamma_\alpha$.

What can you say about $G_\alpha$ when the enemy gives you only $G_{\alpha}^\Gamma$?

A transitive permutation group $L$ is said to be semiprimitive if every normal subgroup of $L$ is either semiregular or transitive.

**Theorem (P.S.)**

*If $\Gamma$ is $G$-arc-transitive, $G_\alpha$ is finite and $G_{\alpha}^\Gamma$ is semiprimitive, then $G^{[1]}_{\alpha\beta}$ is a $p$-group, for some prime $p$. (here $\Gamma$ is not necessarily finite).*

This theorem suggests that highly transitive graphs have vertex stabilizers which are nearly $p$-groups. This result has a counterpart for totally disconnected locally compact groups acting on trees: the vertex stabilizers are virtually pro $p$-groups.
An application: Suppose that $\Gamma$ is $G$-arc-transitive, and that $G^\Gamma_\alpha$ and $G^\Gamma_{\alpha\beta}$ are both non-abelian simple groups. Then $G^\Gamma_{\alpha\beta}[1] = 1$.

Proof.
The group $G^\Gamma_\alpha$ semiprimitive. Suppose that $G^\Gamma_{\alpha\beta} \neq 1$. Then it is a non trivial $p$-group. Now $(O_p(G_{\alpha\beta}))^\Gamma_\alpha$ is normal in $G^\Gamma_\alpha$. So it is trivial! Hence $O_p(G_{\alpha\beta}) \leq G^\Gamma_\alpha$. Thus $O_p(G_{\alpha\beta}) = O_p(G^\Gamma_{\alpha\beta})$, which is normal in $G^\Gamma_{\alpha\beta}$. So it is normal in $G = \langle G_\alpha, G^\Gamma_{\alpha\beta} \rangle$. Thus $O_p(G_{\alpha\beta}) = 1$, contradicting $G^\Gamma_{\alpha\beta}[1]$ being a non-trivial $p$-group. \qed
Conjecture (Richard Weiss)

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\Gamma$ is $G$-vertex-transitive of valency $d$ and $G^{\Gamma}_\alpha$ is primitive, then $|G_\alpha| \leq f(d)$.

Conjecture (Cheryl Praeger)

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\Gamma$ is $G$-vertex-transitive of valency $d$ and $G^{\Gamma}_\alpha$ is quasiprimitive, then $|G_\alpha| \leq f(d)$.

Conjecture (Primoz Potocnik, P.S., Gabriel Verret)

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\Gamma$ is $G$-vertex-transitive of valency $d$ and $G^{\Gamma}_\alpha$ is semiprimitive, then $|G_\alpha| \leq f(d)$.
Fix the valency

Theorem (Gabriel Verret, Primoz Potocnik, P.S.)

Let $\Gamma$ be a cubic $G$-vertex-transitive graph. Then either

$$2|G_\omega| \log_2(|G_\alpha|/2) \leq |V\Gamma| \text{ or } \Gamma \text{ is classified.}$$

Same pattern as before. Either there are not too many symmetries (this time as a function of $|V\Gamma|$) or $\Gamma$ can be described explicitly.
A census of small connected cubic vertex-transitive graphs

by Primož Potočnik, Pablo Spiga and Gabriel Verret

www.matapp.unimib.it/~spiga/

Pablo Spiga
Università degli studi di Milano

How vertex stabilizers grow?
Fix nothing

Conjecture

There exists a function \( f : \mathbb{N} \to \mathbb{N} \) such that if \( \Gamma \) is a \( G \)-vertex-transitive graph \( \Gamma \) of valency \( d \), with \( G_\omega \) finite for each vertex \( \omega \), then \( G \) is generated by at most \( f(d) \) elements.

True for valency 3. No clue in how to prove it in general. Here is a bonus (a direct consequence of this conjecture)

Theorem (Gabriel Verret, Primoz Potocnik, P.S.)

The number of cubic vertex-transitive graphs on \( n \) vertices is roughly \( \exp((\log n)^2) \).
The mother of all problems

Conjecture

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $\Gamma$ is a $G$-vertex-transitive graph $\Gamma$ of valency $d$, then $G_{\alpha}$ has exponent at most $f(d)$.

If this is true, then (because of the restricted Burnside problem) $G_{\alpha}$ can be big only if it has a large number of generators.