HOROCYCLIC PRODUCTS OF TREES AND HYPERBOLIC SPACES

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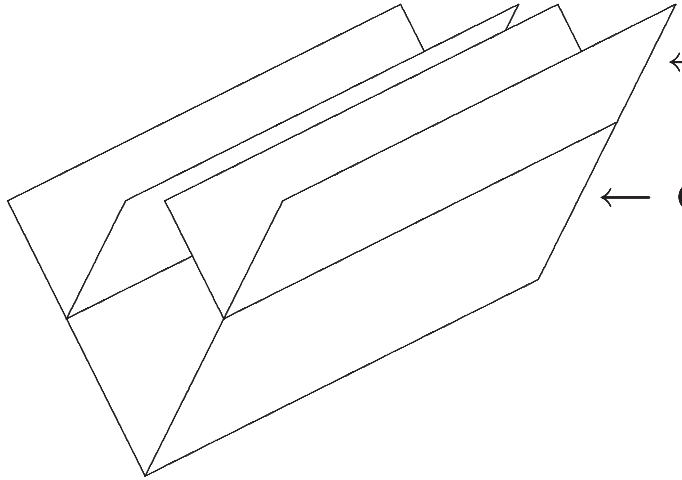
(No kangaroos in Austria !)

1. Treebolic spaces

$\mathbb{H} = \{x + \mathfrak{i}\, y : x \in \mathbb{R}\,,\ y > 0\}$ hyperbolic upper half space,

$\mathbb{T} = \mathbb{T}_{\mathsf{q}}$ homogeneous tree, degree $\mathsf{q} \geq 2$, a 1-complex.

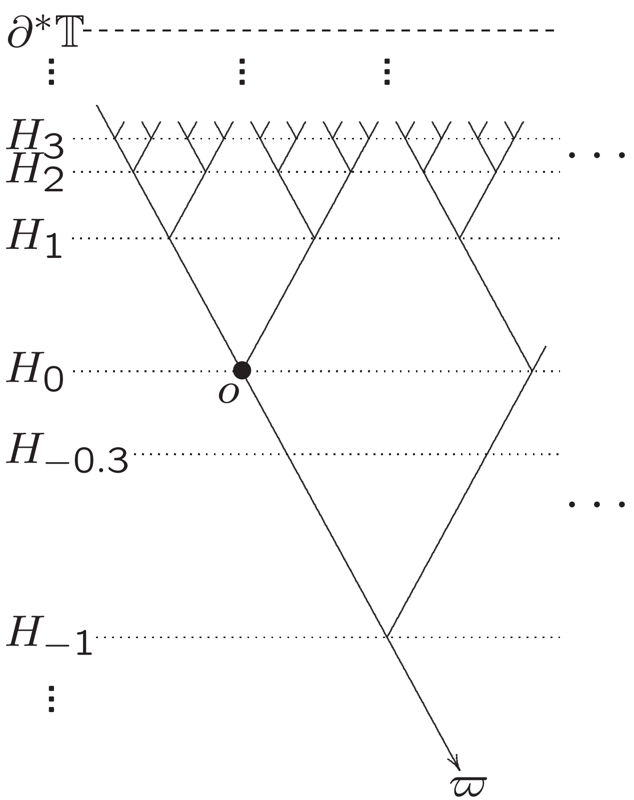
Treebolic space: Riemannian 2-complex, horocyclic product of $\mathbb{H}$ and $\mathbb{T}$.

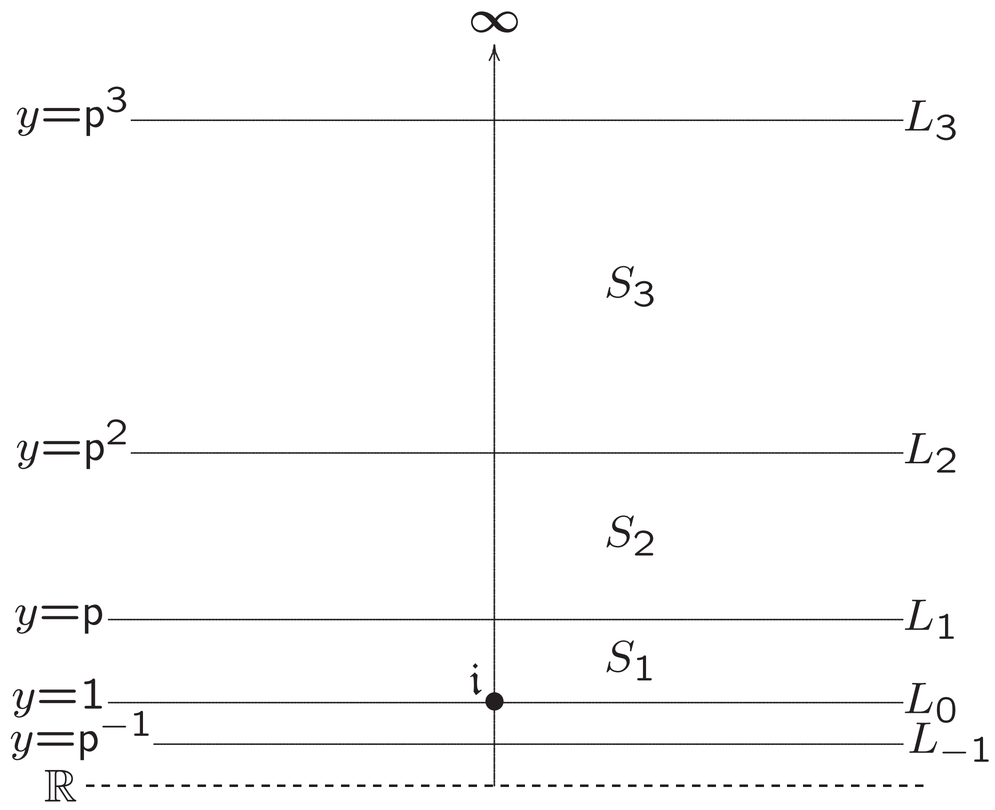


Strips $S_k = \{x + \mathfrak{i}\, y : x \in \mathbb{R}\,,\ \mathsf{p}^{k-1} \leq y \leq \mathsf{p}^k\}$ in $\mathbb{H}$ $(k \in \mathbb{Z})$, $\mathsf{p} > 1$.

Boundary lines L_{k-1} and L_k, where $L_k = \{x + \mathfrak{i}\, \mathsf{p}^k : x \in \mathbb{R}\}$,

horocycles with respect to "upper" boundary point ∞.





Busemann functions

$\mathfrak{h}_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}$ and $\mathfrak{h}_{\mathbb{H}(\mathsf{p})} : \mathbb{H} \to \mathbb{R}\,, \quad \mathfrak{h}_{\mathbb{H}(\mathsf{p})}(z) = \log_{\mathsf{p}} Im(z)\,.$

$$\mathsf{HT}(\mathsf{p},\mathsf{q}) = \left\{ \mathfrak{z} = (z,w) \in \mathbb{H} \times \mathbb{T}_{\mathsf{q}} \;:\; \mathfrak{h}_{\mathbb{T}}(w) = \mathfrak{h}_{\mathbb{H}(\mathsf{p})}(z) \right\}.$$

Metric on HT : hyperbolic length of $\mathbb{H}$ extends naturally.

Measure on HT : restricted to each strip, it is hyperbolic area,
$d\mathfrak{z} = y^{-2}dx\,dy$ for $\mathfrak{z} = (z,w) \in \mathsf{HT}$ with $z = x + \mathfrak{i}\,y \in \mathbb{H}\,,\ w \in \mathbb{T}$.
Lines L_k have measure 0.

Isometries of HT : Consider first the group of affine transformations of $\mathbb{H}$

$$\mathsf{Aff}(\mathbb{H},\mathsf{p}) = \{g = (\mathsf{p}^n, b) : n \in \mathbb{Z}\,,\ b \in \mathbb{R}\}$$

acting by $gz = \mathsf{p}^n z + b\,,\ z \in \mathbb{H}\,.$

It leaves the set of lines L_k, (where $y = \mathsf{p}^k$), $k \in \mathbb{Z}$, invariant.

$\mathsf{Aff}(\mathbb{H},\mathsf{p})$ is locally compact, left Haar measure and modular function are

$$dg = \mathsf{p}^{-n}\,dn\,db \quad \text{and} \quad \Delta_{\mathbb{H}}(g) = \mathsf{p}^{-n}\,, \quad \text{if} \quad g = (\mathsf{p}^n, b)\,,$$

dn counting measure on $\mathbb{Z}$, db Lebesgue measure on $\mathbb{R}$.

Affine group of $\mathbb{T}_{\mathsf{q}}$ is

$$\mathsf{Aff}(\mathbb{T}_{\mathsf{q}}) = \{\gamma \in \mathsf{Aut}(\mathbb{T}_{\mathsf{q}}) : \gamma\varpi = \varpi\}\,.$$

Locally compact, totally disconnected, compactly generated, acts transitively on vertex set of $\mathbb{T}$.

$\gamma \in \mathsf{Aff}(\mathbb{T}) \iff \gamma(v^-) = (\gamma v)^-$ for every vertex v of $\mathbb{T}$.

$$\Phi_{\mathbb{T}} : \mathsf{Aff}(\mathbb{T}) \to \mathbb{Z}\,, \quad \gamma \mapsto \mathfrak{h}(\gamma w) - \mathfrak{h}(w)$$

is independent of $w \in \mathbb{T}$ and a homomorphism.

$\gamma(H_t) = H_{t+k}$ if $\Phi_{\mathbb{T}}(\gamma) = k$. Modular function on $\mathsf{Aff}(\mathbb{T})$ is

$$\Delta_{\mathbb{T}}(\gamma) = \mathsf{q}^{\Phi(\gamma)}\,.$$

Proposition 1. The group

$$\mathcal{A}_{\mathsf{HT}}(\mathsf{p},\mathsf{q}) = \Big\{(g,\gamma) \in \mathsf{Aff}(\mathbb{H},\mathsf{p}) \times \mathsf{Aff}(\mathbb{T}_{\mathsf{q}}) \;:\; \log_{\mathsf{p}} \Delta_{\mathbb{H}}(g) + \log_{\mathsf{q}} \Delta_{\mathbb{T}}(\gamma) = 0\Big\}$$

acts on $\mathsf{HT}(\mathsf{p},\mathsf{q})$ by isometries $(g,\gamma)(z,w) = (gz,\gamma w)$.
It is the semidirect product

$\mathcal{A}_{\mathsf{HT}} = \mathsf{Aff}(\mathbb{T}) \ltimes \mathbb{R}$ with respect to the action $b \mapsto \mathsf{p}^{\Phi(\gamma)} b$, $\gamma \in \mathsf{Aff}(\mathbb{T})$, $b \in \mathbb{R}$.
The full group of isometries of $\mathsf{HT}(\mathsf{p},\mathsf{q})$ is generated by $\mathcal{A}(\mathsf{p},\mathsf{q})$ and the reflection

$$\mathfrak{s}(x + \mathrm{i}\, y, w) = (-x + \mathrm{i}\, y, w)\,,$$

it acts on $\mathsf{HT}(\mathsf{p},\mathsf{q})$ with compact quotient isomorphic with the circle of length $\log \mathsf{p}$, and it leaves the area element of HT invariant.

$\mathcal{A}_{\mathsf{HT}}$ is locally compact, compactly generated and amenable, its modular function is

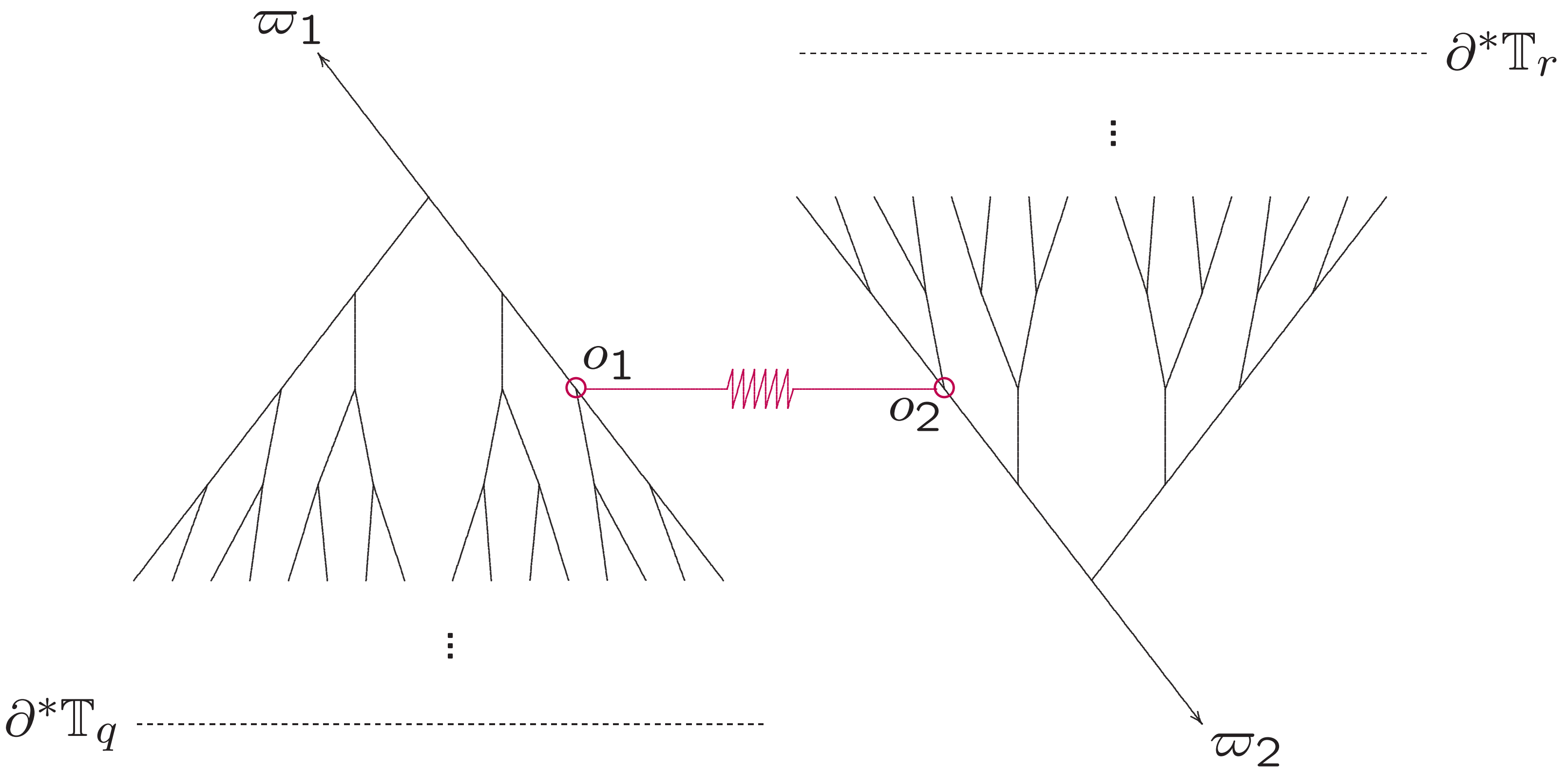
$$\Delta_{\mathcal{A}}(g,\gamma) = \Delta_{\mathbb{H}}(g)\, \Delta_{\mathbb{T}}(\gamma) = (\mathsf{q}/\mathsf{p})^{\Phi(\gamma)}\,.$$

2. Diestel-Leader graphs

Here $\mathbb{T}_p$ and $\mathbb{T}_q$ are discrete (consist of vertices only).

$$\mathrm{DL}(p,q) = \left\{x_1x_2 \in \mathbb{T}_q \times \mathbb{T}_p \ : \ \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\right\}.$$

Neighbourhood is given by $x_1x_2 \sim y_1y_2 \iff x_1 \sim y_1$ and $x_2 \sim y_2$.



Proposition 2. The group

$$\mathcal{A}_{\mathrm{DL}}(\mathsf{p},\mathsf{q}) = \Big\{(\gamma_1,\gamma_2) \in \mathrm{Aff}(\mathbb{T}_{\mathsf{p}}) \times \mathrm{Aff}(\mathbb{T}_{\mathsf{q}}) \;:\; \underbrace{\log_{\mathsf{p}} \Delta_{\mathbb{T}_{\mathsf{p}}}(\gamma_1) + \log_{\mathsf{q}} \Delta_{\mathbb{T}_{\mathsf{q}}}(\gamma_2)}_{=\,\Phi(\gamma_1)+\Phi(\gamma_2)} = 0\Big\}$$

acts transitively on (the vertex set of) $\mathrm{DL}(\mathsf{p},\mathsf{q})$ by graph isometries

$$(\gamma_1,\gamma_2)(x_1,x_2) = (\gamma_1 x_1, \gamma_2 x_2)\,.$$

It is the full group of graph isometries of $\mathrm{DL}(\mathsf{p},\mathsf{q})$ when $\mathsf{p} \neq \mathsf{q}$.
If $\mathsf{p} = \mathsf{q}$ then the full isometry group is generated by $\mathcal{A}_{\mathrm{DL}}(\mathsf{p},\mathsf{p})$ and the reflection

$$\mathfrak{s}(x_1 x_2) = (x_2 x_1)\,.$$

$\mathcal{A}_{\mathrm{DL}}$ is locally compact, compactly generated and amenable, its modular function is

$$\Delta_{\mathcal{A}}(\gamma_1,\gamma_2) = \Delta_{\mathbb{T}_{\mathsf{p}}}(\gamma_1)\Big/\Delta_{\mathbb{T}_{\mathsf{q}}}(\gamma_2) = (\mathsf{p}/\mathsf{q})^{\Phi(\gamma_1)}\,.$$

3. Sol-manifolds

$\mathbb{H}(\mathsf{p}) = \{x + \mathrm{i}\,\mathsf{p}^z \,:\, x, z \in \mathbb{R}\}$ hyperbolic plane, curvature $-(\log \mathsf{p})^2$.

Sol(p, q) is the horocyclic product of hyperbolic planes $\mathbb{H}(\mathsf{p})$ and $\mathbb{H}(\mathsf{q})$: it consists of all pairs

$$(x + \mathrm{i}\,\mathsf{p}^z, y + \mathrm{i}\,\mathsf{q}^{-z}), \quad x, y, z \in \mathbb{R}.$$

Here, $\mathsf{p}, \mathsf{q} > 1$. Topologically, Sol(p, q) is $\mathbb{R}^3$, length element

$$ds^2 = d_{\mathsf{p},\mathsf{q}}s^2 = \mathsf{p}^{-2z}\,dx^2 + \mathsf{q}^{2z}\,dy^2 + dz^2 .$$

Volume element $(\mathsf{q}/\mathsf{p})^z\,dx\,dy\,dz$.

Natural projections onto $\mathbb{H}(\mathsf{p})$ $\mathbb{H}(\mathsf{p})$ and $\mathbb{R}$:

$$\pi_1(x,y,z) = x + \mathrm{i}\,\mathsf{p}^z, \quad \pi_2(x,y,z) = y + \mathrm{i}\,\mathsf{q}^{-z}, \quad \tilde{\pi}(x,y,z) = z .$$

The Sol-manifold is also a Lie group, usually also denoted $\mathrm{Sol}(\mathsf{p},\mathsf{q})$. Here, we write

$$\mathcal{A}_{\mathsf{Sol}}(\mathsf{p},\mathsf{q}) = \left\{ \mathfrak{g} = \begin{pmatrix} \mathsf{p}^z & x & 0 \\ 0 & 1 & 0 \\ 0 & y & \mathsf{q}^{-z} \end{pmatrix} \ : \ x,y,z \in \mathbb{R} \right\}.$$

Can be identified with the manifold, has natural 1st and 2nd projections to affine groups acting on $\mathbb{H}(\mathsf{p})$ and $\mathbb{H}(\mathsf{q})$, and projection on translation group $\mathbb{R}$.

Left Haar measure $\equiv$ volume element.

Modular function $\quad \Delta(\mathfrak{g}) = (\mathsf{q}/\mathsf{p})^z$.

4. Discrete group actions and rough isometries

Rough isometry (quasi-isometry) between metric spaces (X_1, d_1) and (X_1, d_1) is $\varphi : X_1 \to X_2$ with

$$A^{-1} d_1(x_1, y_1) - B \leq d_2(\varphi x_1, \varphi y_1) \leq A d_1(x_1, y_1) + B \quad \forall\, x_1, x_2 \in X_1$$
$$d(y_2, \varphi X_1) \leq B \quad \forall\, y_2 \in X_2\,.$$

For a finitely generated group Γ with finite generating set $S = S^{-1}$, its Cayley graph has vertex set Γ, and

$$g \sim h \iff h = gs\,, \quad s \in S\,.$$

Rough isometry of Γ refers to Cayley graph metric.

- If $p = q \in \mathbb{N}$, the amenable Baumslag-Solitar group

$$\left\langle a, b \mid ab = b^{p} a \right\rangle$$

acts on $HT(p, p)$ by isometries and with compact quotient.

- If $p \neq q$, then no discrete group can act with finite point stabilizers and compact quotient on $HT(p, q)$, since the full isometry group is non-unimodular.

It probably follows from work & methods of [Farb and Mosher, 1998], [Eskin, Fisher and Whyte, 2006-07] that $HT(p, q)$ is not roughly isometric with any finitely generated group.

• If $p = q \in \mathbb{N}$, then $DL(p, p)$ is a Cayley graph of the lamplighter group

$$\mathbb{Z}_p \wr \mathbb{Z} ,$$

observation by [R. Möller and P. Neumann, 2001].

• $DL(2, 3)$ was invented by [Diestel and Leader, 1992-2001] in the attempt to answer a question of [Woess, 1990]:

Is there a vertex-transitive graph that is not roughly isometric with a (Cayley graph of a) finitely generated group ?

• If $p \neq q$, then no discrete group can act with finite point stabilizers and finitely many orbits on $DL(p, q)$, since the full isometry group is non-unimodular.

[Eskin, Fisher and Whyte, 2006-07]:
$DL(p, q)$ is not roughly isometric with any finitely generated group.

- If $p = q > 0$, then $\mathcal{A}_{\mathrm{Sol}}(p, p)$ contains a co-compact lattice.

Stated by [Eskin, Fisher and Whyte, 2006-07].

- If $p \neq q$, then no discrete group can act with finite point stabilizers and compact quotien on $\mathrm{Sol}(p, q)$, since the full isometry group is non-unimodular.

[Eskin, Fisher and Whyte, 2006-07]:
$\mathrm{Sol}(p, q)$ is not roughly isometric with any finitely generated group.

5. Random walks, Brownian motion, harmonic functions

General program: study these issues by strong use of the fact that our spaces are horocyclic products of trees an hyperbolic spaces.

- Very complete body of work exists already for $DL(p, q)$.

Rate of escape, central limit theorem : [Bertacchi, 2001]

Spectrum of simple random walk (with or without drift) :
[Grigorchuk and Zuk, 2001] $(p = q = 2)$,
[Dicks and Schick, 2002] $(p = q \geq 2)$,
[Bartholdi and Woess, 2005] (p , q arbitrary).

Precise asymptotics of return probabilities :
[Revelle, 2003] $(p = q)$,
[Bartholdi and Woess, 2005] (p , q arbitrary).

Positive harmonic functions, Martin boundary :
[Woess, 2005], [Brofferio and Woess, 2006, 2007]

- Ongoing work on the other two.

- HT$(\mathfrak{p}, \mathfrak{q})$ Rigorous construction of Laplace operator, positive harmonic functions:
[Bendikov, Saloff-Coste, Salvatori and Woess]

- Sol$(\mathfrak{p}, \mathfrak{q})$ Positive harmonic functions:
[Brofferio, Salvatori and Woess]

6. Positive harmonic functions on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$

The Laplace operator with vertical drift parameter $\mathsf{a} \in \mathbb{R}$ on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is

$$\mathfrak{L}_{\mathsf{a}} = \mathfrak{L}_{\mathsf{a}}^{\mathsf{Sol}(\mathsf{p},\mathsf{q})} = \mathsf{p}^{2z} \frac{\partial^2}{\partial x^2} + \mathsf{q}^{-2z} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \mathsf{a} \frac{\partial}{\partial z}\,.$$

The Laplace-Beltrami operator arises for $\mathsf{a} = \mathsf{q} - \mathsf{p}$.

Under the projection π_1, the Laplacian $\mathfrak{L}_{\mathsf{a}}$ projects onto the Laplacian with drift on $\mathbb{H}(\mathsf{p})$ given by

$$\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})} = \mathsf{p}^{2z} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \mathsf{a} \frac{\partial}{\partial z}\,.$$

That is, for a C^2-function f_1 on $\mathbb{H}(\mathsf{p})$,

$$\mathfrak{L}_{\mathsf{a}}(f_1 \circ \pi_1) = (\mathfrak{L}_{\mathsf{a}}^{\mathbb{H}(\mathsf{p})} f_1) \circ \pi_1\,.$$

This is the Laplace-Beltrami operator on $\mathbb{H}(\mathsf{p})$ when $\mathsf{a} = -\mathsf{p}$.

Analogously, under π_2 (where the sign of z is changed), $\mathfrak{L}_a$ projects onto the Laplacian with drift $\mathbb{H}(\mathsf{q})$ given by

$$\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})} = e^{2\mathsf{q}z}\frac{\partial^2}{\partial y^2} + + \frac{\partial^2}{\partial z^2} - \mathsf{a}\,\frac{\partial}{\partial z}\,.$$

Theorem. Every positive $\mathfrak{L}_\mathsf{a}$-eigenfunction on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ has the form

$$h(x,y,z) = h_1(x,z) + h_2(y,-z)\,,$$

where h_1 is a non-negative $\mathfrak{L}_\mathsf{a}^{\mathbb{H}(\mathsf{p})}$-eigenfunction on $\mathbb{H}(\mathsf{p})$ and h_2 is non-negative $\mathfrak{L}_{-\mathsf{a}}^{\mathbb{H}(\mathsf{q})}$-eigenfunction on $\mathbb{H}(\mathsf{q})$, both with the same eigenvalue as h.

7. What else ?

Horocyclic products of more than two trees and/or hyperbolic spaces. [Bartholdi, Neuhauser and Woess, in print]

Horocyclic products of affine buildings. Onging work by [Parkinson]