Sierpiński graphs as spanning subgraphs of Hanoi graphs

Sara Sabrina Zemljič

Andreas M. Hinz **Sandi Klavžar**

Institute of Mathematics, Physics and Mechanics

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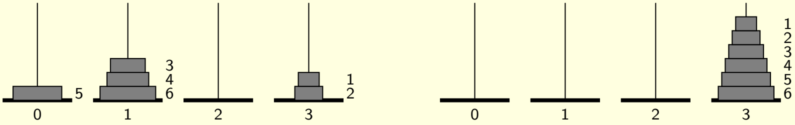
Hanoi graphs

Tower of Hanoi puzzle



- *divine rule:* no larger disc is (put) on a smaller disc
- *regular state:* distribution of disc obeying the divine rule
- *perfect state:* regular state with all discs on one peg

Hanoi graph H_p^n ...

... represents the Tower of Hanoi puzzle with p pegs and n discs.

- **vertices** = regular states $\Rightarrow$ $V(H_p^n) = [p]_0^n$
- two vertices are **adjacent** if one can be obtained form the other by a legal move

Hanoi graphs

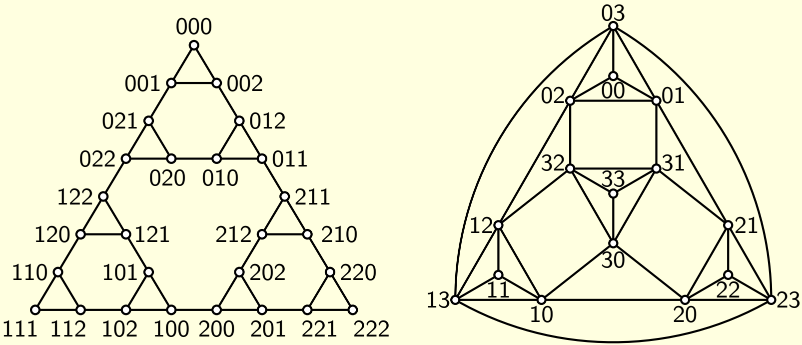


Figure: The Hanoi graphs H_3^3 and H_4^2

Sierpiński graphs

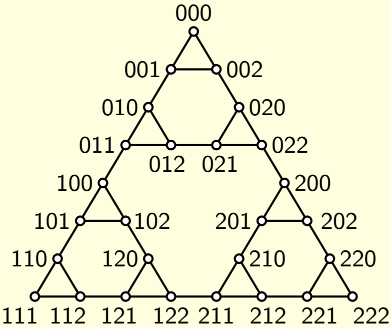
Sierpiński graph S_p^n

- $V(S_p^n) = [p]_0^n$
- two vertices $s_n \ldots s_1$ and $r_n \ldots r_1$ are **adjacent** if there exists an index δ, such that
  - (i) $s_\ell = r_\ell$, for $\ell = n, \ldots, \delta + 1$;
  - (ii) $s_\delta \neq r_\delta$; and
  - (iii) $s_\ell = r_\delta$ and $r_\ell = s_\delta$ for $\ell = 1, \ldots, \delta - 1$.

Equivalently:

$$E(S_p^1) = \big\{\{i,j\} \mid i \neq j \in \{0, 1 \ldots, p-1\}\big\}$$
$$E(S_p^n) = \big\{\{is, ir\} \mid i = 0, 1, \ldots, p-1\,, \{s,r\} \in E(S_p^{n-1})\big\} \cup \big\{\{ij^{n-1}, ji^{n-1}\} \mid i \neq j \in \{0, 1 \ldots, p-1\}\big\} \quad (n \geq 2)$$

Sierpiński graphs



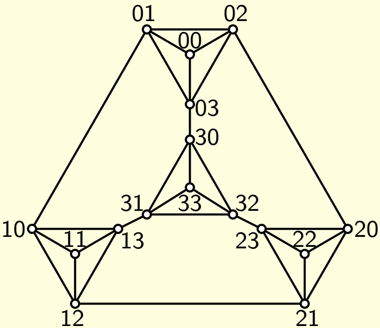


Figure: The Sierpiński graphs S_3^3 and S_4^2

Notations

$ii\ldots i = i^n$ – extreme vertex in S_p^n or perfect vertex in H_p^n

- p extreme vertices in S_p^n
- degree of any extreme vertex is $p-1$,
  all other vertices have degree p
- p perfect vertices in H_p^n
- degree of any perfect vertex is $p-1$,
  all other vertices have degree $\geq 2p-3$ (for $n \geq 2$)

$$s_n \ldots s_{r+1} S_p^r = \{ s_n \ldots s_{r+1} \overline{s} \,|\, s \in S_p^r \} \simeq S_p^r$$

$$s_n \ldots s_{r+1} H_p^r = \{ s_n \ldots s_{r+1} \overline{s} \,|\, s \in H_p^r \} \simeq H_p^r$$

Theorem

Theorem

Let $p, n \in \mathbb{N}$. Then S_p^n can be embedded isomorphically into H_p^n if and only if p is odd or $n = 1$.

- $n = 1$: $H_p^1 = S_p^1 = K_p$
- p odd:
  - $H_1^n = S_1^n = K_1$
  - For $p \geq 3$ define for each $k \in [p]_0$

$$\pi_k(i) = \frac{1}{2}\big(k(p+1) - i(p-1)\big) \bmod p \quad \text{... permutation of } [p]_0 \,,$$

$$\pi_k^n(s_n \dots s_1) = \pi_k(s_n)\dots\pi_k(s_1) \quad \text{... permutation of } [p]_0^n \,.$$

The embedding $\iota_{n+1} : V(S_p^{n+1}) \to V(H_p^{n+1})$ is defined by

$$\iota_{n+1}(ks) = k\pi_k^n\left(\iota_n(s)\right) \,.$$

Proof

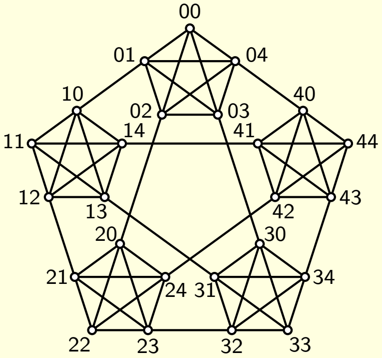


Figure: Isomorphic embedding ι_2 from S_5^2 into H_5^2

Proof - p even

Theorem

Let $p, n \in \mathbb{N}$. Then S_p^n can be embedded isomorphically into H_p^n if and only if p is odd or $n = 1$.

- p even:
  - H_2^n ... $n-1$ disjoint K_2
    $S_2^n = P_{2^n}$ ... path on 2^n vertices
  - $p \geq 4$

Lemma

Every complete subgraph of H_p^n, $p, n \in \mathbb{N}$, is induced by edges corresponding to moves of one and the same disc.
In particular, $\omega(H_p^n) = p$ and the only p-cliques of H_p^n are of the form $s_n \ldots s_2 H_p^1$.

Proof - p even

Theorem

Let $p, n \in \mathbb{N}$. Then S_p^n can be embedded isomorphically into H_p^n if and only if p is odd or $n = 1$.

- $p \geq 4$ even:
  - $n = 2$: extreme to perfect vertices and p-cliques to p-cliques remaining: $\binom{p}{2}$ non-incident edges in S_p^2, but there are only $p\lfloor \frac{p-1}{2} \rfloor$ non-incident edges in H_p^2
  - $n \geq 3$: subgraph $i^{n-2}S_p^2$ has to be mapped to some $j^{n-2}H_p^2$, a contradition

Proof - p even

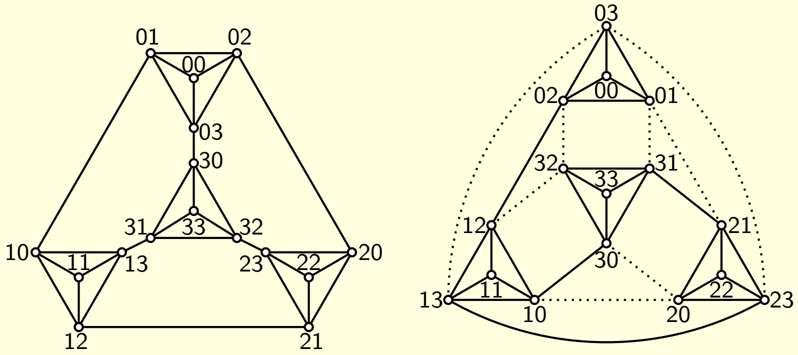


Figure: Graphs S_4^2 and H_4^2

Corollary

Hamming graph ...

... is the graph with vertex set $[r_1] \times [r_2] \times \ldots \times [r_n]$, where two vertices are adjacent if they differ in precisely one coordinate.
... is the Cartesian product of complete graphs $K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$.

$K_p^n = K_p \Box K_p \Box \ldots \Box K_p$

Corollary

Let p be odd. Then for any n, S_p^n is a spanning subgraph of the Hamming graph K_p^n.