Linear maps preserving semi-Fredholm operators and generalized invertibility\

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Abstract

Let $H$ be an infinite-dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. We characterize surjective linear maps $\phi : B(H) \to B(H)$ preserving semi-Fredholm operators in both directions. As an application we substantionally improve a recently obtained characterization of linear preservers of generalized invertibility [13]. The new proof given in this note is not only more efficient, but also much shorter and simpler.

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1 Introduction and statement of the main results

Over the last few decades there has been a considerable interest in the so called linear preserver problems (see the survey papers [3, 5, 9, 10, 17]). The goal is to

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describe the general form of linear maps between two Banach algebras that preserve a certain property, or a certain class of elements, or a certain relation. One of the most famous problems in this direction is Kaplansky’s problem [8] asking whether every surjective unital invertibility preserving linear map between two semi-simple Banach algebras is a Jordan homomorphism? In the commutative case the well-known Gleason-Kahane-Żelazko theorem provides the affirmative answer. In the non-commutative case the best known results so far are due to Aupetit [2] and Sourour [18]. Aupetit showed that the answer to the Kaplan-sky’s question is in the affirmative in the case of von Neumann algebras. The main idea of his proof is to show that bijective unital linear invertibility preserving maps on von Neumann algebras preserve idempotents. Sourour showed that bijective unital linear invertibility preserving maps acting on the algebra of all bounded operators on a Banach space preserve rank one operators and then used the structural result for rank one preservers to solve the Kaplansky’s problem in this special case.

In [13] the problem of characterizing surjective linear maps $\phi : B(H) \to B(H)$ preserving generalized invertibility in both directions was treated. Here, $H$ is a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. Throughout our paper (except in Final remarks) we will always assume that $H$ is separable. Yet, although this problem is closely related to the Kaplansky’s problem, it is essentially different in the sense that it is trivial in the finite-dimensional case. Namely, if $H$ is finite-dimensional, then every operator from $B(H)$ is generalized invertible, and consequently, in this special case every linear map on $B(H)$ preserves generalized invertibility in both directions. Also, in the infinite-dimensional case every finite rank perturbation of a generalized invertible operator is generalized invertible, and therefore we can describe the action of $\phi$ only up to finite rank perturbations. The closure of the ideal of finite rank operators is the ideal of all compact operators. This means that we have to factorize generalized invertibility preserving linear maps through the Calkin algebra $\mathcal{C}(H) = B(H)/\mathcal{K}(H)$ if we want to get a nice structural result.

For a more precise description of the main result from [13] we need a few definitions. Let $H$ be an infinite-dimensional complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on $H$, and $\mathcal{F}(H)$, $\mathcal{K}(H) \subset B(H)$ the ideal of all finite rank operators, and the ideal of all compact operators, respectively. Let $A \in B(H)$. If there exists $B \in B(H)$ such that $ABA = A$ then $B$ is called a generalized inverse of $A$. We denote by $\mathcal{G}(H) \subset B(H)$ the subset of all operators which have a generalized inverse. Note that $A \in \mathcal{G}(H)$ if and only if the image of $A$ is closed (see [15]). We say that a linear map $\phi : B(H) \to B(H)$ preserves generalized invertibility in both directions if for every $A \in B(H)$ the operator $A$ has a generalized inverse if and only if $\phi(A)$ has a generalized inverse. It is well-known (and easy to check) that if $A \in \mathcal{G}(H)$, then $A + F \in \mathcal{G}(H)$ for every $F \in \mathcal{F}(H)$. Hence, if $\psi : B(H) \to \mathcal{F}(H)$ is an arbitrary linear map, then $\phi : B(H) \to \mathcal{F}(H)$ preserves generalized invertibility in both directions if and only if the map $A \mapsto \phi(A) + \psi(A)$, $A \in B(H)$, preserves generalized invertibility.
in both directions. Thus, one can expect to get a nice structural result for linear preservers of generalized invertibility only up to an arbitrary linear perturbation $\psi$ mapping into $\mathcal{F}(H)$. As we shall see later, the ideal $\mathcal{K}(H)$ is invariant under every surjective linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserving generalized invertibility in both directions. Thus, every such map factorizes through the Calkin algebra $\mathcal{C}(H)$. This factorization annihilates all linear perturbations $\psi$ mapping into $\mathcal{F}(H)$ and thus it is natural to ask if the induced map has a nice structure. It was proved in [13] that if $H$ is separable and $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a bijective continuous unital linear map preserving generalized invertibility in both directions, then $\phi(\mathcal{F}(H)) = \mathcal{F}(H)$, $\phi(\mathcal{K}(H)) = \mathcal{K}(H)$, and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$, $\varphi(A + \mathcal{K}(H)) = \phi(A) + \mathcal{K}(H)$, $A \in \mathcal{B}(H)$, is either an automorphism, or an anti-automorphism. The proof was rather long and complicated. To explain the idea we recall that an operator $A \in \mathcal{B}(H)$ is said to be Fredholm if its image is closed and both its kernel and cokernel are finite-dimensional and is semi-Fredholm if its image is closed and its kernel or its cokernel is finite-dimensional. We denote by $\mathcal{SF}(H) \subset \mathcal{B}(H)$ the subset of all semi-Fredholm operators (see [4, 14]). If $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a bijective continuous unital linear map preserving generalized invertibility in both directions, then it is easy to see that $\phi(\mathcal{F}(H)) = \mathcal{F}(H)$ and then, by continuity, $\phi(\mathcal{K}(H)) = \mathcal{K}(H)$. The next step was to show that $\phi$ preserves the set of operators that are generalized invertible and are not semi-Fredholm (they have the infinite-dimensional kernel and the infinite-dimensional cokernel). A certain natural partial order was introduced on the set of such operators. Some rather complicated properties of this order together with the assumption that $\phi$ is unital were used to show that the induced linear map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ preserves idempotents. And then one can prove using standard techniques that this induced map is either an automorphism, or an anti-automorphism. It was conjectured that the same conclusion holds true without the continuity assumption and that an analogous result can be obtained without assuming that $\phi$ is unital. Moreover, as the assumption of preserving generalized invertibility is not affected by linear perturbations mapping into $\mathcal{F}(H)$ it is natural to replace the bijectivity assumption by the surjectivity up to finite rank operators. We say that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is surjective up to finite rank operators if for every $A \in \mathcal{B}(H)$ there exists $B \in \mathcal{B}(H)$ such that $A - \phi(B) \in \mathcal{F}(H)$, or equivalently, $\mathcal{B}(H) = \text{Im} \phi + \mathcal{F}(H)$. Can we relax the assumptions of the main theorem from [13] as suggested above? One of the two main results of this note answers this question in the affirmative.

**Theorem 1.1** Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a linear map preserving generalized invertibility in both directions. Assume that $\phi$ is surjective up to finite rank operators. Then $\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$ and there exist an invertible element $a \in \mathcal{C}(H)$ and either an automorphism $\tau : \mathcal{C}(H) \to \mathcal{C}(H)$ or an anti-automorphism $\tau : \mathcal{C}(H) \to \mathcal{C}(H)$ such that the
induced map $\varphi : C(H) \to C(H)$, $\varphi(A + \mathcal{K}(H)) = \phi(A) + \mathcal{K}(H)$, $A \in \mathcal{B}(H)$, is of the form

$$\varphi(x) = a\tau(x), \quad x \in C(H).$$

The automorphism or anti-automorphism $\tau$ appearing in the conclusion of the above theorem is continuous. This follows from the proof but also from the well-known Johnson’s theorem [7] on the automatic continuity of epimorphisms of Banach algebras onto semisimple Banach algebras. It should be mentioned here that recently the existence of outer automorphisms of $C(H)$ has been proved by Phillips and Weaver [16]. In fact, they proved that there exists a $*$-automorphism $\tau$ of $C(H)$ that is not an inner automorphism induced by a unitary element of $C(H)$. It is then easy to see that $\tau$ cannot be of the form $\tau(x) = sx{s}^{-1}$, $x \in C(H)$, with $s$ being an invertible element of $C(H)$.

On our way to Theorem 1.1 we obtain the following result that is of independent interest. We say that a map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves semi-Fredholm operators in both directions if for every $A \in \mathcal{B}(H)$ the operator $\phi(A)$ is semi-Fredholm.

**Theorem 1.2** Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a linear map preserving semi-Fredholm operators in both directions. Assume that $\phi$ is surjective up to compact operators (i.e., $\mathcal{B}(H) = \text{Im} \phi + \mathcal{K}(H)$). Then

$$\phi(\mathcal{K}(H)) \subseteq \mathcal{K}(H)$$

and the induced map $\varphi : C(H) \to C(H)$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in C(H)$.

For a Fredholm operator $T \in \mathcal{B}(H)$ we define the index of $T$ by

$$\text{ind}(T) = \dim \ker T - \text{codim} \text{Im} T \in \mathbb{Z}.$$

**Theorem 1.3** Under the same hypothesis and notation as in Theorem 1.2, the following statements hold true:

(i) $\phi$ preserves Fredholm operators in both directions;

(ii) there is an $n \in \mathbb{Z}$ such that either

$$\text{ind}(\phi(T)) = n + \text{ind}(T)$$

for every Fredholm operator $T$, or

$$\text{ind}(\phi(T)) = n - \text{ind}(T)$$

for every Fredholm operator $T$.

Let us mention at the end of the introductory section that Hou and Cui [6] were dealing with similar problems independently.
2 Preliminary results

Let $H$ be an infinite-dimensional complex separable Hilbert space. We first note that if $A \in \mathcal{G}(H)$, then there exists $C \in \mathcal{B}(H)$ such that $ACA = A$ and $CAC = C$. Indeed, if $ABA = A$, then the choice $C = BAB$ works. Denote by $\pi$ the quotient map of $\mathcal{B}(H)$ onto $\mathcal{C}(H)$. Recall that $A \in \mathcal{B}(H)$ is Fredholm if and only if $\pi(A)$ is invertible in $\mathcal{C}(H)$. Moreover, $A$ is semi-Fredholm if and only if $\pi(A)$ has a right inverse or a left inverse. It is easy to see that any left invertible element of any unital algebra multiplied by an invertible element on the right side or on the left side is again left invertible. The same holds true for right invertible elements. It follows that if $A \in \mathcal{B}(H)$ is Fredholm, then for every $B \in \mathcal{B}(H)$ we have $B \in SF(H)$ if and only if $AB \in SF(H)$ if and only if $BA \in SF(H)$.

It is well-known that in any unital Banach algebra $\mathcal{A}$ both the left spectrum and the right spectrum are \(\partial\)-spectra. Here, a \(\partial\)-spectrum $\Lambda$ is defined as any map from $\mathcal{A}$ into the set of non-empty closed subsets of the complex plane satisfying

$$\partial \sigma(a) \subset \Lambda(a) \subset \sigma(a)$$

for every $a \in \mathcal{A}$. Thus, if we define the semi-Fredholm spectrum of $A \in \mathcal{B}(H)$ as

$$\sigma_{SF}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \not\in SF(H)\},$$

then

$$\sigma_{SF}(A) = \sigma_l(\pi(A)) \cap \sigma_r(\pi(A)) \supset \partial \sigma(\pi(A)) \neq \emptyset.$$ The left-right spectrum is defined as the intersection of the left and the right spectrum, $\sigma_{lr}(\cdot) = \sigma_l(\cdot) \cap \sigma_r(\cdot)$. Of course, the left-right spectrum is again \(\partial\)-spectrum.

We start with the following lemma.

**Lemma 2.1** Let $A$ be a unital Banach algebra and $\Lambda$ a \(\partial\)-spectrum on $\mathcal{A}$. Assume that for $a \in \mathcal{A}$ we have

$$\Lambda(a + b) = \Lambda(b)$$

for every $b \in \mathcal{A}$. Then $a$ belongs to the radical of $\mathcal{A}$.

**Proof.** For every $b \in \mathcal{A}$ we have

$$\partial \sigma(a + b) \subseteq \Lambda(a + b) = \Lambda(b) \subset \sigma(b).$$

In particular, for every quasinilpotent $b \in \mathcal{A}$ we have

$$\partial \sigma(a + b) = \{0\}.$$ Thus, the spectral radius of $a + b$ is zero for every quasinilpotent $b \in \mathcal{A}$, and by the Zemanek’s theorem (see [1, Theorem 5.3.1] or [19]), $a \in \text{rad}(\mathcal{A})$. \qed
Let $A$ be any unital Banach algebra. It is easy to see that the set of all left (right) invertible elements is open in $A$. Indeed, assume that $a, b \in A$ satisfy $ab = 1$. Let $c \in A$ be any element with $\|c\| < \|a\|^{-1}$. Then $a(b + c) = 1 + ac$ is invertible, and hence $b + c$ has a left inverse.

**Lemma 2.2** Let $A \in \mathcal{B}(H)$. Then the following are equivalent:

- $A$ is semi-Fredholm,
- for every $B \in \mathcal{B}(H)$ there exists $\delta > 0$ such that $A + \lambda B \in \mathcal{G}(H)$ for every complex $\lambda$ with $|\lambda| < \delta$.

**Proof.** Assume first that $A \in \mathcal{S}_F(H)$. We want to prove that the second condition is fulfilled. In fact, even more is true. Namely, from the above remark, the fact that an operator is semi-Fredholm if and only if its $\pi$-image has a one-sided inverse, and the continuity of $\pi$, one can easily conclude that the set of semi-Fredholm operators is open in $\mathcal{B}(H)$. As $\mathcal{S}_F(H) \subset \mathcal{G}(H)$, the second condition follows trivially.

Assume now that the second condition holds true but $A \not\in \mathcal{S}_F(H)$. Applying the second condition with $B = 0$ we conclude that $A \in \mathcal{G}(H)$. As it is not semi-Fredholm, we have $\dim \ker A = \operatorname{codim} \operatorname{im} A = \infty$. Choose a non-finite rank compact operator $L : \ker A \to \ker A^*$. Extend $L$ to a compact operator $K : H \to H$ by setting $Kx = 0$ for every $x \in (\ker A)^\perp$. Then the image of $A + \lambda K$ is not closed for every nonzero scalar $\lambda$. \hfill $\Box$

**Lemma 2.3** Let $K \in \mathcal{B}(H)$. Then the following are equivalent:

- $K$ is compact,
- for every $B \in \mathcal{S}_F(H)$ we have $B + K \in \mathcal{S}_F(H)$.

**Proof.** Assume that $K \in \mathcal{K}(H)$ and $B \in \mathcal{S}_F(H)$. Then $\pi(B) = \pi(B + K)$ is left or right invertible in $\mathcal{C}(H)$. It follows that $B + K \in \mathcal{S}_F(H)$.

If $K \in \mathcal{B}(H)$ satisfies the second condition then $\sigma_{SF}(B + K) = \sigma_{SF}(B)$ for every $B \in \mathcal{B}(H)$. Thus, $\sigma_{lr}(\pi(B) + \pi(K)) = \sigma_{lr}(\pi(B))$ for all $\pi(B) \in \mathcal{C}(H)$. By Lemma 2.1, $\pi(K) \in \operatorname{rad}(\mathcal{C}(H))$. As $\mathcal{C}(H)$ is semi-simple, we have $\pi(K) = 0$, and thus, $K$ is compact. \hfill $\Box$

Recall that an operator $A \in \mathcal{B}(H)$ is called upper semi-Fredholm if its image is closed and its kernel is finite-dimensional. It is called lower semi-Fredholm if its image is closed and of a finite codimension.

**Lemma 2.4** Let $A$ be lower (resp. upper) semi-Fredholm. If $A$ is not Fredholm, then we can find a lower (resp. upper) semi-Fredholm operator $B$ such that every non-trivial linear combination $\lambda A + \mu B$, $\lambda \neq 0$ or $\mu \neq 0$, is lower (resp. upper) semi-Fredholm.

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Proof. Assume that $A$ is lower semi-Fredholm. Since $A$ is not Fredholm, the kernel of $A$ is infinite-dimensional. Hence we can find an operator $B \in \mathcal{B}(H)$ which maps $\ker A$ bijectively onto $\im A$ and satisfies $Bx = 0$ for every $x \in (\ker A)^\perp$. It follows that for every pair $\lambda, \mu \in \mathbb{C}$ we have

$$\im (\lambda A + \mu B) \subseteq \im A + \im B = \im A.$$ 

If $\lambda \neq 0$, then for every $y \in \im A$ we can find a unique $x \in (\ker A)^\perp$ such that $Ax = y$. Then

$$y = \frac{1}{\lambda} (\lambda A + \mu B)x,$$

and consequently, $\im (\lambda A + \mu B) = \im A$. Thus, $\im (\lambda A + \mu B)$ is closed and of finite-codimension, and consequently, $\lambda A + \mu B$ is lower semi-Fredholm, as desired. In the case that $\lambda = 0$ we need to consider only the case when $\mu \neq 0$. As $\im B = \im A$ we have $\mu B$ is lower semi-Fredholm in this case as well.

Since, $A$ is upper semi-Fredholm if and only if $A^*$ is lower semi-Fredholm, the rest of proof follows by duality. \qed

3 Proofs of main theorems

Proof of Theorem 1.2. Denote $\phi(I) = T$. We will show that $T$ is Fredholm. Assume on the contrary that this is not the case. Since $I$ is semi-Fredholm, $T$ must be semi-Fredholm as well. Then by Lemma 2.4 we can find $S \in \mathcal{B}(H)$ such that $\lambda T - S$ is semi-Fredholm for every $\lambda \in \mathbb{C}$. We can further find $R \in \mathcal{B}(H)$ such that $\phi(R) = S + K$ for some $K \in \mathcal{K}(H)$. Any compact perturbation of a semi-Fredholm operator is semi-Fredholm (to see this use the fact that $\pi(C + \mathcal{K}(H)) = \pi(C)$ for every $C \in \mathcal{B}(H)$), and thus,

$$\lambda T - \phi(R) = \phi(\lambda I - R)$$

is semi-Fredholm for every $\lambda \in \mathbb{C}$. As $\phi$ preservers semi-Fredholness in both directions we get from here that $\sigma_{SF}(R) = \emptyset$, a contradiction.

Thus, $T$ is a Fredholm operator. It follows that there exists $A \in \mathcal{B}(H)$ such that $AT$ is an idempotent with a finite-dimensional kernel. Define a new map $\psi : \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\psi(X) = A\phi(X), \quad X \in \mathcal{B}(H).$$

Obviously, $\psi$ is a linear map and by one of the remarks given at the begining of the second section, it preserves semi-Fredholm operators in both directions.

Let $K \in \mathcal{K}(H)$ and assume that $B \in SF(H)$. Then there exist $C \in \mathcal{B}(H)$ and $D \in \mathcal{K}(H)$ such that $\phi(C) = B + D$. By our assumption, $C$ is semi-Fredholm and by Lemma 2.3, $C + K$ must be semi-Fredholm as well. It follows that $\phi(C + K) = B + \phi(K) + D \in SF(H)$, and thus $B + \phi(K) \in SF(H)$.
As $B$ was an arbitrary semi-Fredholm operator, Lemma 2.3 yields that $\phi(K)$ is compact. Hence, 
$$\phi(K(H)) \subseteq K(H).$$

Thus, $\phi$ induces a map $\varphi$ on $C(H)$. For every $K \in B(H)$, the product $AK$ is compact if and only if $K$ is compact (to see this use the fact that $\pi$ maps Fredholm operators into invertible elements of the Calkin algebra). Hence, the map $\psi$ can also be factorized through the Calkin algebra. Denote by $\tau$ the induced map. Then $\varphi(x) = a\tau(x)$, $x \in C(H)$. Here, $a = \pi(A)^{-1}$.

So, to complete the proof we have to show that $\tau$ is either an automorphism, or an anti-automorphism of $C(H)$. It is clear that $\varphi$ is surjective. The same must be then true for $\tau$. Moreover, $\tau$ is unital and satisfies $\sigma_{lr}(\tau(x)) = \sigma_{lr}(x)$, $x \in C(H)$. By [12, Proposition 3.4], $\tau$ is a bijective unital continuous linear idempotent preserving map. Applying the fact that $C(H)$ is a prime $C^*$-algebra of real rank zero one can prove using the standard arguments that $\tau$ is either an automorphism, or an anti-automorphism (for the details of this step of the proof see the first two paragraphs of the proof of [13, Theorem 3.1]).

2 Proof of Theorem 1.1. Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : B(H) \to B(H)$ a linear map preserving generalized invertibility in both directions. Assume that $\phi$ is surjective up to finite rank operators. Using the fact that $G(H)$ is invariant under finite rank perturbations and Lemma 2.2 one can easily see that $\phi$ preserves the set of semi-Fredholm operators in both directions. Thus, the proof can be completed by applying Theorem 1.2.

2 Proof of Theorem 1.3. We use the same notation as in the proof of Theorem 1.2. We have 
$$\pi(\phi(T)) = \varphi(\pi(T)) = \pi(A)^{-1}\tau(\pi(T)).$$
Since $T$ is Fredholm if and only if $\pi(T)$ invertible in $C(H)$, we conclude that $T$ is Fredholm if and only if $\phi(T)$ is Fredholm. Thus (i) is proved.

Now, by [11, Theorem 3.3], either $\text{ind}(A\phi(T)) = \text{ind}(\psi(T)) = \text{ind}(T)$, or $\text{ind}(A\phi(T)) = \text{ind}(\psi(T)) = -\text{ind}(T)$ for every Fredholm operator $T$. Thus, if we set $n = -\text{ind}(A) = \text{ind}(\phi(I))$, then either 
$$\text{ind}(\phi(T)) = n + \text{ind}(T)$$
for every Fredholm operator $T$, or 
$$\text{ind}(\phi(T)) = n - \text{ind}(T)$$
for every Fredholm operator $T$. 

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4 Final remarks

We first observe that Theorems 1.1 and 1.2 are essentially different in the sense that the converse of Theorem 1.2 holds true, while this is not the case with Theorem 1.1.

The assumption of surjectivity up to finite rank operators is indispensable in our characterization of linear preservers of generalized invertibility. To see this note that $H$ is isomorphic to $H \oplus H$. Hence, every map defined on $\mathcal{B}(H)$ can be considered as a map from $\mathcal{B}(H)$ into $\mathcal{B}(H \oplus H)$. The elements of $\mathcal{B}(H \oplus H)$ can be represented as $2 \times 2$ matrices with entries in $\mathcal{B}(H)$. Let $\psi : \mathcal{B}(H) \to \mathcal{B}(H)$ be any linear map with the property that $\psi(A) \in \mathcal{G}(H)$ for every $A \in \mathcal{B}(H)$. Then the linear map

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & \psi(A) \end{bmatrix}$$

preserves generalized invertibility in both directions.

The assumption of separability has been used only in Lemma 2.4 and this lemma was used only when proving that $\phi(I)$ is a Fredholm operator. So, if we had assumed from the very beginning that $\phi(I) = I$, then we would have got our main results without the separability assumption. Hence we have the following result.

**Theorem 4.1** Let $H$ be an infinite-dimensional Hilbert space and $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a unital linear map satisfying one of the following two conditions:

- $\phi$ preserves semi-Fredholm operators in both directions,
- $\phi$ preserves generalized invertibility in both directions.

Assume that $\phi$ is surjective up to finite rank operators. Then

$$\phi(K(H)) \subseteq K(H)$$

and the induced map $\varphi : \mathcal{C}(H) \to \mathcal{C}(H)$ is either an automorphism, or an anti-automorphism.

We could formulate a slightly more general result by replacing the assumption that $\phi$ is unital by the weaker assumption that $\phi(I)$ is Fredholm. Then, of course, we would have to change also the conclusion ($\varphi$ would then be an automorphism or an anti-automorphism multiplied by an invertible element of the Calkin algebra).

In the following theorem, we characterize linear maps preserving lower (resp. upper) semi-Fredholm operators in both directions.

**Theorem 4.2** Let $H$ be an infinite-dimensional separable Hilbert space and $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ a linear map surjective up to compact operators. Then the following are equivalent:
(i) φ preserves upper semi-Fredholm operators in both directions;
(ii) φ preserves lower semi-Fredholm operators in both directions;
(iii) φ(K(H)) ⊆ K(H) and the induced map φ : C(H) → C(H) is an automorphism multiplied by an invertible element a ∈ C(H).

Proof. Just a slight modification of the proof of Theorem 1.2 shows that (i) or (ii) implies that τ is either an automorphism, or an anti-automorphism (here τ is as in the proof of Theorem 1.2).

Let us show that τ cannot be an anti-automorphism. Assume to the contrary that τ is an anti-automorphism and suppose that φ satisfies the condition (i). Let T be an upper semi-Fredholm and not Fredholm operator. Then there is an S ∈ B(H) such that π(S)π(T) = π(I) and π(T)π(S) = π(I). It is easy to see, from the condition (i), that τ(π(T)) is left invertible. On the other hand, π(I) = τ(π(ST)) = τ(π(T))τ(π(S)). Consequently, τ(π(T)) is invertible. Thus π(I) = τ(π(S))τ(π(T)) = τ(π(T)π(S)) = τ(π(I)), and since τ is injective, we get that π(T)π(S) = π(I), a contradiction. Hence, (i) implies (iii). In the same manner we can see that (ii) implies (iii).

It is clear that (iii) implies (i) and (ii).

As in the case of Theorem 4.1 we can omit the separability assumption in the above result under the additional assumption that φ is unital.

5 Problems

Let X be a Banach space and B(X) the algebra of bounded linear operators on X. Suppose that the Calkin algebra C(X) is semi-simple. Then we can prove the following. Let φ : B(X) → B(X) be a unital linear map surjective up to finite rank operators. Assume that φ preserves generalized invertibility in both directions. Then φ(K(X)) ⊆ K(X) and the induced map φ : C(X) → C(X) is a bijective unital continuous linear idempotent preserving map.

Note that C(X) is semisimple if and only if K(X) = In(X) (inessential ideal). This is the case for X = ℓ_p(1 ≤ p < ∞) and c_0, and more generally, for Banach spaces X which have a basis satisfying certain conditions (see [4] Theorem 5.4.20). It is not the case for X = L_p(0,1), p ≠ 2, or X = C[0,1].

Question 5.1 It would be interesting to know if Theorem 1.1, Theorem 1.2, and Theorem 4.2 hold true in the context of the Banach algebra of bounded linear operators on a complex Banach space.

Let T ∈ B(H) be generalized invertible and C ∈ B(H) a Fredholm operator. Then both CT and TC belong to G(H). Indeed, it is enough to show only that CT is generalized invertible since both the set of all Fredholm operators and G(H) are invariant under the ∗-operation. Let P be the orthogonal projection onto (Ker C)^⊥. We have CT = (CP)T = C(PT). The operator C maps the
image of $P$ topologically isomorphically onto the closed subspace $\text{Im} C$. So, in order to prove that $CT$ is generalized invertible, that is, the image of $CT$ is closed, we have to show that the image of $PT$ is closed. This is true because $PT = T - (I - P)T$ is a finite rank perturbation of a generalized invertible operator $T$.

Thus, for fixed Fredholm operators $A, B \in \mathcal{B}(H)$ and an arbitrary linear map $\chi : \mathcal{B}(H) \to \mathcal{F}(H)$, both linear maps $T \mapsto ATB + \chi(T), T \in \mathcal{B}(H)$, and $T \mapsto AT^*B + \chi(T), T \in \mathcal{B}(H)$, preserve generalized invertibility in both directions. Here, $A^t$ denotes the transpose of $A$ with respect to some arbitrary, but fixed orthonormal basis.

**Question 5.2** Does a map $\phi$ satisfying the hypothesis of Theorem 1.1 has one of the above two forms?

For fixed Fredholm operators $A, B \in \mathcal{B}(H)$ and an arbitrary linear map $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$, both linear maps $T \mapsto ATB + \chi(T), T \in \mathcal{B}(H)$, and $T \mapsto AT^*B + \chi(T), T \in \mathcal{B}(H)$, preserve Fredholm operators in both directions.

**Question 5.3** Does a map $\phi$ satisfying the hypothesis of Theorem 1.2 has one of the above two forms?

For fixed Fredholm operators $A, B \in \mathcal{B}(H)$ and an arbitrary linear map $\chi : \mathcal{B}(H) \to \mathcal{K}(H)$, the linear map $T \mapsto ATB + \chi(T), T \in \mathcal{B}(H)$, preserves lower (resp. upper) semi-Fredholm operators in both directions.

**Question 5.4** Does a map $\phi$ satisfying the hypothesis of Theorem 4.2 has the above described form?

**References**


