Minimal Rank and Reflexivity of Operator Spaces

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Abstract

Let $S$ be an $n$-dimensional space of linear operators between the linear spaces $U$ and $V$ over an algebraically closed field $F$. Improving results of Larson, Ding, and Li and Pan we show the following

Theorem: Let $S_1, \ldots, S_n$ be a basis of $S$. Assume that every nonzero operator in $S$ has rank larger than $n$. Then a linear operator $T : U \to V$ belongs to $S$ if and only if for every $u \in U$, $Tu$ is a linear combination of $S_1u, \ldots, S_nu$.

1 Introduction

Let $U$ and $V$ be linear spaces over a field $F$, and let $\mathcal{L}(U, V)$ denote the space of all linear operators from $U$ to $V$. Let $S \subset \mathcal{L}(U, V)$ be a linear subspace of operators. An operator $T \in \mathcal{L}(U, V)$ is locally linearly dependent on $S$, if $Tu \in Su = \{Su : S \in S\}$ for all $u \in U$. The reflexive closure of $S$ is the space $\text{ref}(S)$ of all $T \in \mathcal{L}(U, V)$ that are locally linearly dependent on $S$.

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The space $\mathcal{S}$ is (algebraically) reflexive if $\text{ref}(\mathcal{S}) = \mathcal{S}$. Various aspects of reflexive operator spaces were studied in [5, 6, 7, 8, 10]. In this paper we are concerned with the relation between minimal rank and reflexivity. Let

$$\rho(\mathcal{S}) = \min\{\text{rank}(A) : 0 \neq A \in \mathcal{S}\}$$

and let

$$f_F(n) = \sup\{\rho(\mathcal{S}) : \dim \mathcal{S} = n, \text{ref}(\mathcal{S}) \neq \mathcal{S}\}.$$  

Larson [7] proved that if $\mathcal{S}$ is nonreflexive then $\rho(\mathcal{S}) < \infty$. Ding [6] gave a quantitative version of Larson’s result by showing that $f_F(n) \leq n^2$. This was further improved in the complex case by Li and Pan [8] who proved $f_C(n) \leq 2n - 2$. In [10] it was shown that if $|F| \geq n + 3$, then any $n$-dimensional nonreflexive space either contains a nonzero operator of rank $\leq 2n - 3$, or all of its nonzero members have rank $2n - 2$. In particular $f_F(n) \leq 2n - 2$.

The study of reflexivity of finite dimensional operator spaces is closely related to questions concerning locally linearly dependent operator spaces. An $n$-dimensional subspace $\mathcal{R} \subset \mathcal{L}(U,V)$ is locally linearly dependent if $\dim \mathcal{R}u < n$ for all $u \in U$. Locally linearly dependent spaces and reflexive spaces have many applications in operator theory and in ring theory (see e.g. [1, 4, 9]). In [9] it was shown that such $\mathcal{R}$ must contain a nonzero operator of rank at most $n - 1$. Let $\mathcal{S}$ be a nonreflexive $n$-dimensional space of operators and assume that $T \in \text{ref}(\mathcal{S}) - \mathcal{S}$. Then clearly $\mathcal{R} = \text{Span}\{T\} + \mathcal{S}$ is locally linearly dependent. In [10] we used this fact together with a structural result for minimal locally linearly dependent operator spaces to obtain the above mentioned $f_F(n) \leq 2n - 2$ bound. However, one cannot expect this approach to yield sharp estimates, since the condition $T \in \text{ref}(\mathcal{S}) - \mathcal{S}$ is in general stronger then local linear dependence of $\text{Span}\{T\} + \mathcal{S}$. In fact, we conjecture that every $n$-dimensional nonreflexive space contains a nonzero operator of rank at most $n$. Our main result establishes this conjecture for algebraically closed fields.

**Theorem 1.1.** If $F$ is algebraically closed then

$$\left[\frac{n}{2} - \sqrt{n}\right] \leq f_F(n) \leq n \quad (1)$$

The upper bound in Theorem 1.1 is proved in section 2. In section 3 we prove the lower bound, and in addition give a simple construction which in particular shows that if $F$ is finite or $F = \mathbb{Q}$, then $f_F(n) \geq n$. 

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2 The Upper Bound

Let $\mathbb{F}$ be algebraically closed and let $S$ be an $n$-dimensional subspace of $\mathcal{L}(U, V)$ such that $\text{ref}(S) \neq S$. We have to show that there exists an $0 \neq S \in S$ such that $\text{rank} S \leq n$. By standard considerations (see e.g. Proposition 2.1 in [9]) we may assume that $U$ and $V$ are finite dimensional. We thus write $U = \mathbb{F}^k$, $V = \mathbb{F}^\ell$ and identify $\mathcal{L}(U, V)$ with $M_{\ell \times k}(\mathbb{F})$ - the space of $\ell \times k$ matrices over $\mathbb{F}$. Let $S_1, \ldots, S_n$ be a basis of $S$. Let $u = (u_1, \ldots, u_k)$ be a vector of $k$ independent variables and let $K$ denote the function field $\mathbb{F}(u_1, \ldots, u_k)$. For a column vector $v \in K^\ell$ and a row vector $w \in K^k$ we write $v \otimes w$ for the matrix product $v \cdot w \in M_{\ell \times k}(K)$.

Choose an operator $T \in \text{ref}(S) - S$. If $\dim S_a < \dim S = n$ for all $a \in \mathbb{F}^k$, then by [3, 9], $S$ contains a nonzero matrix of rank at most $n - 1$. We thus assume that $\dim S_a = n$ for some $a \in \mathbb{F}^k$. This implies that $S_1u, \ldots, S_nu$ are linearly independent vectors in $K^\ell$. Since $Ta \in \text{Span}\{S_1a, \ldots, S_na\}$ for all $a \in \mathbb{F}^k$, it follows (by the infinitude of $\mathbb{F}$) that $Tu, S_1u, \ldots, S_nu$ are linearly dependent in $K^\ell$. Thus, by Cramer rule, there exist homogeneous polynomials $p(u), p_1(u), \ldots, p_n(u)$ of the same degree $d$, such that $p(u) \neq 0$ and such that

$$p(u)Tu = \sum_{i=1}^n p_i(u)S_iu \quad (2)$$

We may assume that $\gcd(p(u), p_1(u), \ldots, p_n(u)) = 1$. Furthermore, $T \notin S$ implies that $d \geq 1$. Differentiating (2) we obtain

$$p(u)T + Tu \otimes dp(u) = \sum_{i=1}^n p_i(u)S_i + \sum_{i=1}^n S_iu \otimes dp_i(u) \quad (3)$$

where $df(u) = (\frac{\partial f}{\partial u_1}(u), \ldots, \frac{\partial f}{\partial u_k}(u))$. Therefore

$$p(u)T - \sum_{i=1}^n p_i(u)S_i = \sum_{i=1}^n S_iu \otimes (dp_i(u) - \frac{p_i(u)}{p(u)} dp(u)) \quad (4)$$

The righthand side is a sum of at most $n$ rank one matrices in $M_{\ell \times k}(K)$, hence

$$\text{rank}(p(u)T - \sum_{i=1}^n p_i(u)S_i) \leq n \quad (5)$$
If \((p_1(a), \ldots, p_n(a)) = 0\) for all \(a \in \mathbb{F}^k\) such that \(p(a) = 0\), then by the Nullstellensatz, \(p(u)\) must divide some power of each of the \(p_i(u)\)'s. Since \(d \geq 1\), this implies that \(p(u), p_1(u), \ldots, p_n(u)\) have a nontrivial common factor, a contradiction. Therefore, there exists an \(a \in \mathbb{F}^k\) such that \(p(a) = 0\) but \((p_1(a), \ldots, p_n(a)) \neq 0\). By (5)

\[
0 < \text{rank}(\sum_{i=1}^{n} p_i(a)S_i) \leq n .
\]

\[\square\]

3 The Lower Bound

We shall need a characterization of reflexivity due essentially to Azoff [2]. Consider the standard bilinear form on \(M_{\ell \times k}(\mathbb{F})\) given by \((A, B) = \text{tr}(AB^T)\).

For a subspace \(S \subset M_{\ell \times k}(\mathbb{F})\) let

\[
S^\perp = \{B \in M_{\ell \times k}(\mathbb{F}) : (A, B) = 0 \text{ for all } A \in S\} .
\]

Let \(R_d = \{A \in M_{\ell \times k}(\mathbb{F}) : \text{rank}A \leq d\}\. The following result is a variant of Proposition 2.2 in [2].

Claim 3.1. For any subspace \(S \subset M_{\ell \times k}(\mathbb{F})\)

\[
\text{ref}(S)/S \cong S^\perp/\text{Span}(S^\perp \cap R_1) .
\]

In particular, \(S\) is reflexive iff \(S^\perp\) is generated by rank one elements.

Proof: For \(u \in \mathbb{F}^k\) let \(z(u) = \{A \in M_{\ell \times k}(\mathbb{F}) : Au = 0\}\. Clearly \(z(u) = \mathbb{F}^\ell \otimes u^\perp\). It follows that

\[
\text{ref}(S) = \bigcap_{u \in \mathbb{F}^k} (S + z(u)) = \bigcap_{u \in \mathbb{F}^k} (S + \mathbb{F}^\ell \otimes u^\perp) .
\]

Therefore

\[
\text{ref}(S)^\perp = \sum_{u \in \mathbb{F}^k} (S + \mathbb{F}^\ell \otimes u^\perp)^\perp =
\sum_{u \in \mathbb{F}^k} (S^\perp \cap \mathbb{F}^\ell \otimes \text{Span}\{u\}) = \text{Span}(S^\perp \cap R_1) .
\]
It follows that
\[ S^\perp / \text{Span}(S^\perp \cap R_1) = S^\perp / \text{ref}(S)^\perp \cong \text{ref}(S)/S. \]

Let \( F \) be algebraically closed. To prove the lower bound in Theorem 1.1, we show that there exists an \( n \)-dimensional nonreflexive operator space \( S \) such that \( \rho(S) \geq \lfloor \frac{n}{2} - \sqrt{n} \rfloor = r \). Let \( k = \lfloor \frac{n}{2} \rfloor \) and let \( S \) be a generic \( n \)-dimensional subspace of \( M_k(F) \). Recall that \( R_d = \{ A \in M_k(F) : \text{rank} A \leq d \} \) is an irreducible affine variety of codimension \((k - d)^2\) in \( M_k(F) \).

Since \( \text{dim} S\perp = k^2 - n < (k - 1)^2 = \text{codim} R_1 \)

it follows that \( S^\perp \cap R_1 = 0 \), hence \( S \) is nonreflexive by Claim 3.1.

On the other hand
\[ \text{dim} S = n < (k - r + 1)^2 = \text{codim} R_{r-1} \]

Hence \( \rho(S) \geq r \).

For certain non algebraically closed fields, such as \( F = F_q \) or \( F = \mathbb{Q} \), the lower bound can be significantly improved.

**Claim 3.2.** Let \( n > 1 \) and suppose there exists a (not necessarily associative) \( n \)-dimensional division algebra \( A \) over the field \( F \). Then \( f_F(n) \geq n \).

**Proof:** For \( a \in A \) let \( T_a \in \mathcal{L}(A) \) be given by \( T_a(b) = ab \). Let \( S = \{ T_a : a \in A \} \subset \mathcal{L}(A) \). Clearly \( \rho(S) = \text{dim} S = n \). On the other hand \( Sb = A \) for all \( 0 \neq b \in A \) implies that \( \text{ref}(S) = \mathcal{L}(A) \), hence \( S \) is nonreflexive.

**References**


