ZERO PRODUCT PRESERVING MAPS ON MATRIX RINGS OVER DIVISION RINGS

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Abstract. Let $\mathbb{D}$ be a division ring and let $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, $n \geq 2$, be a (not necessarily additive) map satisfying $\phi(A)\phi(B) = 0$ whenever $AB = 0$. We describe the form of ϕ under various assumptions on ϕ, n or $\mathbb{D}$, and provide examples showing that these assumptions are necessary.

1. Introduction

Let R be a ring. We say that a map $\phi : R \to R$ *preserves zero products* if for all $a, b \in R$, $ab = 0$ implies $\phi(a)\phi(b) = 0$. The goal is to describe the form of ϕ. This problem has been considered by many authors over many years, but mostly in the case where R is an algebra and ϕ is linear. In this setting, the usual conclusion is that ϕ is close to an algebra homomorphism – for example, it may be equal to an algebra homomorphism multiplied by a fixed central element. We refer to [1] for a historical account on this topic.

The problem is, of course, much more difficult if we do not assume the linearity (or at least the additivity) of ϕ. We are aware of only one result in this direction [9]. It concerns the case where $R = B(X)$, the algebra of all bounded linear operators on an infinite-dimensional Banach space X (see Theorem 2.1). We will consider the (more entangled!) case where X is finite-dimensional. In fact, in most of our results we will consider a more general situation where $R = M_n(\mathbb{D})$, the ring of $n \times n$ matrices over a possibly noncommutative division ring $\mathbb{D}$. Optimal results will be obtained under the following assumptions:

(a) $n \geq 3$, ϕ preserves zero products in both directions, and either $\mathbb{D}$ is not isomorphic to any of its proper subrings or ϕ is bijective (Theorems 3.4 and 3.5).
(b) $n \geq 3$, $\mathbb{D} = \mathbb{R}$ or $\mathbb{D} = \mathbb{C}$, and ϕ is continuous and bijective (Theorem 4.1).
(c) ϕ is additive (Theorem 5.2).

We will provide several examples illustrating the theorems and justifying the assumptions that are imposed.

Different assumptions require different methods. In the case of (a) and (b), the proofs depend upon a version of the fundamental theorem of projective geometry. In the case of (b), we also make use of the invariance of domain theorem. The notion of a zero product determined ring is the concept behind the proof in the case of (c).

Section 2 is devoted to notation, terminology, and other preliminaries. The case (a) is considered in Section 3, (b) in Section 4, and (c) in Section 5.

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2. Preliminaries

We begin by formulating the result from [9]. To this end, we have to introduce some notation. Let X be a Banach space and $B(X)$ the algebra of all bounded linear operators on X. For a bounded linear operator $A : X \to X$ we denote by $\operatorname{Im} A$ and $\operatorname{Ker} A$ the image of A and the kernel of A, respectively. We say that $\theta : B(X) \to B(X)$ is a *kernel-image preserving map* if

$$\operatorname{Ker}\theta(A) = \operatorname{Ker} A \quad \text{and} \quad \overline{\operatorname{Im}\theta(A)} = \overline{\operatorname{Im} A}$$

for all $A \in B(X)$. Obviously, for any pair $A, B \in B(X)$ we have

$$AB = 0 \iff \operatorname{Im} B \subset \operatorname{Ker} A \iff \overline{\operatorname{Im} B} \subset \operatorname{Ker} A.$$

It is then clear that any kernel-image preserving map $\theta : B(X) \to B(X)$ preserves zero products in both directions, that is, for every pair $A, B \in B(X)$ we have

$$AB = 0 \iff \theta(A)\theta(B) = 0.$$

The following statement was proved in [9].

Theorem 2.1. *Let X be an infinite-dimensional real or complex Banach space and $\phi : B(X) \to B(X)$ a bijective map preserving zero products in both directions. Then there exist a bijective kernel-image preserving map $\theta : B(X) \to B(X)$ and a bounded bijective linear or (in the complex case) conjugate-linear operator $T : X \to X$ such that*

$$\phi(A) = T\theta(A)T^{-1}$$

for all $A \in B(X)$.

The structure of bijective kernel-image preserving maps can be easily described. We define an equivalence relation $\sim$ on $B(X)$ by

$$A \sim B \iff \operatorname{Ker} A = \operatorname{Ker} B \quad \text{and} \quad \overline{\operatorname{Im} A} = \overline{\operatorname{Im} B}.$$

It is straightforward to verify that $\theta : B(X) \to B(X)$ is a bijective kernel-image preserving map if and only if θ maps every equivalence class into itself and the restriction of θ to each equivalence class is a bijection of this class onto itself.

In this paper, we will be interested in the finite-dimensional case. In some sense, this case is more complicated than the infinite-dimensional one. Namely, the above theorem can be reformulated by saying that if X is an infinite-dimensional Banach space, then every bijective map on $B(X)$ preserving zero products in both directions is a product of a bijective kernel-image preserving map and a bounded linear or conjugate-linear spatial automorphism of $B(X)$. It is known that this is not true in the finite-dimensional case. Namely, if X is a finite-dimensional complex Banach space, then there exist discontinuous ring automorphisms of $B(X)$. By a ring automorphism we mean a bijective additive and multiplicative map. Clearly, such maps preserve zero products in both directions.

Nevertheless, we may ask whether an appropriate modification of Theorem 2.1 holds under the assumption that X is finite-dimensional. Moreover, we may ask if in the finite-dimensional case we have an analogue of this statement in the absence of the bijectivity assumption and/or under the weaker assumption that zero products are preserved in one direction only.

In the finite-dimensional case we can identify operators with matrices. As this paper is more abstract algebra than operator theory oriented, we will be interested not only in real or complex matrices, but in matrices over arbitrary division rings. Therefore some more notation is needed. When working with vector spaces over a division ring we have to distinguish between left and right vector spaces. In this regard, we will follow the conventions that are more standard in abstract algebra than in linear algebra.

Let $\mathbb{D}$ be a division ring. We denote by $M_n(\mathbb{D})$ the ring of all $n \times n$ matrices over $\mathbb{D}$. We will always consider $\mathbb{D}^n$, the set of all $1 \times n$ matrices, as a left vector space over $\mathbb{D}$. Correspondingly, we have the right vector space of all $n \times 1$ matrices ${}^t\mathbb{D}^n$. For $A \in M_n(\mathbb{D})$ we take the row space of A, that is the left vector subspace of $\mathbb{D}^n$ generated by the rows of A, and define the row rank of A to be the dimension of this subspace. Similarly, the column rank of A is the dimension of the right vector space generated by the columns of A. This space is called the column space of A. It turns out that these two ranks are equal for every matrix over $\mathbb{D}$ and this common value is called the *rank* of a matrix.

Let $a \in \mathbb{D}^n$ and ${}^tb \in {}^t\mathbb{D}^n$ be any nonzero vectors. Then ${}^tba = ({}^tb)a$ is a matrix of rank one. For a nonzero $x \in \mathbb{D}^n$ and a nonzero ${}^ty \in {}^t\mathbb{D}^n$ we denote by $R(x)$ and $L({}^ty)$ the subsets of $M_n(\mathbb{D})$ defined by

$$R(x) = \{ {}^tux \, : \, {}^tu \in {}^t\mathbb{D}^n \}$$

and

$$L({}^ty) = \{ {}^tyv \, : \, v \in \mathbb{D}^n \}.$$

Clearly, all the elements of these two sets are of rank at most one.

As always, we will identify $n \times n$ matrices with linear transformations mapping $\mathbb{D}^n$ into $\mathbb{D}^n$. Namely, each $n \times n$ matrix A gives rise to a linear operator defined by $x \mapsto xA$, $x \in \mathbb{D}^n$. The rank of the matrix A is equal to the dimension of the image $\operatorname{Im} A$ of the corresponding operator A. The kernel of an operator A is defined as $\operatorname{Ker} A = \{x \in \mathbb{D}^n \, : \, xA = 0\}$. It is the set of all vectors $x \in \mathbb{D}^n$ satisfying $x({}^ty) = 0$ for every ${}^ty$ from the column space of A.

By $\mathbb{P}(\mathbb{D}^n)$ and $\mathbb{P}({}^t\mathbb{D}^n)$ we denote the projective spaces over the left vector space $\mathbb{D}^n$ and the right vector space ${}^t\mathbb{D}^n$, respectively, $\mathbb{P}(\mathbb{D}^n) = \{[x] \, : \, x \in \mathbb{D}^n \setminus \{0\}\}$ and $\mathbb{P}({}^t\mathbb{D}^n) = \{[{}^ty] \, : \, {}^ty \in {}^t\mathbb{D}^n \setminus \{0\}\}$. Here, $[x]$ and $[{}^ty]$ denote the one-dimensional left vector subspace of $\mathbb{D}^n$ generated by x and the one-dimensional right vector subspace of ${}^t\mathbb{D}^n$ generated by ${}^ty$, respectively.

One of the main tools in our proofs will be a relatively recently obtained *non-surjective version of the fundamental theorem of projective geometry* [6]. We will present a slightly weaker version. We start with the projective space over the left vector space $\mathbb{D}^n$. A map $f : \mathbb{D}^n \to \mathbb{D}^n$ is called *semilinear* if we have

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{D}^n$ and there exists a (not necessarily surjective) ring endomorphism $\tau : \mathbb{D} \to \mathbb{D}$ such that

$$f(\lambda x) = \tau(\lambda) f(x)$$

for every $\lambda \in \mathbb{D}$ and $x \in \mathbb{D}^n$. For a map $\alpha : \mathbb{P}(\mathbb{D}^n) \to \mathbb{P}(\mathbb{D}^n)$ we say that its image is *contained in a line* if there exist nonzero vectors $u, v \in \mathbb{D}^n$ such that $\alpha([x]) \subset [u] + [v]$ for every nonzero $x \in \mathbb{D}^n$. The non-surjective version of the fundamental theorem of projective geometry can be formulated in the following way. Let $\alpha : \mathbb{P}(\mathbb{D}^n) \to \mathbb{P}(\mathbb{D}^n)$ be an injective map whose image is not contained in a line. Assume that for all $x, y, z \in \mathbb{D}^n \setminus \{0\}$,

$$[x] \subset [y] + [z] \implies \alpha([x]) \subset \alpha([y]) + \alpha([z]).$$

Then there is an injective semilinear map $f : \mathbb{D}^n \to \mathbb{D}^n$ such that

$$\alpha([x]) = [f(x)], \quad x \in \mathbb{D}^n \setminus \{0\}.$$

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{D}^n$. We define A to be the $n \times n$ matrix whose k-th row is $f(e_k)$, $k = 1, \dots, n$. We claim that

$$f(x) = x_\tau A, \quad x \in \mathbb{D}^n,$$

where

$$x_\tau = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}_\tau = \begin{bmatrix} \tau(x_1) & \dots & \tau(x_n) \end{bmatrix}.$$

Indeed, this is trivially true for $x \in \{e_1, \dots, e_n\}$. For all other vectors $x \in \mathbb{D}^n$ the above equality is a straightforwad consequence of the semilinearity of f. It is important to observe that A is not invertible in general. However, the map $x \mapsto x_\tau A$, $x \in \mathbb{D}^n$, is injective, that is, $x_\tau A \neq 0$ for all nonzero vectors x.

One can now easily formulate an analogous statement for maps on the projective space over the right vector space ${}^t\mathbb{D}^n$.

Let n be a positive integer. In what follows we will always identify matrices $A \in M_n(\mathbb{D})$ with operators $A : \mathbb{D}^n \to \mathbb{D}^n$. Let τ be an endomorphism of the division ring $\mathbb{D}$. For a matrix $A \in M_n(\mathbb{D})$ we denote by A_τ the matrix obtained from $A = [a_{ij}]$ by applying τ entrywise,

$$A_\tau = [a_{ij}]_\tau = [\tau(a_{ij})].$$

Then we will say that $\xi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is a *τ-kernel-image preserving map* if for every $A \in M_n(\mathbb{D})$ we have

$$\operatorname{Im} \xi(A) = \operatorname{Im} A_\tau \quad \text{and} \quad \operatorname{Ker} \xi(A) = \operatorname{Ker} A_\tau.$$

Note first that for every $A \in M_n(\mathbb{D})$ we have $\operatorname{rank} A = \operatorname{rank} A_\tau$, that is, $\dim \operatorname{Im} A = \dim \operatorname{Im} A_\tau$. The easiest way to verify this is to use the well-known fact that $\operatorname{rank} A = r$ if and only if there exist invertible matrices $T, S \in M_n(\mathbb{D})$ such that

$$A = T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S.$$

Here, I_r is the $r \times r$ identity matrix and the zeros stand for zero matrices of the appropriate sizes. Then clearly,

$$A_\tau = T_\tau \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S_\tau$$

and T_τ and S_τ are invertible with $(T_\tau)^{-1} = (T^{-1})_\tau$ and $(S_\tau)^{-1} = (S^{-1})_\tau$. Moreover, since the equality $n = \dim \operatorname{Im} A + \dim \operatorname{Ker} A$ holds for matrices over an arbitrary division ring, we have also $\dim \operatorname{Ker} A = \dim \operatorname{Ker} A_\tau$.

We claim that ξ preserves zero products in both directions. Indeed, for $A, B \in M_n(\mathbb{D})$ we have

$$AB = 0 \iff A_\tau B_\tau = 0 \iff \operatorname{Im} A_\tau \subset \operatorname{Ker} B_\tau$$
$$\iff \operatorname{Im} \xi(A) \subset \operatorname{Ker} \xi(B) \iff \xi(A)\xi(B) = 0.$$

Let $\Gamma : \mathbb{D}^n \to \mathbb{D}^n$ be the map defined by

$$\Gamma\left(\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}\right) = \begin{bmatrix} \tau(a_1) & \tau(a_2) & \dots & \tau(a_n) \end{bmatrix}.$$

For a subspace $U \subset \mathbb{D}^n$ we denote by U_τ the linear span of its Γ-image, that is,

$$U_\tau = \operatorname{span} \Gamma(U)$$

(note that we have $U_\tau = \Gamma(U)$ if τ is an automorphism, not just an endomorphism). For every pair of subspaces $U, V \subset \mathbb{D}^n$ with $\dim U + \dim V = n$ we set

$$\mathcal{S}(U, V) = \{A \in M_n(\mathbb{D}) : \operatorname{Im} A = U \text{ and } \operatorname{Ker} A = V\}.$$

Clearly, $M_n(\mathbb{D})$ is a disjoint union of such subsets. A map $\xi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is τ-kernel-image preserving if and only if for every pair of subspaces $U, V \subset \mathbb{D}^n$ with $\dim U + \dim V = n$ we have

$$\xi\left(\mathcal{S}(U,V)\right) \subset \mathcal{S}(U_\tau, V_\tau).$$

All the above observations will be used frequently without reference in what follows.

3. The general case

In this section, we consider the general situation where ϕ is an arbitrary map.

When starting to work on optimal finite-dimensional analogues of Theorem 2.1, our initial conjecture was that if $n \geq 3$ is an integer, $\mathbb{D}$ any division ring, and $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ a map such that for every pair $A, B \in M_n(\mathbb{D})$ we have $AB = 0$ if and only if $\phi(A)\phi(B) = 0$, then there exist an endomorphism $\tau : \mathbb{D} \to \mathbb{D}$, a τ-kernel-image preserving map $\xi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, and an invertible matrix $T \in M_n(\mathbb{D})$ such that $\phi(A) = T\xi(A)T^{-1}$ for every $A \in M_n(\mathbb{D})$.

Clearly, the converse is true, that is, every map $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ of the form $\phi(A) = T\xi(A)T^{-1}$, $A \in M_n(\mathbb{D})$, preserves zero products in both directions.

It should be explained why the assumption $n \geq 3$ is necessary in the above conjecture. Of course, we need it in order to apply the fundamental theorem of projective geometry. But it turns out that it is indispensable. Namely, in the following example we construct a bijective map $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ preserving zero products in both directions that is not of the form as described in the conjecture.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be any bijective function with $f(0) = 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(0) = 0$ and

$$g(x) = -\frac{1}{f\left(-\frac{1}{x}\right)}$$

whenever $x \neq 0$. We now define $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $\phi(A) = A$ whenever A is invertible or $A = 0$. It remains to define $\phi(A)$ for all matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

of rank one. If A is such a matrix with $a_1 \neq 0$, then we define

$$\phi(A) = a_1 \begin{bmatrix} 1 & g\left(\frac{a_2}{a_1}\right) \\ f\left(\frac{a_3}{a_1}\right) & f\left(\frac{a_3}{a_1}\right) g\left(\frac{a_2}{a_1}\right) \end{bmatrix}.$$

In the case that A is of rank one, $a_1 = 0$, and $a_2 \neq 0$ (then, clearly, $a_3 = 0$) we set

$$\phi(A) = a_2 \begin{bmatrix} 0 & 1 \\ 0 & f\left(\frac{a_4}{a_2}\right) \end{bmatrix}.$$

Further, when A is of rank one, $a_1 = a_2 = 0$, and $a_3 \neq 0$ we define

$$\phi(A) = a_3 \begin{bmatrix} 0 & 0 \\ 1 & g\left(\frac{a_4}{a_3}\right) \end{bmatrix}.$$

And finally, for every $a_4 \neq 0$ we set

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix}.$$

Since we can choose any bijective function fixing 0 for f, in general ϕ is not of the form $\phi(A) = T\xi(A)T^{-1}$ as in the above conjecture. To check that ϕ is bijective we first note that ϕ is a bijection of the set of all invertible matrices onto itself and it maps the zero matrix to itself. Thus we need to see that it maps the set of all rank one matrices bijectively onto itself. We write the set of all rank one matrices as the disjoint union of four sets:

- $\mathcal{S}_1 \subset M_2(\mathbb{R})$ is the set of all rank one matrices with a nonzero $(1,1)$-entry,
- $\mathcal{S}_2 \subset M_2(\mathbb{R})$ is the set of all rank one matrices with the zero $(1,1)$ entry and a nonzero $(1,2)$-entry,
- $\mathcal{S}_3 \subset M_2(\mathbb{R})$ is the set of all rank one matrices with the zero $(1,1)$ and $(1,2)$ entries and a nonzero $(2,1)$-entry,
- $\mathcal{S}_4 \subset M_2(\mathbb{R})$ is the set of all rank one matrices whose all entries are zero but the $(2,2)$-entry.

Clearly, if

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

is a rank one matrix with $a_1 \neq 0$, then $a_4 = \frac{a_2 a_3}{a_1}$. It follows that $\phi(\mathcal{S}_j) \subset \mathcal{S}_j$, $j = 1, 2, 3, 4$, and we need to show that ϕ maps each $\mathcal{S}_j$ bijectively onto itself. Using the fact that both $f, g : \mathbb{R} \to \mathbb{R}$ are bijective functions one can easily complete the proof of the bijectivity of ϕ.

It remains to verify that for every pair $A, B \in M_2(\mathbb{R})$ we have

$$AB = 0 \iff \phi(A)\phi(B) = 0.$$

Assume first that A is invertible. Then $AB = 0$ if and only if $B = 0$ and $\phi(A)\phi(B) = 0 = A\phi(B)$ if and only if $\phi(B) = 0$ which is equivalent to $B = 0$. It is similarly easy to check that the above equivalence holds true when B is invertible or $A = 0$ or $B = 0$.

Hence, we can assume that both A and B are of rank one. We have to distinguish all possible cases, that is, $A \in \mathcal{S}_i$ and $B \in \mathcal{S}_j$, $i, j \in \{1, 2, 3, 4\}$. We will consider here only the first possibility and leave the others (which are easy exercises) to the reader. So, let A, B both belong to $\mathcal{S}_1$. Then we have

$$A = a \begin{bmatrix} 1 & x \\ y & xy \end{bmatrix} \quad \text{and} \quad B = b \begin{bmatrix} 1 & u \\ v & uv \end{bmatrix}$$

for some real numbers a, b, x, y, u, v with both a and b nonzero. Obviously, $AB = 0$ if and only if $xv = -1$. By the definition of ϕ we have

$$\phi(A) = a \begin{bmatrix} 1 & g(x) \\ f(y) & g(x)f(y) \end{bmatrix} \quad \text{and} \quad \phi(B) = b \begin{bmatrix} 1 & g(u) \\ f(v) & g(u)f(v) \end{bmatrix},$$

and thus, $\phi(A)\phi(B) = 0$ if and only if $1 + g(x)f(v) = 0$. Clearly this happens if and only if $1 + xv = 0$, that is, $AB = 0$.

The reader may have observed that instead of defining $\phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ with formulas and verifying the required properties with somewhat lengthy (but trivial) computations, we could have used a more conceptual approach with two maps on projective spaces $\alpha : \mathbb{P}(\mathbb{R}^2) \to \mathbb{P}(\mathbb{R}^2)$ and $\beta : \mathbb{P}({}^t\mathbb{R}^2) \to \mathbb{P}({}^t\mathbb{R}^2)$ having a certain orthogonality property: if for $x \in \mathbb{R}^2 \setminus \{0\}$ and ${}^ty \in {}^t\mathbb{R}^2 \setminus \{0\}$ we denote $\alpha([x]) = [u]$ and $\beta([{}^ty]) = [{}^tv]$, then the requirement is that $x\,{}^ty = 0$ if and only if $u\,{}^tv = 0$. In the $n = 2$ case there are plenty of such pairs of maps α, β and they can be used to define the restriction of ϕ to the set of all rank one matrices (some special attention has to be paid to the bijectivity of a map ϕ defined in this way).

The following lemma seemingly supports the aforementioned conjecture. We denote by $M_n^{\leq 1}(\mathbb{D})$ the set of all $n \times n$ matrices with rank at most one.

Lemma 3.2. *Let $n \geq 3$ be an integer and $\mathbb{D}$ any division ring. Let $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be a map such that for every pair $A, B \in M_n(\mathbb{D})$,*

$$AB = 0 \iff \phi(A)\phi(B) = 0.$$

Then there exist an endomorphism $\tau : \mathbb{D} \to \mathbb{D}$, a τ-kernel-image preserving map $\xi : M_n^{\leq 1}(\mathbb{D}) \to M_n^{\leq 1}(\mathbb{D})$, and an invertible matrix $T \in M_n(\mathbb{D})$ such that

$$\phi(A) = T\xi(A)T^{-1}$$

for every $A \in M_n^{\leq 1}(\mathbb{D})$.

Proof. Let ${}^t x_1, \dots, {}^t x_n \in {}^t\mathbb{D}^n$ and $y_1, \dots, y_n \in \mathbb{D}^n$ be any two (linearly independent) n-tuples of vectors satisfying

$$y_i \, {}^t x_j = \delta_{ij}, \quad i, j = 1, \dots, n.$$

If we denote $A_i = {}^t x_i y_i \in M_n(\mathbb{D})$, $i = 1, \dots, n$, then clearly $A_i^2 = A_i$, $i = 1, \dots, n$, and $A_i A_j = 0$ whenever $i \neq j$. Hence, we have

$$\phi(A_i)\phi(0) = 0$$

for all $i = 1, \dots, n$,

$$\phi(A_i)\phi(A_i) \neq 0$$

for all $i = 1, \dots, n$, and

$$\phi(A_i)\phi(A_j) = 0$$

for all $i, j \in \{1, \dots, n\}$, $i \neq j$. In particular, each of the matrices (operators) A_i, $i = 1, \dots, n$, is nonzero. We further conclude that $\{0\} \neq \operatorname{Im} \phi(A_i) \subset \operatorname{Ker} \phi(A_j)$ whenever $i \neq j$. Therefore

$$\sum_{i \neq j} \operatorname{Im} \phi(A_i) \subset \operatorname{Ker} \phi(A_j)$$

and since $\operatorname{Im} \phi(A_j) \not\subset \operatorname{Ker} \phi(A_j)$, we have

$$\operatorname{Im} \phi(A_j) \not\subset \sum_{i \neq j} \operatorname{Im} \phi(A_i)$$

for all $j = 1, \dots, n$. It follows easily that for every $j \in \{1, \dots, n\}$ the subspace $\sum_{i \neq j} \operatorname{Im} \phi(A_i)$ has dimension at least $(n-1)$, and since $\operatorname{Ker} \phi(A_j) \neq \mathbb{D}^n$, we infer that $\sum_{i \neq j} \operatorname{Im} \phi(A_i) = \operatorname{Ker} A_j$ is of dimension $n-1$. We conclude that all $\phi(A_j)$'s are of rank one. Hence, all the images of operators $\phi(A_i)$ are one-dimensional subspaces and they are linearly independent. Thus, we have

$$\phi(A_i) = {}^t u_i v_i$$

for some nonzero vectors ${}^t u_1, \dots, {}^t u_n \in {}^t\mathbb{D}^n$ and $v_1, \dots, v_n \in \mathbb{D}^n$ and we further know that $v_1, \dots, v_n \in \mathbb{D}^n$ are linearly independent. Similarly, ${}^t u_1, \dots, {}^t u_n \in {}^t\mathbb{D}^n$ are linearly independent. It follows that $\phi(0) = 0$.

Let ${}^t x$, ${}^t y \in {}^t\mathbb{D}^n$ be any pair of linearly independent vectors. We choose vectors ${}^t x = {}^t x_1, {}^t y = {}^t x_2, {}^t x_3, \dots, {}^t x_n \in {}^t\mathbb{D}^n$ and $y_1, \dots, y_n \in \mathbb{D}^n$ as above. Further let matrices A_j, $j = 1, \dots, n$, and vectors ${}^t u_1, \dots, {}^t u_n \in {}^t\mathbb{D}^n$ and $v_1, \dots, v_n \in \mathbb{D}^n$ be defined as in the previous paragraph.

Take any nonzero $w \in \mathbb{D}^n$ and observe that

$$A_j\,{}^t xw = 0, \quad j = 2, \ldots, n,$$

and

$$A_1\,{}^t xw \neq 0,$$

and therefore

$${}^t u_j v_j \phi({}^t xw) = 0$$

for every $j = 2, \ldots, n$. It follows that

$$v_j \phi({}^t xw) = 0$$

for every $j = 2, \ldots, n$, and

$$v_1 \phi({}^t xw) \neq 0.$$

Since $v_2, \ldots, v_n$ are linearly independent, we conclude that $\phi({}^t xw)$ is of rank one, that is, $\phi({}^t xw) = {}^t ab$ for some nonzero vectors ${}^t a$ and b (note that, in particular, we have verified that each rank one matrix is mapped into a matrix of rank one). Using

$$v_j\,{}^t u_1 = 0 \quad \text{and} \quad v_j\,{}^t a = 0, \quad j = 2, \ldots, n,$$

we see that u_1 and a are linearly dependent. In other words, for every $w \in \mathbb{D}^n$ there exists a vector $c \in \mathbb{D}^n$ such that

$$\phi({}^t xw) = {}^t u_1 c,$$

that is,

$$\phi(L({}^t x)) \subset L({}^t u_1).$$

As we have started with an arbitrary nonzero vector ${}^t x$ we see that for every nonzero ${}^t e \in {}^t\mathbb{D}^n$ there exists a nonzero ${}^t d \in {}^t\mathbb{D}^n$ such that

$$\phi(L({}^t e)) \subset L({}^t d).$$

Clearly, $L({}^t e) = L({}^t e')$ if and only if ${}^t e$ and ${}^t e'$ are linearly dependent. If, on the other hand, ${}^t e$ and ${}^t e'$ are linearly independent, then

$$L({}^t e) \cap L({}^t e') = \{0\}.$$

Thus, ϕ induces a map $\beta : \mathbb{P}({}^t\mathbb{D}^n) \to \mathbb{P}({}^t\mathbb{D}^n)$ such that

$$\beta([{}^t e]) = [{}^t d] \iff \phi(L({}^t e)) \subset L({}^t d).$$

Note that β is injective. Indeed, we have shown that for an arbitrary pair of linearly independent vectors ${}^t x$, ${}^t y$ we have $\beta([{}^t x]) \neq \beta([{}^t y])$.

Our next claim is that if for some nonzero ${}^t f \in {}^t\mathbb{D}^n$ we have $[{}^t f] \subset [{}^t x] + [{}^t y]$ then $\beta([{}^t f]) \subset \beta([{}^t x]) + \beta([{}^t y])$. Indeed, from $[{}^t f] \subset [{}^t x] + [{}^t y]$ we conlude that

$$A_j\,{}^t f = 0$$

for all $j = 3, \ldots, n$, implying that

$$\phi(A_j)\phi({}^t fg) = 0, \quad j = 3, \ldots, n,$$

where $g \in \mathbb{D}^n$ is any nonzero vector. Since $\phi(A_j) = {}^t u_j v_j$ and $\phi({}^t fg) = {}^t f' g'$ for some nonzero ${}^t f' \in \beta([{}^t f])$ and some nonzero $g' \in \mathbb{D}^n$, we see that

$$v_j\,{}^t f' = 0, \quad j = 3, \ldots, n.$$

We know that $v_1, \ldots, v_n$ are linearly independent, ${}^t u_1$ and ${}^t u_2$ are linearly independent, $v_j\,{}^t u_i = 0$, $i = 1, 2$, $j = 3, \ldots, n$, and consequently, ${}^t f'$ belongs to the linear span of ${}^t u_1$ and ${}^t u_2$. Since $\beta([{}^t f]) = [{}^t f']$, $\beta([{}^t x]) = [{}^t u_1]$ and $\beta([{}^t y]) = [{}^t u_2]$, we finally conclude that $\beta([{}^t f]) \subset \beta([{}^t x]) + \beta([{}^t y])$.

We have proved that for any pair of linearly independent vectors ${}^t x$ and ${}^t y$ we have

$$[{}^t f] \subset [{}^t x] + [{}^t y] \Rightarrow \beta([{}^t f]) \subset \beta([{}^t x]) + \beta([{}^t y]).$$

As the range of β is not contained in a line (the range of β contains elements $[{}^t u_1], \ldots, [{}^t u_n]$) we can apply the fundamental theorem of projective geometry to conclude that there exist a matrix $T \in M_n(\mathbb{D})$ and an endomorphism $\tau : \mathbb{D} \to \mathbb{D}$ such that

$$\beta([{}^t x]) = [T\,{}^t x_\tau], \quad {}^t x \in {}^t D^n \setminus \{0\}.$$

Since the range of β contains elements $[{}^t u_1], \ldots, [{}^t u_n]$, the matrix T is invertible. Using the definition of the map β we arrive at

$$\phi(L({}^t x)) \subset L(T\,{}^t x_\tau)$$

for every nonzero ${}^t x \in {}^t\mathbb{D}^n$. In the same way we see that there exist an invertible matrix S and an endomorphism $\sigma : \mathbb{D} \to \mathbb{D}$ such that

$$\phi(R(y)) \subset R(y_\sigma S)$$

for every nonzero $y \in \mathbb{D}^n$.

Hence, for every rank one matrix ${}^t xy \in M_n(\mathbb{D})$ there exists a nonzero $\lambda \in \mathbb{D}$ (depending on ${}^t xy$) such that

$$\phi({}^t xy) = T\,{}^t x_\tau \lambda y_\sigma S.$$

The zero product preserving property then yields that for any pair of vectors $y \in \mathbb{D}^n$ and ${}^t u \in {}^t\mathbb{D}^n$ we have

$$y\,{}^t u = 0 \iff y_\sigma ST\,{}^t u_\tau = 0.$$

If we denote $C = ST$ then for any pair of n-tuples $y_1, \ldots, y_n \in \mathbb{D}$ and $u_1, \ldots, u_n \in \mathbb{D}$, the following is true:

$$\sum_{k=1}^n y_k u_k = 0 \iff \begin{bmatrix} \sigma(y_1) & \ldots & \sigma(y_n) \end{bmatrix} \; C \begin{bmatrix} \tau(u_1) \\ \vdots \\ \tau(u_n) \end{bmatrix} = 0.$$

Choosing $y_1 = u_2 = 1$ and $y_2 = \ldots = y_n = u_1 = u_3 = \ldots = u_n = 0$ we see that the $(1,2)$-entry of C is zero. In the same way we show that all off-diagonal entries of C are zero. Thus, C is a diagonal matrix with diagonal entries $c_1, \ldots, c_n$ and then the above equivalence can be rewritten as

$$\sum_{k=1}^n y_k u_k = 0 \iff \sum_{k=1}^n \sigma(y_k) c_k \tau(u_k) = 0.$$

Choosing $y_1 = u_1 = 1$, $y_2 = \lambda$, $u_2 = -\frac{1}{\lambda}$, and $y_3 = \ldots = y_n = u_3 = \ldots = u_n = 0$, where $\lambda \in \mathbb{D}$ is any nonzero element, we arrive at

$$c_1 - \sigma(\lambda) c_2 \tau(\lambda)^{-1} = 0$$

for every nonzero $\lambda \in \mathbb{D}$. If we set $\lambda = 1$ we conclude that $c_1 = c_2$, and then similarly, $c_1 = \ldots = c_n = c$. In particular, $ST = cI$ and

$$\sigma(\lambda) = c\tau(\lambda)c^{-1}, \quad \lambda \in \mathbb{D}.$$

It follows that for every rank one matrix ${}^t xy \in M_n(\mathbb{D})$ there exists a nonzero $\lambda \in \mathbb{D}$ (depending on ${}^t xy$) such that

$$\phi({}^t xy) = T\,{}^t x_\tau \lambda (c y_\tau c^{-1})(cT^{-1}) = T\,{}^t x_\tau \mu y_\tau T^{-1},$$

where we have denoted $\mu = \lambda c$. Clearly,

$$\mathrm{Im}\, {}^t x_\tau \mu y_\tau = \mathrm{Im}\,({}^t xy)_\tau \quad \text{and} \quad \mathrm{Ker}\, {}^t x_\tau \mu y_\tau = \mathrm{Ker}\,({}^t xy)_\tau .$$

Thus, the map $\xi : M_n^{\leq 1}(\mathbb{D}) \to M_n^{\leq 1}(\mathbb{D})$ given by $\xi({}^t xy) = {}^t x_\tau \mu y_\tau$ has the desired properties. □

In spite of the fact that the above statement shows that the behavior of ϕ on the set of all rank one matrices is as expected, the conjecture turns out to be false.

Example 3.3. Recall (see [7]) that there exists an endomorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that $\tau(\mathbb{C})$ is a proper subfield of $\mathbb{C}$. Even more, it is possible to find an endomorphism τ such that $\mathbb{C}$ is an infinite-dimensional vector space over $\tau(\mathbb{C})$. In particular, it is possible to find complex numbers $c_1, \ldots, c_n$ that are linearly independent in the vector space $\mathbb{C}$ over the field $\tau(\mathbb{C})$, and, moreover, $c_1^2 + \cdots + c_n^2 \neq 0$. We define $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\phi(A) = A_\tau$$

for all $A \in M_n(\mathbb{C}) \setminus \{I\}$ and

$$\phi(I) = {}^t cc,$$

where $c = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix}$. Using the fact ${}^t ccB \neq 0$ and $B\,{}^t cc \neq 0$ for every nonzero complex $n \times n$ matrix B with all entries in $\tau(\mathbb{C})$ we can easily verify that ϕ preserves zero products in both directions, but is not of the form as in the above conjecture.

Nevertheless, the conjecture is true for a large class of division rings. We call a division ring $\mathbb{D}$ an EAS division ring if every ring endomorphism $\tau : \mathbb{D} \to \mathbb{D}$ is automatically surjective. In other words, $\mathbb{D}$ is EAS if it is not isomorphic to any of its proper subrings. The field of real numbers and the field of rational numbers are well-known to be EAS. Obviously, every finite field is EAS. Another example is the division ring of quaternions (see, for example, [8]). In the above counterexample we have used the fact that the field of complex numbers is not EAS.

Theorem 3.4. *Let $n \geq 3$ be an integer and $\mathbb{D}$ any EAS division ring. Let $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be a map such that for every pair $A, B \in M_n(\mathbb{D})$,*

$$AB = 0 \iff \phi(A)\phi(B) = 0.$$

Then there exist an automorphism $\tau : \mathbb{D} \to \mathbb{D}$, a τ-kernel-image preserving map $\xi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, and an invertible matrix $T \in M_n(\mathbb{D})$ such that

$$\phi(A) = T\xi(A)T^{-1}$$

for every $A \in M_n(\mathbb{D})$.

Proof. We apply Lemma 3.2. Because of the EAS assumption τ must be an automorphism. After composing ϕ first by a similarity transformation $A \mapsto T^{-1}AT$ and then by the map $A \mapsto A_{\tau^{-1}}$ we can assume with no loss of generality that every matrix ${}^t xy$ of rank one is mapped by ϕ to ${}^t x\lambda y$ for some nonzero $\lambda \in \mathbb{D}$ depending on ${}^t xy$. From

$${}^t xyA = 0 \iff \phi({}^t xy)\phi(A) = 0 \quad \text{and} \quad A\,{}^t xy = 0 \iff \phi(A)\phi({}^t xy) = 0$$

we conclude that for every $A \in M_n(\mathbb{D})$, ${}^t x \in {}^t\mathbb{D}^n$, and every $y \in \mathbb{D}^n$ we have

$$yA = 0 \iff y\phi(A) = 0 \quad \text{and} \quad A\,{}^t x = 0 \iff \phi(A)\,{}^t x = 0$$

yielding that

$$\operatorname{Im} A = \operatorname{Im} \phi(A) \quad \text{and} \quad \operatorname{Ker} A = \operatorname{Ker} \phi(A).$$

□

For general division rings we need to add the bijectivity assumption to get the desired conclusion.

Theorem 3.5. *Let $n \geq 3$ be an integer and $\mathbb{D}$ any division ring. Let $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be a bijective map such that for every pair $A, B \in M_n(\mathbb{D})$ we have*

$$AB = 0 \iff \phi(A)\phi(B) = 0.$$

Then there exist an automorphism $\tau : \mathbb{D} \to \mathbb{D}$, a bijective τ-kernel-image preserving map $\xi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, and an invertible matrix $T \in M_n(\mathbb{D})$ such that

$$\phi(A) = T\xi(A)T^{-1}$$

for every $A \in M_n(\mathbb{D})$.

Proof. Again we apply Lemma 3.2. As before there is no loss of generality in assuming that T is the identity matrix. We need to show that τ is bijective. Once we will prove this, the proof can be completed in exactly the same way as the proof of the previous theorem.

We observe that ϕ^{-1} has exactly the same properties as ϕ. Thus, ϕ^{-1} maps rank one matrices to rank one matrices. Let $c \in \mathbb{D}$ be any nonzero element. By bijectivity, there exists a rank one matrix ${}^t xy$ that is mapped by ϕ into ${}^t e_1(e_1 + ce_2)$, that is,

$${}^t x_\tau \lambda y_\tau = {}^t e_1(e_1 + ce_2)$$

for some nonzero element λ which further implies that

$$y_\tau = \begin{bmatrix} \tau(y_1) & \tau(y_2) & \ldots & \tau(y_n) \end{bmatrix}$$

and $e_1 + ce_2$ are linearly dependent. Consequently, c belongs to the range of τ. Hence, τ is surjective, as desired. □

One may ask whether we can prove the above theorem under the weaker assumption that zero products are preserved in one direction only. The next example shows that the answer is, in general, negative.

Example 3.6. We need to construct a bijective map $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ satisfying $\phi(A)\phi(B) = 0$ whenever $AB = 0$ that is not of the form as in the conclusion of the above theorem. We will consider the special case when $\mathbb{D} = \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. As usual, we write E_{11} for the matrix unit all of whose entries are zero but the $(1,1)$ entry that is equal to 1, and I for the identity matrix. Define $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by $\phi(A) = A$ whenever

$$A \notin \{E_{11}, I, 2E_{11}, 2I, 3E_{11}, 3I, \ldots\},$$

$$\phi(I) = E_{11},$$

$$\phi(nI) = (n-1)I, \quad n = 2, 3, \ldots,$$

and

$$\phi(nE_{11}) = (n+1)E_{11}, \quad n = 1, 2, \ldots$$

It is easy to check that ϕ is bijective. So, assume that $AB = 0$ for some $A, B \in M_n(\mathbb{F})$ and we need to show that then $\phi(A)\phi(B) = 0$. This is obviously true when $A \neq I$ and $B \neq I$ since in this case we have $\phi(A) = \lambda A$ and $\phi(B) = \mu B$ for some nonzero scalars λ, μ. If $A = I$ or $B = I$ then $AB = 0$ implies that the other one must be the zero matrix and thus, $\phi(A)\phi(B) = 0$ in this case as well.

Of course, using similar ideas one can construct more complicated examples. Moreover, the composition of two bijective maps preserving zero products in one direction is again a bijective map with the same preserving property, which yields a variety of further examples.

4. The case where ϕ is continuous

The map ϕ in Example 3.6 is not continuous. It is perhaps somewhat surprising that in the presence of the continuity assumption we can get a nice result on maps preserving zero products in one direction only. Let $\mathbb{F}$ be either the field of real numbers or the field of complex numbers. As before we say that a map $\theta : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a kernel-image preserving map if

$$\operatorname{Ker}\theta(A) = \operatorname{Ker} A \quad \text{and} \quad \operatorname{Im}\theta(A) = \operatorname{Im} A$$

for all $A \in M_n(\mathbb{F})$. In the complex case we say that a map $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a conjugate-kernel-image preserving map if

$$\operatorname{Ker}\theta(A) = \operatorname{Ker}\overline{A} \quad \text{and} \quad \operatorname{Im}\theta(A) = \operatorname{Im}\overline{A}$$

for all $A \in M_n(\mathbb{C})$. Here, $\overline{A} = \overline{[a_{ij}]} = [\overline{a_{ij}}]$ is the matrix obtained from A by applying the complex conjugation entrywise.

The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $\mathbb{F}$ be either the field of real numbers or the field of complex numbers and $n \geq 3$ a positive integer. Assume that $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a bijective continuous map such that for every pair $A, B \in M_n(\mathbb{F})$,*

$$AB = 0 \implies \phi(A)\phi(B) = 0.$$

Then either there exist a continuous bijective kernel-image preserving map $\theta : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ and an invertible matrix $T \in M_n(\mathbb{F})$ such that

$$\phi(A) = T\theta(A)T^{-1}$$

for every $A \in M_n(\mathbb{F})$, or $\mathbb{F} = \mathbb{C}$ and there exist a continuous bijective conjugate-kernel-image preserving map $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and an invertible matrix $T \in M_n(\mathbb{C})$ such that

$$\phi(A) = T\theta(A)T^{-1}$$

for every $A \in M_n(\mathbb{C})$.

Proof. Obviously, $\phi(0) = 0$. Indeed, we have $\phi(A)\phi(0) = 0$ for every $A \in M_n(\mathbb{F})$ and therefore, by the surjectivity of ϕ we see that $B\phi(0) = 0$ for every $B \in M_n(\mathbb{F})$, and thus, $\phi(0) = 0$.

Our next claim is that for every nonzero $x \in \mathbb{F}^n$ there exists a nonzero $y \in \mathbb{F}^n$ such that

$$\phi(R(x)) \subset R(y).$$

Without loss of generality we can assume that $x = e_1$, that is, $R(x) = R(e_1)$ is the set of all matrices of the form

$$\begin{bmatrix} * & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{bmatrix}.$$

We denote by $\mathcal{V}$ the linear subspace of $M_n(\mathbb{F})$ consisting of all matrices of the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{bmatrix}.$$

Clearly, $AB = 0$ for every $A \in R(e_1)$ and every $B \in \mathcal{V}$.

Let $A_0 \in R(e_1)$ be any nonzero matrix. Since ϕ is injective we have $\phi(A_0) \neq 0$. We can find invertible matrices $P, Q \in M_n(\mathbb{F})$ such that

$$P\phi(A_0)Q = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where $r > 0$ is the rank of A_0. Since $\phi(A_0)\phi(B) = 0$ for every $B \in \mathcal{V}$ we have

$$Q^{-1}\phi(B) = \begin{bmatrix} 0_r & 0 \\ * & * \end{bmatrix}, \quad B \in \mathcal{V},$$

where 0_r stands for the $r \times r$ zero matrix. Hence, the $n(n-1)$-dimensional space $\mathcal{V}$ is mapped by ϕ injectively and continuously into some $n(n-r)$-dimensional subspace of $M_n(\mathbb{F})$. By the invariance of domain theorem this is possible only if $r = 1$.

Hence, the map $X \mapsto Q^{-1}\phi(X)$ is an injective continuous map from $\mathcal{V}$ into itself, and since the zero matrix is mapped into itself and this map is open by the invariance of domain theorem, we can find a positive real number ε and $B_0 \in \mathcal{V}$ such that

$$Q^{-1}\phi(B_0) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \varepsilon & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varepsilon & 0 \end{bmatrix}.$$

Since for every $A \in R(e_1)$ we have $\phi(A)QQ^{-1}\phi(B_0) = 0$ we conclude that $\phi(A)Q \in R(e_1)$. In other words, $\phi(R(e_1)) \subset R(e_1Q^{-1})$.

Thus, we have shown that for every nonzero $x \in \mathbb{F}^n$ there exists a nonzero $y \in \mathbb{F}^n$ such that $\phi(R(x)) \subset R(y)$, and similarly, for every nonzero ${}^t u \in {}^t\mathbb{F}^n$ there exists a nonzero ${}^t v \in {}^t\mathbb{F}^n$ such that $\phi(L({}^t u)) \subset L({}^t v)$.

Let $x, u \in \mathbb{F}^n$ be any linearly independent vectors. Assume that we have $\phi(R(x)) \subset R(y)$ and $\phi(R(u)) \subset R(y)$ for some nonzero $y \in \mathbb{F}^n$. All three subspaces $R(x)$, $R(y)$, and $R(u)$ are homeomorphic to $\mathbb{F}^n$ and thus, by the invariance of domain theorem, both $\phi(R(x))$ and $\phi(R(u))$ are open subsets in $R(y)$ and both of these open subsets contain the zero matrix. But then $\phi(R(x)) \cap \phi(R(u))$ is a nonempty open set contradicting the bijectivity of ϕ and the fact that

$$R(x) \cap R(u) = \{0\}.$$

Hence, ϕ induces an injective map $\alpha : \mathbb{P}(\mathbb{F}^n) \to \mathbb{P}(\mathbb{F}^n)$ such that for any nonzero $x, y \in \mathbb{F}^n$ we have

$$\alpha([x]) = ([y]) \iff \phi(R(x)) \subset R(y).$$

Similarly, there is an injective map $\beta : \mathbb{P}({}^t\mathbb{F}^n) \to \mathbb{P}({}^t\mathbb{F}^n)$ such that for any nonzero ${}^t u$, ${}^t v \in {}^t\mathbb{F}^n$ we have

$$\beta([{}^t u]) = ([{}^t v]) \iff \phi(L({}^t u)) \subset L({}^t v).$$

Let $x, y \in \mathbb{F}^n$ be nonzero vectors such that $\phi(R(x)) \subset R(y)$, and let k be an integer, $1 \le k \le n$. Assume that $U \subset R(x)$ is a linear subspace of dimension k. Then we can find vectors ${}^t u_1, \dots, {}^t u_k \in {}^t\mathbb{F}^n$ such that

$$ {}^t u_1 x, \dots, {}^t u_k x \in U, $$

and if we denote

$$ \phi({}^t u_j x) = {}^t v_j y, \quad j = 1, \dots, k, $$

then ${}^t v_1, \dots, {}^t v_k$ are linearly independent. Indeed, if this was not the case then $\phi(U)$ would be contained in some $(k-1)$-dimensional subspace of $R(y)$ contradicting the invariance of domain theorem.

In the next step we will prove that for any nonzero $x, y, z \in \mathbb{F}^n$ we have

$$ [x] \subset [y] + [z] \implies \alpha([x]) \subset \alpha([y]) + \alpha([z]). $$

The case when y and z are linearly dependent is trivial. So, assume they are linearly independent. Denote $\alpha([x]) = [x']$, $\alpha([y]) = [y']$, and $\alpha([z]) = [z']$. Choose nonzero vectors $w, t \in \mathbb{F}^n$ satisfying $\phi(R(w)) \subset R(t)$. Then the linear subspace $W \subset R(w)$ of all matrices A satisfying $yA = zA = 0$ is of dimension $n-2$, and by the previous paragraph we can find linearly independent vectors ${}^t a_1, \dots, {}^t a_{n-2}$ such that

$$ {}^t a_1 t, \dots, {}^t a_{n-2} t \in \phi(W). $$

From $yA = zA = 0$, $A \in W$, we conclude that $xA = 0$ for every $A \in W$, and consequently,

$$ x' \, {}^t a_j = y' \, {}^t a_j = z' \, {}^t a_j = 0, \quad j = 1, \dots, n-2. $$

Since ${}^t a_1, \dots, {}^t a_{n-2}$ are linearly independent and y' and z' are linearly independent we conclude that x' belongs to the linear span of y' and z', as desired.

Thus, we can apply the fundamental theorem of projective geometry to find a field endomorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible $n \times n$ matrix T (the invertibility follows from the fact descibed in the paragraph before the previous one) such that

$$ \alpha([x]) = [x_\tau T], \quad x \in \mathbb{F}^n \setminus \{0\}. $$

We repeat the same arguments for β and then conclude that there exists another field endomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and an invertible $n \times n$ matrix S such that for every rank one matrix ${}^t yx \in M_n(\mathbb{F})$ we have

$$ \phi({}^t yx) = \lambda S \, {}^t y_\sigma x_\tau T $$

for some nonzero scalar λ (note that rank one matrix cannot be mapped to the zero matrix because ϕ is bijective).

Take any sequence (μ_n) of complex numbers converging to 0. Then

$$ {}^t e_1(e_1 + \mu_n e_2) \to {}^t e_1 e_1, $$

and if we denote

$$ \phi({}^t e_1 e_1) = \lambda_0 S \, {}^t e_1 e_1 T, $$

then

$$ \phi({}^t e_1(e_1 + \mu_n e_2)) = \lambda_n (S \, {}^t e_1)(e_1 + \tau(\mu_n) e_2) T \to \lambda_0 S \, {}^t e_1 e_1 T, $$

yielding that $\lambda_n \to \lambda_0$, which together with

$$ \lambda_n \tau(\mu_n) \to 0 $$

implies that $\tau(\mu_n)$ converges to 0. This yields that τ is continuous at 0, and hence continuous everywhere. It follows that τ is either the identity or the complex conjugation (of course, in the real case the only endomorphism is the identity and therefore this last paragraph could be omitted if we were interested in the real case only). Similarly, σ is either the identity or the complex conjugation.

Using the same ideas as in the proof of Lemma 3.2 we show that either both τ and σ are the identity, or they are both the complex conjugation, and $S = cT^{-1}$ for some nonzero scalar c. After composing ϕ with a similarity transformation $X \mapsto TXT^{-1}$, and the map $X \mapsto \overline{X}$, if necessary, we may assume with no loss of generality that for every rank one matrix R there exists a nonzero scalar λ (depending on R) such that

$$\phi(R) = \lambda R.$$

It remains to show that for any pair of subspaces $U, V \in \mathbb{F}^n$ satisfying $\dim U + \dim V = n$ we have

$$\phi(\mathcal{M}(U, V)) = \mathcal{M}(U, V),$$

where

$$\mathcal{M}(U, V) = \{A \in M_n(\mathbb{D}) \, : \, \operatorname{Im} A \subset U \text{ and } V \subset \operatorname{Ker} A\}.$$

Once we know this, an easy inductive argument shows that then actually

$$\phi(\mathcal{S}(U, V)) = \mathcal{S}(U, V),$$

which completes the proof.

Thus, let $U, V \in \mathbb{F}^n$ be subspaces satisfying $\dim U + \dim V = n$. Set $\dim U = r$ and choose invertible matrices $P, Q \in M_n(\mathbb{F})$ such that $UQ = \operatorname{span}\{e_1, \ldots, e_r\}$ and $VP^{-1} = \operatorname{span}\{e_{r+1}, \ldots, e_n\}$. Then $A \in \mathcal{M}(U, V)$ if and only if

$$\operatorname{Im}(PAQ) = \operatorname{Im}(AQ) \subset UQ = \operatorname{span}\{e_1, \ldots, e_r\}$$

and

$$\operatorname{Ker}(PAQ) = \operatorname{Ker}(PA) \supset VP^{-1} = \operatorname{span}\{e_{r+1}, \ldots, e_n\},$$

that is, we have $A \in \mathcal{M}(U, V)$ if and only if the matrix PAQ has the block form

$$PAQ = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix},$$

where $*$ stands for some $r \times r$ matrix. If $A \in \mathcal{M}(U, V)$ and R is any rank one matrix of the form

$$R = \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix},$$

where the blocks are of the same size as in the block matrix representation of PAQ, then $PAQR = 0$, and therefore $AQR = 0$ which further yields that $0 = \phi(A)\phi(QR) = \phi(A)\lambda(QR)$ for some nonzero scalar λ. Thus, $(P\phi(A)Q)R = 0$ for every such rank one matrix R yielding that $P\phi(A)Q$ is of the form

$$P\phi(A)Q = \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}.$$

We repeat the same trick with rank one matrices but this time with multiplication on the left side to conclude that

$$P\phi(A)Q = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}.$$

In other words, we have

$$\phi(\mathcal{M}(U, V)) \subset \mathcal{M}(U, V).$$

Clearly, $\mathcal{M}(U,V)$ is homeomorphic to $\mathbb{F}^{r^2}$. By the invariance of domain theorem, $\phi(\mathcal{M}(U,V))$ is open in $\mathcal{M}(U,V)$. We need to prove that it is also closed. Once we will do this we will know that we have the equality

$$\phi(\mathcal{M}(U,V)) = \mathcal{M}(U,V),$$

as desired.

Thus, the last step in the proof is to show that $\phi(\mathcal{M}(U,V))$ is closed in $\mathcal{M}(U,V)$. By the invariance of domain theorem the bijective map ϕ maps open sets into open sets, and therefore closed sets into closed sets, and thus $\phi(\mathcal{M}(U,V))$ is closed in $M_n(\mathbb{F})$. It follows that $\phi(\mathcal{M}(U,V)) \subset \mathcal{M}(U,V)$ is closed in $\mathcal{M}(U,V)$. □

5. The case where ϕ is additive

In this final section, we consider the case where $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, with $n \geq 2$ and $\mathbb{D}$ a division ring, is an *additive* zero product preserving map. There are two obvious types of such maps. The first type consists of additive maps ϕ satisfying

$$\phi(A)\phi(B) = 0$$

for all $A, B \in M_n(\mathbb{D})$. The second type consists of maps of the form

$$\phi(A) = C\phi_0(A)$$

where ϕ_0 is a ring endomorphism of $M_n(\mathbb{D})$ (i.e., a multiplicative additive map from $M_n(\mathbb{D})$ to itself) and C is a matrix commuting with each $\phi_0(A)$. Our goal is to show that there are no other types than these two.

Some comments concerning the second type are in order. First, every ring endomorphism ϕ_0 of $M_n(\mathbb{D})$ is of the form

$$\phi_0(A) = TA_\tau T^{-1},$$

where T is an invertible matrix in $M_n(\mathbb{D})$ and τ is a ring endomorphism of $\mathbb{D}$ (and, as always, $A_\tau = [a_{ij}]_\tau = [\tau(a_{ij})]$). We believe this is a folklore result; in any case, it can be proved by standard methods. From this it follows easily that $T^{-1}CT$ is a scalar matrix, $T^{-1}CT = cI$, where $c \in D$ satisfies $c\tau(a) = \tau(a)c$ for all $a \in \mathbb{D}$. As the next example shows, this does not necessarily mean that c lies in the center of $\mathbb{D}$ (and hence C does not necessarily lie in the center of $M_n(\mathbb{D})$).

Example 5.1. Let W be the Weyl algebra over a characteristic 0 field F in countably infinitely many variables. That is, W is the algebra generated by x_i, y_i, $i = 1, 2, \ldots$, and relations

$$x_iy_j - y_jx_i = \delta_{ij}, \quad x_ix_j = x_jx_i, \quad y_iy_j = y_jy_i$$

for all i, j. It is well-known that a Weyl algebra in finitely many variables is a (right and left) Ore domain [3, Example 7.21]; since every pair of elements in W lies in such a (sub)algebra, W is an Ore domain as well. The classical (right or left) ring of quotients of W is therefore a division ring, which we denote by $\mathbb{D}$ (see [3, Section 7.3] for details). Let $\tau_0 : W \mapsto W$ be the algebra endomorphism determined by

$$\tau_0(x_i) = x_{i+1} \quad \text{and} \quad \tau_0(y_i) = y_{i+1}.$$

We can extend τ_0 to an endomorphism τ of $\mathbb{D}$ [3, Proposition 7.14]. It is clear that $c = x_1$ commutes with every element in $\tau(\mathbb{D})$. However, c does not commute with y_1 and so does not belong to the center of $\mathbb{D}$.

Our approach is based on the concept of a *zero product determined ring* [4]. This is a ring, let us call it R, with the following property: if G is an additive Abelian group and $\beta : R \times R \to G$ is a biadditive map satisfying $\beta(x, y) = 0$ whenever $xy = 0$, then there exists an additive map $\alpha : R \to G$ such that $\beta(x, y) = \alpha(xy)$ for all $x, y \in R$. If S is any unital ring and $n \geq 2$, then $R = M_n(S)$ is a zero product determined ring [4, Theorem 2.1] (see also [2, Section 4] for a more conceptual proof). This is clearly applicable to our problem. Indeed, if $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$, $n \geq 2$, is an additive zero product preserving map, then the map $\beta : M_n(\mathbb{D}) \times M_n(\mathbb{D}) \to M_n(\mathbb{D})$ given by $\beta(A, B) = \phi(A)\phi(B)$ satisfies $\beta(A, B) = 0$ whenever $AB = 0$, and hence there exists an additive map $\alpha : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ such that

$$\phi(A)\phi(B) = \beta(A, B) = \alpha(AB)$$

for all $A, B \in M_n(\mathbb{D})$. (It should be mentioned that the existence of a map α satisfying the last formula can be also extracted from the arguments in [5, Section 2].)

Our last theorem reads as follows.

Theorem 5.2. *Let $n \geq 2$ be an integer and $\mathbb{D}$ any division ring. Let $\phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be an additive map such that for every pair $A, B \in M_n(\mathbb{D})$,*

$$AB = 0 \implies \phi(A)\phi(B) = 0.$$

Then either

$$\phi(A)\phi(B) = 0$$

for all $A, B \in M_n(\mathbb{D})$ or there exist $C \in M_n(\mathbb{D})$ and a ring endomorphism ϕ_0 of $M_n(\mathbb{D})$ such that

$$\phi(A) = C\phi_0(A) = \phi_0(A)C$$

for all $A \in M_n(\mathbb{D})$.

Proof. As we have just explained, there exists an additive map $\alpha : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ such that

$$\alpha(AB) = \phi(A)\phi(B)$$

for all $A, B \in M_n(\mathbb{D})$. Setting

$$C = \phi(I)$$

we clearly have

$$\alpha(A) = \phi(A)C = C\phi(A)$$

for all $A \in M_n(\mathbb{D})$, and hence

$$C\phi(AB) = \phi(A)\phi(B)$$

for all $A, B \in M_n(\mathbb{D})$.

If C is invertible then

$$\phi_0(A) = C^{-1}\phi(A)$$

defines a ring endomorphism of $M_n(\mathbb{D})$ that clearly satisfies the conditions from the statement of the theorem. Assume, therefore, that C is not invertible. Then C has a nontrivial kernel. From $C\phi(AB) = \phi(A)\phi(B)$ we thus see that the intersection of the kernels of all $\phi(A)$ is nontrivial. Choosing an appropriate basis we may therefore assume that the n-th row of each matrix $\phi(A)$ is zero. That is, writing

$$\phi(A) = [\phi_{ij}(A)]$$

where ϕ_{ij} are additive maps from $M_n(\mathbb{D})$ to $\mathbb{D}$, we have $\phi_{nj} = 0$ for every j. Similarly, let us write

$$\alpha(A) = [\alpha_{ij}(A)]$$

where $\alpha_{ij} : M_n(\mathbb{D}) \to \mathbb{D}$ are additive maps. Suppose $\alpha \neq 0$. Then $\alpha_{rs} \neq 0$ for some r and s. From $\phi(A)\phi(B) = \alpha(AB)$ we infer that

$$\sum_{k=1}^{n-1} \phi_{rk}(A)\phi_{ks}(B) = \alpha_{rs}(AB).$$

Since α_{rs} is additive, there exists a matrix unit E_{pq} and $d \in \mathbb{D}$ such that $\alpha_{rs}(dE_{pq}) \neq 0$. Writing dE_{pi} for A and E_{jq} for B in the above identity we thus obtain

$$\sum_{k=1}^{n-1} \phi_{rk}(dE_{pi})\phi_{ks}(E_{jq}) = \delta_{ij}\alpha_{rs}(dE_{pq}).$$

That is, the vectors

$$x_i = \begin{bmatrix}\phi_{r1}(dE_{pi}) & \phi_{r2}(dE_{pi}) & \dots & \phi_{r,n-1}(dE_{pi})\end{bmatrix} \in \mathbb{D}^{n-1},\ i = 1, \dots, n,$$

and

$$y_j = \begin{bmatrix}\phi_{1s}(E_{jq}) & \phi_{2s}(E_{jq}) & \dots & \phi_{n-1,s}(E_{jq})\end{bmatrix} \in \mathbb{D}^{n-1},\ j = 1, \dots, n,$$

satisfy

$$x_i{}^t y_j = 0 \iff i \neq j$$

for all $i, j = 1, \dots, n$. However, this is impossible since $x_1, \dots, x_n$, being vectors in an $(n-1)$-dimensional vector space, are linearly dependent. This contradiction shows that $\alpha = 0$, yielding that $\phi(A)\phi(B) = 0$ for all $A, B \in M_n(\mathbb{D})$. □

Theorem 5.2 is a generalization of [5, Corollary 2.4] and [2, Corollary 5.2].

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