Non-linear commutativity preserving maps on hermitian matrices *

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Abstract

Let $H_n$, $n \geq 3$, be the space of all $n \times n$ hermitian matrices. Assume that a map $\phi : H_n \rightarrow H_n$ preserves commutativity in both directions (no linearity or bijectivity of $\phi$ is assumed). Then $\phi$ is a unitarily similarity transformation composed with a locally polynomial map possibly composed with the transposition. The same result holds for injective continuous maps on $H_n$ preserving commutativity in one direction only. We give counterexamples showing that these two theorems cannot be improved or extended to the infinite-dimensional case.

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1 Introduction and statement of the main results

Let $H$ be a separable complex Hilbert space, $\dim H \geq 3$. We denote by $B(H)$ and $B_s(H)$ the algebra of all bounded linear operators on $H$ and the real subspace of all hermitian (self-adjoint) operators. The elements of $B_s(H)$ describe bounded observable physical quantities in quantum mechanics. Therefore they are usually called bounded observables. The bounded observables are compatible if they can be measured jointly. Transformations on quantum structures

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that preserve certain physically relevant relations or operations or properties are
called symmetries. Such maps have been extensively studied by mathematical
physicists [4].

Two bounded observables are compatible if and only if the corresponding
hermitian operators commute. Thus, the problem of describing the general
form of symmetries with respect to compatibility can be reformulated in
the mathematical language as the problem of characterizing bijective maps
Φ : \( B_+(H) \to B_+(H) \) preserving commutativity in both directions, i.e., the
bijective maps with the property that
\[
AB = BA \iff Φ(A)Φ(B) = Φ(B)Φ(A)
\]
for every pair \( A, B \in B_+(H) \). We say that \( Φ : B_+(H) \to B_+(H) \)
preserves commutativity if \( Φ(A) \) and \( Φ(B) \) commute for every pair of commuting operators
\( A, B \in B_+(H) \).

Let \( U \) be a unitary or antiunitary (conjugate-linear bijective isometry) op-
erator on \( H \). Then obviously, the map \( A \mapsto UAU^*, A \in B_+(H) \), is a bijection
of \( B_+(H) \) onto itself preserving commutativity in both directions. Let us choose
for every \( A \in B_+(H) \) a real-valued bounded Borel function \( f_A \) defined on the
spectrum of \( A \) and define a map \( ϕ : B_+(H) \to B_+(H) \) by \( ϕ(A) = f_A(A) \). Clearly,
every such map preserves commutativity (and is bijective and preserves com-
mutativity in both directions if certain additional conditions are fulfilled). In
[9] it was proved that every bijective map on \( B_+(H) \) preserving commutativity
in both directions is a product of two maps of the above two types. In the
finite-dimensional case this result reads as follows.

**Theorem 1.1** [9] Let \( H_n, n \geq 3 \), be the real space of all \( n \times n \) hermitian ma-
trices and let \( ϕ : H_n \to H_n \) be a bijective map. Then \( ϕ \) preserves commutativity
in both directions if and only if there exists a unitary \( n \times n \) matrix \( U \) and for
every \( A \in H_n \) there is an injective real-valued function \( f_A \) on \( σ(A) \) such that either
\[
ϕ(A) = Uf_A(A)U^*
\]
for every \( A \in H_n \), or
\[
ϕ(A) = Uf_A(A^t)U^*
\]
for every \( A \in H_n \). Here, \( σ(A) \) denotes the set of all eigenvalues of \( A \) and \( A^t \)
denotes the transpose of \( A \).

Note that for every hermitian matrix \( A \) we have \( A^t = \overline{A} \), where \( \overline{A} \) denotes
the matrix obtained from \( A \) by applying the complex conjugation entrywise. If
\( f_A \) is a real-valued function defined on the spectrum of \( A \) and \( A = \sum_{t \in σ(A)} tE(t) \)
is the spectral decomposition of \( A \) (for every \( t \in σ(A) \) the symbol \( E(t) \) stands
for the orthogonal projection on the eigenspace corresponding to the eigenvalue
\( t \)), then \( f_A(A) = \sum_{t \in σ(A)} f_A(t)E(t) \). Of course, \( f_A(A) = p_A(A) \) for some real
polynomial $p_A$, and thus, the maps of the form $A \mapsto f_A(A)$ are called locally polynomial maps. Every map on $H_n$ of the form (1.1) or (1.2) will be called a standard commutativity preserving map.

We started with a physical motivation for studying commutativity preserving maps. There are other motivations for studying the above problem. Two operators $A$ and $B$ commute if and only if their Lie product $[A, B] = AB - BA$ is equal to zero. Hence, the study of such maps is closely related to the study of Lie isomorphisms (see [2]). This is one of the reasons that the problem of characterizing linear maps preserving commutativity on operator and more general algebras has been extensively studied in the last few decades. For a very recent result we refer to [3], where further references can be found.

The problem of characterizing linear maps preserving commutativity belongs to a large group of the so called linear preserver problems [7]. The theory of linear preservers is well-developed. Recently, the first results on more difficult non-linear preserver problems have been obtained [1, 8, 10, 11]. Our work is another contribution in this direction.

There are two natural questions when thinking of possible improvements of the above mentioned characterization of bijective maps on $B_s(H)$ preserving commutativity in both directions. Can we obtain the same conclusion under the weaker assumption of preserving commutativity in one direction only? What can be said about maps preserving commutativity in both directions in the absence of the bijectivity assumption?

Let us start with the second question. As we will see in the next section there exist rather wild maps on $B_s(H)$ preserving commutativity in both directions. However, in the finite-dimensional case Theorem 1.1 can be improved as follows.

**Theorem 1.2** Let $H_n$, $n \geq 3$, be the real space of all $n \times n$ hermitian matrices. A map $\phi : H_n \to H_n$ preserves commutativity in both directions if and only if there exists a unitary $n \times n$ matrix $U$ and for every $A \in H_n$ there is a real polynomial $p_A$ which is injective on $\sigma(A)$ such that either

\[
\phi(A) = U p_A(A) U^* 
\]

for every $A \in H_n$, or

\[
\phi(A) = U p_A(A^t) U^* 
\]

for every $A \in H_n$.

It is somewhat surprising that bijective maps on $H_n$ preserving commutativity in one direction only need not be standard. The counter-example given in the next section is discontinuous. It turns out that the discontinuity is essential when constructing non-standard bijective maps preserving commutativity in one direction. Namely, we will prove the following result.

**Theorem 1.3** Let $H_n$, $n \geq 3$, be the real space of all $n \times n$ hermitian matrices. Assume that $\phi : H_n \to H_n$ is an injective continuous map such that $\phi(A)\phi(B) =$
\( \phi(B)\phi(A) \) for every pair of commuting matrices \( A,B \in H_n \). Then there exists a unitary \( n \times n \) matrix \( U \) and for every \( A \in H_n \) there is a real polynomial \( p_A \) such that either
\[
\phi(A) = U p_A(A) U^* \quad \text{for every } A \in H_n,
\]
or
\[
\phi(A) = U p_A(A^t) U^* \quad \text{for every } A \in H_n.
\]

To prove our results we will use essentially different methods than those in [9]. In particular, the proof of Theorem 1.3 is based on the invariance of domain theorem [6, p.344] stating that if \( U \) is an open subset of \( \mathbb{R}^n \) and \( F : U \rightarrow \mathbb{R}^n \) a continuous injective map, then \( F(U) \) is open. It should be mentioned here that the first counter-example given in the next section shows that the finite-dimensionality assumption is indispensable in Theorem 1.3.

### 2 Counter-examples

Let \( H \) be an infinite-dimensional complex Hilbert space. We will first show that there exist “rather wild” injective continuous maps on \( B_s(H) \) preserving commutativity in both directions, thus showing that the finite-dimensionality assumption is essential in our results. Observe that \( B_s(H) \) can be identified with \( B_s(H \oplus H) \). Define a map \( \phi : B_s(H) \rightarrow B_s(H \oplus H) \) by
\[
\phi(A) = \begin{bmatrix} A & 0 \\ 0 & \varphi(A) \end{bmatrix}, \quad A \in B_s(H),
\]
where \( \varphi : B_s(H) \rightarrow B_s(H) \) is a continuous map preserving commutativity. Obviously, \( \phi \) is injective, continuous, and it preserves commutativity in both directions. There are plenty of continuous maps \( \varphi \) on \( B_s(H) \) preserving commutativity. For an easy example, one can take any continuous map whose range consists of pairwise commuting operators. As another example take a pair of non-commuting operators \( A_0, B_0 \in B_s(H) \) and let \( U, V \) be open neighbourhoods of \( A_0 \) and \( B_0 \), respectively, such that \( U \cap V = \emptyset \) and \( CD - DC \neq 0 \) for every pair of operators \( C \in U \) and \( D \in V \). Let \( f, g : B_s(H) \rightarrow \mathbb{R} \) be non-zero continuous functions whose supports are contained in \( U \) and \( V \), respectively. Choose any pair of non-commuting operators \( T, S \in B_s(H) \) and define
\[
\varphi(A) = f(A) T + g(A) S, \quad A \in B_s(H).
\]
Then \( \varphi \) is a continuous map on \( B_s(H) \) preserving commutativity, and the range of \( \varphi \) is not commutative. One can construct further examples using the above ideas and the fact that a direct sum of commutativity preserving continuous maps is a continuous commutativity preserving map. Further, composing two
continuous commutativity preserving maps we again arrive at a map of the same type. It is now clear that there is no nice description of injective continuous maps on $B_b(H)$ preserving commutativity in both directions.

To construct a bijective non-standard map on $H_n$, $n \geq 2$, preserving commutativity in one direction only we need a few definitions. Let $\mathcal{P} = \{P_1, \ldots, P_n\} \subset H_n$ be a maximal set of pairwise orthogonal rank one projections. We will denote by $\Pi$ the family of all such sets. We will further denote by $H^1_n$ the set of all $n \times n$ hermitian matrices whose all eigenvalues have multiplicity one. Thus, $H^1_n$ is the set of all $n \times n$ hermitian matrices with $n$ distinct eigenvalues. For every $\mathcal{P} = \{P_1, \ldots, P_n\} \in \Pi$ we denote by $\mathcal{D}_\mathcal{P}$ the set of all hermitian matrices of the form $\sum_{k=1}^n t_k P_k$, where the $t_k$'s are arbitrary real numbers. The set of all non-scalar hermitian matrices that are diagonalizable with respect to $\mathcal{P}$ will be denoted by $\mathcal{E}_\mathcal{P}$,

$$\mathcal{E}_\mathcal{P} = \mathcal{D}_\mathcal{P} \setminus \mathbb{R} I.$$ 

And finally, set

$$\mathcal{F}_\mathcal{P} = \mathcal{D}_\mathcal{P} \cap H^1_n.$$ 

It is easy to find $\mathcal{P}, \mathcal{Q} \in \Pi$ such that

$$\mathcal{E}_\mathcal{P} \cap \mathcal{E}_\mathcal{Q} = \emptyset.$$ 

Indeed, all we have to do is to find $\mathcal{P} = \{P_1, \ldots, P_n\}, \mathcal{Q} = \{Q_1, \ldots, Q_n\} \in \Pi$ such that $P_1, \ldots, P_n, Q_1, \ldots, Q_{n-1}$ are linearly independent. Then $\mathcal{D}_\mathcal{P} \cap \mathcal{D}_\mathcal{Q}$ is at most one-dimensional real space, and therefore, this intersection is the space of all scalar hermitian matrices. The desired equality $\mathcal{E}_\mathcal{P} \cap \mathcal{E}_\mathcal{Q} = \emptyset$ follows trivially. Choose a bijective map $\phi : \mathcal{F}_\mathcal{P} \cup \mathcal{F}_\mathcal{Q} \to \mathcal{E}_\mathcal{P} \cup \mathcal{E}_\mathcal{Q}$ such that

$$\phi(\mathcal{F}_\mathcal{P}) = \mathcal{E}_\mathcal{Q} \quad \text{and} \quad \phi(\mathcal{F}_\mathcal{Q}) = \mathcal{E}_\mathcal{P}.$$ 

Assume that $\Delta$ is a subset of $\Pi$ containing $\mathcal{P}$ and $\mathcal{Q}$. Suppose further that

$$\psi : \bigcup_{\mathcal{R} \in \Delta} \mathcal{F}_\mathcal{R} \to \bigcup_{\mathcal{R} \in \Delta} \mathcal{E}_\mathcal{R}$$

is a bijective extension of $\phi$ with the property that

$$\mathcal{F}_\mathcal{R} \subset \psi(\mathcal{F}_\mathcal{R}) \subset \mathcal{E}_\mathcal{R} \quad (2.1)$$

whenever $\mathcal{R} \neq \mathcal{P}, \mathcal{Q}$. Let $S \in \Pi \setminus \Delta$. Then, clearly,

$$\mathcal{F}_S \cap \left( \bigcup_{\mathcal{R} \in \Delta} \mathcal{F}_\mathcal{R} \right) = \emptyset$$

and

$$\mathcal{F}_S \subset \mathcal{E}_S \setminus \left( \bigcup_{\mathcal{R} \in \Delta} \mathcal{E}_\mathcal{R} \right).$$
Thus, we can extend $\psi$ to a bijection

$$\psi : \bigcup_{R \in \Delta \cup \{S\}} F_R \rightarrow \bigcup_{R \in \Delta \cup \{S\}} E_R$$

such that (2.1) holds true.

By Zorn’s lemma, there exists a bijection

$$\phi : \bigcup_{R \in \Pi} F_R \rightarrow \bigcup_{R \in \Pi} E_R$$

such that

$$\phi(F_P) = E_Q \quad \text{and} \quad \phi(F_Q) = E_P \quad (2.2)$$

and

$$F_R \subset \phi(F_R) \subset E_R \quad (2.3)$$

whenever $R \neq P, Q$. In other words, $\phi$ is a bijection of $H_n^1$ onto the set of all non-scalar hermitian matrices. Two matrices $A, B \in H_n^1$ commute if and only if there exists $P \in \Pi$ such that $A, B \in F_P$. So, $\phi$ preserves commutativity and it can be extended to a bijection of $H_n$ onto itself. This extension, which will be denoted by $\phi$ again, maps all matrices with at least one multiple eigenvalue into scalar matrices. So, it preserves commutativity as well. And because of (2.2) and (2.3), it is not of a standard form.

### 3 Preliminary results

For every $A \in H_n$ we denote by $A'$ the commutant of $A$, $A' = \{B \in H_n : AB = BA\}$. Obviously, $A'$ is a real-linear subspace of $H_n$. Let $\phi : H_n \rightarrow H_n$ be any map. If $A \in H_n$, then we denote by $\phi(A)^c$ the set of all matrices from the range of $\phi$ that commute with $\phi(A)$,

$$\phi(A)^c = \{B \in \text{range} \phi : B\phi(A) = \phi(A)B\}.$$

A hermitian matrix $P \in H_n$ is called a projection if $P^2 = P$. Every projection can be identified with its range. Thus, projections of rank $k$ are identified with $k$-dimensional subspaces of $\mathbb{C}^n$. We will denote by $E_{jk}, j, k = 1, \ldots, n$, the $n \times n$ matrix whose all entries are zero except the $(j, k)$-entry which is equal to one.

Let us start with a simple observation.

**Lemma 3.1** Assume that $\phi : H_n \rightarrow H_n$ preserves commutativity in both directions. Then for every pair $A, B \in H_n$ we have

$$A' \subset B' \iff \phi(A)^c \subset \phi(B)^c$$

and

$$\phi(A)' \subset \phi(B)' \Rightarrow A' \subset B'.$$
Proof. The verification of the first statement is trivial. To check the second one note that \( \phi(A)^c = \phi(A)' \cap \mathrm{range} \phi \). Thus, \( \phi(A)' \subset \phi(B)' \) yields \( \phi(A)^c \subset \phi(B)^c \), which, by the first statement, gives the desired inclusion \( A' \subset B' \).

\[ \square \]

Before improving the second statement of the above lemma we will make a few simple observations. It is easy to see that for a diagonal hermitian matrix \( D = \text{diag} (t_1, \ldots, t_n) \) the commutant \( D' \) is the real-linear subspace of \( H_n \) spanned by diagonal matrix units \( E_{kk}, k = 1, \ldots, n \), and all matrices of the form \( E_{kl} + iE_{lk}, iE_{kl} - iE_{lk} \), where \( l, k \) is any pair of integers, \( 1 \leq k, l \leq n \), satisfying \( k \neq l \) and \( t_l = t_k \). In particular, if \( C = \text{diag} (s_1, \ldots, s_n) \), \( D = \text{diag} (t_1, \ldots, t_n) \) \( \in H_n \) are diagonal matrices, then \( C' \subset D' \) if and only if for every pair of integers \( k, l \), \( 1 \leq k, l \leq n \), the equality \( s_k = s_l \) yields \( t_k = t_l \). In this case we have \( \mathrm{card} \sigma(D) \leq \mathrm{card} \sigma(C) \). Here, \( \mathrm{card} \sigma(C) \) denotes the number of distinct eigenvalues of \( C \).

Next, it is trivial to check that if \( \phi : H_n \to H_n \) is a map preserving commutativity in both directions (in one direction), and \( U \) and \( V \) are \( n \times n \) unitary matrices, then the map \( A \mapsto V\phi(UAU^*)V^* \) preserves commutativity in both directions (in one direction) as well.

It is well-known that any family of pairwise commuting hermitian matrices is simultaneously unitarily similar to a subset of diagonal matrices.

And finally, let \( \mathcal{O} \) be a poset (partially ordered set). A map \( \xi : \mathcal{O} \to \mathcal{O} \) is monotone if it preserves order, that is, \( p \leq q \Rightarrow \phi(p) \leq \phi(q) \), \( p, q \in \mathcal{O} \). A bijective map \( \xi : \mathcal{O} \to \mathcal{O} \) is called an automorphism of the poset \( \mathcal{O} \) if both \( \xi \) and \( \xi^{-1} \) are monotone. It is well-known and rather easy to see, that if \( \mathcal{O} \) is a finite poset and \( \xi : \mathcal{O} \to \mathcal{O} \) a bijective monotone map, then \( \xi \) is an automorphism. Indeed, denote for every \( p \in \mathcal{O} \) by \( p_\leq \) the set of all \( q \in \mathcal{O} \) such that \( q \leq p \). Then, clearly, \( \xi(p_\leq) \subset \xi(p)_\leq \) for every \( p \in \mathcal{O} \). Thus, the cardinality of \( \xi(p)_\leq \) is no smaller than the cardinality of \( p_\leq \). As \( \xi \) is bijective and \( \mathcal{O} \) is a finite set, the cardinality of \( \xi(p)_\leq \) and the cardinality of \( p_\leq \) must coincide for every \( p \in \mathcal{O} \). Hence, \( \xi(p_\leq) = \xi(p)_\leq \) for every \( p \in \mathcal{O} \), or equivalently, \( p_\leq \Leftrightarrow \xi(p)_\leq \). \( p, q \in \mathcal{O} \).

**Lemma 3.2** Let \( \phi : H_n \to H_n \) be a map preserving commutativity in both directions. Then for every pair \( A, B \in H_n \) we have

\[
\phi(A)' \subset \phi(B)' \iff A' \subset B'
\]

and

\[
\mathrm{card} \sigma(\phi(A)) = \mathrm{card} \sigma(A).
\]

Proof. Let \( A, B \in H_n \) be any pair of matrices with \( A' \subset B' \). Then \( A \) and \( B \) commute, and therefore, they are simultaneously unitarily similar to diagonal matrices. Thus, after replacing \( \phi \) with \( A \mapsto \phi(UAU^*) \) for an appropriate unitary
matrix $U$, we may assume with no loss of generality that $A$ and $B$ are diagonal matrices with real entries.

Denote by $\mathcal{D}$ the set of all hermitian diagonal matrices. Clearly, $\phi(\mathcal{D})$ is a set of pairwise commuting matrices, and after composing $\phi$ with a unitarily similarity transformation, if necessary, we may, and we will assume that $\phi(\mathcal{D}) \subset \mathcal{D}$.

Let us define an equivalence relation on $\mathcal{D}$ by $D \sim C$ if and only if $D' = C'$. We know that there are finitely many equivalence classes $[C], C \in \mathcal{D}$ (the exact number is not important for us). From the second statement of Lemma 3.1 we know that $\phi(C) \sim \phi(D)$ yields $C \sim D$, or equivalently, $C \not\sim D \Rightarrow \phi(C) \not\sim \phi(D)$. As there are only finitely many equivalence classes, we have necessarily $\phi(C) \sim \phi(D)$ whenever $C \sim D$. Thus, $\phi$ induces a bijection $\varphi$ on the set of equivalence classes.

Moreover, the set of all equivalence classes is a poset with the partial order defined by $[C] \leq [D] \iff C' \subset D'$. The inverse of $\varphi$ is monotone. And because our poset is a finite set, $\varphi$ is an automorphism. Consequently, the relation $A' \subset B'$ yields $\phi(A') \subset \phi(B')$. This completes the proof of the first statement of our corollary.

To prove the second part we note that there is a maximal chain of pairwise distinct equivalence classes

$$[A_n] \leq [A_{n-1}] \leq \ldots \leq [A_1]$$

such that $[A]$ is a member of this chain. Here, clearly, $A_k$ has $k$ distinct eigenvalues, $k = 1, \ldots, n$. We have

$$[\phi(A_n)] \leq [\phi(A_{n-1})] \leq \ldots \leq [\phi(A_1)]$$

and because $\text{card} \sigma(\phi(A_k)) < \text{card} \sigma(\phi(A_{k+1}))$, $k = 1, \ldots, n - 1$, we have necessarily $\text{card} \sigma(\phi(A_k)) = k$, $k = 1, \ldots, n$. In particular, $\text{card} \sigma(\phi(A)) = \text{card} \sigma(A)$.

We denote by $H^k_n$ the set of all $n \times n$ hermitian matrices having exactly $n - k + 1$ distinct eigenvalues one of them being of multiplicity $k$, $k = 1, \ldots, n$. The set $H^1_n$ has been already defined as the set of all hermitian matrices with $n$ distinct eigenvalues, $H^2_n$ is the set of all hermitian matrices with exactly $n - 1$ distinct eigenvalues, $H^3_n$ is the set of all hermitian matrices with one eigenvalue of multiplicity 3 and all others of multiplicity one,..., and $H^n_n = \mathbb{R}I$. Let $A \in H^k_n$, $k \geq 3$. Then $A$ is unitarily similar to a diagonal matrix $\text{diag}(t_1, \ldots, t_1, t_2, \ldots, t_{n-k+1})$, where $t_j \neq t_m$ whenever $j \neq m$. With no loss of generality we may assume that $A$ has this diagonal form. Then we can find matrices $B_1, B_2, B_3 \in H^{k-1}_n$ such that $B_j' \subset A'$, $j = 1, 2, 3$, $B_jB'_m = B'_mB_j$, $1 \leq m, j \leq 3$, and $B_j' \neq B'_m$, $1 \leq j, m \leq 3$, $m \neq j$. Indeed, take

$$B_1 = \text{diag}(s, t_1, \ldots, t_1, t_2, \ldots, t_{n-k+1}),$$
\[
B_2 = \text{diag} \{t_1, s, t_1, \ldots, t_1, t_2, \ldots, t_{n-k+1}\},
\]
and
\[
B_3 = \text{diag} \{t_1, t_1, s, \ldots, t_1, t_2, \ldots, t_{n-k+1}\}.
\]

Here, \(s \not\in \{t_1, \ldots, t_{n-k+1}\}\).

We will prove that this simple observation leads to a characterization of the elements of \(H_n^k\).

**Lemma 3.3** Let \(k\) be an integer, \(3 \leq k \leq n\), and let \(A \in H_n\) be a matrix with exactly \(n-k+1\) distinct eigenvalues. Then the following two statements are equivalent:

- \(A \in H_n^k\).
- There exist matrices \(B_1, B_2, B_3 \in H_{n-1}^k\) such that
  1. \(B'_j \subset A', j = 1, 2, 3\),
  2. \(B_jB_m = B_mB_j, 1 \leq m, j \leq 3\), and
  3. \(B'_j \neq B'_m, 1 \leq j, m \leq 3, m \neq j\).

**Proof.** We already know that the first statement implies the second one. So, assume that \(A\) with exactly \(n-k+1\) eigenvalues satisfies the second condition. It follows from \(B'_j \subset A'\) that either \(A \in H_n^k\), or \(A\) has an eigenvalue of multiplicity \(k-1\), another one of multiplicity 2, while all other eigenvalues are of multiplicity 1. In the first case we are done. So, assume we have the second case. Then with no loss of generality we have

\[
A = \text{diag} \{t_1, \ldots, t_1, t_2, t_2, t_3, \ldots, t_{n-k+1}\}
\]

with \(t_j \neq t_m\) whenever \(j \neq m\). Here, of course, \(t_1\) appears \(k-1\) times. Assume first that \(k > 3\). Then every \(B \in H_{n-1}^k\) with the property that \(B' \subset A'\) must be of the form

\[
B = \text{diag} (s_1, \ldots, s_1) \oplus C \oplus \text{diag} (s_4, \ldots, s_{n-k+2}).
\]

Here, \(s_j \neq s_m\) whenever \(j \neq m\), and \(C\) is a \(2 \times 2\) hermitian matrix with exactly two eigenvalues none of them belonging to \(\{s_1, s_4, \ldots, s_{n-k+2}\}\). Thus, after applying an appropriate unitarily similarity transformation we may assume that \(A\) has the above form and

\[
B_1 = \text{diag} (s_1, \ldots, s_1, s_2, s_3, s_4, \ldots, s_{n-k+2})
\]

with \(s_j \neq s_m\) whenever \(j \neq m\). From \(B_1B_2 = B_2B_1\) and \(B'_2 \subset A'\) we conclude that

\[
B_2 = \text{diag} (r_1, \ldots, r_1, r_2, r_3, r_4, \ldots, r_{n-k+2})
\]

with \(r_j \neq r_m\) whenever \(j \neq m\). But then \(B'_1 = B'_2\), a contradiction.
In the case when \( k = 3 \) we apply similar arguments to conclude that with no loss of generality

\[
A = \text{diag}(t_1, t_1, t_2, t_3, \ldots, t_{n-2}),
\]

\[
B_1 = \text{diag}(s_1, s_1, s_2, s_3, s_4, \ldots, s_{n-1}),
\]

and

\[
B_2 = \text{diag}(r_1, r_2, r_3, r_4, \ldots, r_{n-1})
\]

with \( t_j \neq t_m, s_j \neq s_m, \) and \( r_j \neq r_m \) whenever \( j \neq m \). It is now clear that for every \( B_3 \in H_n^2 \) commuting with both \( B_1 \) and \( B_2 \) and satisfying \( B_3 \subset A' \) we have either \( B_3' = B_1' \) or \( B_3' = B_2' \), a contradiction.

\[\square\]

**Lemma 3.4** Let \( n \geq 3 \) and let \( \phi : H_n \rightarrow H_n \) be a map preserving commutativity. Assume that for every projection \( P \in H_n \) of rank one there exists a rank one projection \( Q \in H_n \) such that \( \phi(RP + RI) \subset HQ + RI \) and \( \phi(RP + RI) \not\subset RI \). Suppose further that for every pair of rank one projections \( P_1 \neq P_2 \) with \( \phi(RP_j + RI) \subset HQ_j + RI, \) \( j = 1, 2 \), we have \( Q_1 \neq Q_2 \). Then there exists a unitary \( n \times n \) matrix \( U \) and for every \( A \in H_n \) there is a real polynomial \( p_A \) such that either

\[
\phi(A) = Up_A(A)U^*
\]

for every \( A \in H_n \), or

\[
\phi(A) = Up_A(A^t)U^*
\]

for every \( A \in H_n \).

**Proof.** According to our assumptions to each rank one projection \( P \) there exists a unique rank one projection \( Q \) such that \( \phi(RP + RI) \subset HQ + RI \). Assume that \( P_1, P_2 \in H_n \) are orthogonal rank one projections, that is, \( P_1P_2 = 0 \) (then, of course, we have also \( P_2P_1 = 0 \)). Let \( Q_1 \) and \( Q_2 \) be the corresponding rank one projections. Then, because any two members of \( RP_1 + RI \) and \( RP_2 + RI \) commute we have \( Q_1Q_2 = Q_2Q_1 \) and \( Q_1 \neq Q_2 \). It follows easily that \( Q_1 \) and \( Q_2 \) are orthogonal.

We denote by \( \mathbb{P}C^n \) the projective space over \( C^n \), that is, the set of all one-dimensional subspaces of \( C^n, \mathbb{P}C^n = \{ [x] : x \in C^n \setminus \{0\} \} \). Here, \( [x] \) denotes the one-dimensional subspace spanned by \( x \). Every projection of rank one can be identified with an element of \( \mathbb{P}C^n \) in a natural way. Thus, the map \( \phi \) induces an orthogonality preserving injective map \( \varphi : \mathbb{P}C^n \rightarrow \mathbb{P}C^n \). We have \( \varphi([x]) = [y] \) if and only if \( [x] \) is the image of a rank one projection \( P, [y] \) is the image of a rank one projection \( Q, \) and \( \phi(RP + RI) \subset HQ + RI \). Let us show that \( [x] \subset [y] + [z] \) yields that \( \varphi([x]) \subset \varphi([y]) + \varphi([z]) \). There is nothing to prove if \( [y] = [z] \). So, assume that \( y \) and \( z \) are linearly independent. Denote by \( P_x, P_y, \) and \( P_z \) the rank one projections onto \( [x], [y], \) and \( [z] \), respectively. We can find
pairwise orthogonal rank one projections $P_3, \ldots, P_n$ such that $P_i P_m = P_m P_i = 0$, $m = 3, \ldots, n$. Then, clearly $P_m P_m = 0$, $m = 3, \ldots, n$. Let $Q_3, \ldots, Q_n$ be rank one projections defined as above. Further, let $Q_x, Q_y$, and $Q_z$ be rank one projections corresponding to $P_x, P_y$, and $P_z$, respectively. It follows that $Q_y Q_m = Q_z Q_m = Q_x Q_m = 0$, $m = 3, \ldots, n$, which further yields that the one-dimensional subspaces $\varphi([x]), \varphi([y]),$ and $\varphi([z])$ are contained in the two-dimensional image of the projection $I - Q_3 - \ldots - Q_n$. Since $\varphi([y]) \neq \varphi([z])$, we have $\varphi([x]) \subset \varphi([y]) + \varphi([z])$, as desired. Thus, we can apply [5, Theorem 4.1] to conclude that there exists a unitary or antiunitary (conjugate-linear isometry) operator $U : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi([x]) = [Ux]$ for every nonzero $x \in \mathbb{C}^n$.

Identifying hermitian matrices with self-adjoint operators on $\mathbb{C}^n$ we see that for every rank one projection $P$ there exist $s_1, t_1, s_2, t_2 \in \mathbb{R}$ with $s_1, s_2 \neq 0$ such that $\phi(s_1 P + t_1 I) = s_2 U P U^* + t_2 I$. If $U$ is antiunitary and $J : \mathbb{C}^n \to \mathbb{C}^n$ is the conjugate-linear involution defined as the entrywise complex conjugation, then $U J$ is unitary, and the matrix that corresponds to the projection $J P J$ is the matrix obtained from $P$ by applying the complex-conjugation entrywise.

Hence, after composing $\phi$ with an appropriate unitarily similarity transformation, and the transposition, if necessary, we may and we will assume that for every rank one projection $P$ there exist $s_1, t_1, s_2, t_2 \in \mathbb{R}$ with $s_1, s_2 \neq 0$ such that

$$\phi(s_1 P + t_1 I) = s_2 P + t_2 I. \quad (3.1)$$

Let now $A$ be any hermitian $n \times n$ matrix. There exists a unitary matrix $V$ such that $V AV^*$ has the block diagonal form

$$V AV^* = \text{diag} (t_1 I_1, \ldots, t_p I_p), \quad (3.2)$$

where the $t_m$’s are pairwise distinct real scalars and $I_1, \ldots, I_p$ are identity matrices of appropriate sizes. It is clear that $A$ commutes with every rank one projection $V^* \text{diag} (0, \ldots, 0, R, 0, \ldots, 0) V$, where the diagonal blocks are of the same sizes as in (3.2). By (3.1) we see that also $\phi(A)$ commutes with every such projection of rank one which yields that

$$V \phi(A) V^* = \text{diag} (s_1 I_1, \ldots, s_p I_p)$$

for some real (not necessarily distinct) numbers $s_1, \ldots, s_p$. Hence, $\phi(A) = p_A(A)$ for some real polynomial $p_A$, as desired.

\[
\square
\]

4 Proofs of the main results

Proof of Theorem 1.2. Assume first that $\phi$ is a map of one of the standard forms described in the statement of Theorem 1.2. We must then show that $\phi$
preserves commutativity in both directions. Every unitarily similarity transformation preserves commutativity in both directions and the same is true for the transposition map $A \mapsto A^t$, $A \in H_n$. In order to complete the proof of the easier implication we have to show that every locally polynomial map $A \mapsto p_A(A)$, $A \in H_n$, where for every $A$ the the restriction of $p_A$ on $\sigma(A)$ is injective, preserves commutativity in both directions. Let $A$ and $B$ be any hermitian matrices with the spectral decompositions $A = \sum_{t \in \sigma(A)} tE(t)$ and $B = \sum_{s \in \sigma(B)} sF(s)$. Then $A$ and $B$ commute if and only if every pair of spectral projections $E(t)$ and $F(s)$ commute. Since $p_A(A) = \sum_{t \in \sigma(A)} p_A(t)E(t)$ and $p_B(B) = \sum_{s \in \sigma(B)} p_B(s)F(s)$ and because $p_A$ and $p_B$ are injective on the spectra of $A$ and $B$, respectively, the spectral projections of $p_A(A)$ and $p_B(B)$ are the same as the spectral projections of $A$ and $B$, respectively. Hence, $A$ and $B$ commute if and only if $p_A(A)$ and $p_B(B)$ commute, as desired.

To prove the nontrivial direction we assume that $\phi$ preserves commutativity in both directions. Applying Lemma 3.2 we see that for every $A \in H_n$ we have

$$A \in H_n^1 \iff \phi(A) \in H_n^1$$

and

$$A \in H_n^2 \iff \phi(A) \in H_n^2.$$ 

It follows then from Lemma 3.3 that $A \in H_n^3 \Rightarrow \phi(A) \in H_n^3$. Applying Lemma 3.3 once more we get the same conclusion with $H_n^4$ instead of $H_n^3$. After finitely many steps we arrive at

$$A \in H_n^{n-1} \Rightarrow \phi(A) \in H_n^{n-1}$$

for every $A \in H_n$. In other words, every matrix of the form $tP + sI$, $t, s \in \mathbb{R}$, $t \neq 0$, where $P$ is a projection of rank one, is mapped into a matrix of the same type. We know that $A' = B' \iff \phi(A)' = \phi(B)'$, $A, B \in H_n$, and consequently, all the assumptions of Lemma 3.4 are satisfied. We apply this lemma and after composing $\phi$ with a unitarily similarity transformation and the transposition map, if necessary, we may assume that $\phi$ is a locally polynomial map

$$\phi(A) = p_A(A), \quad A \in H_n.$$

It remains to show that the assumption of preserving commutativity in both directions yields that $p_A$ is injective on $\sigma(A)$ for every $A \in H_n$. Assume without loss of generality that $A$ is a diagonal matrix. If $p_A$ is not injective on $\sigma(A)$, then we can find two diagonal entries of $A = \text{diag}(t_1, \ldots, t_n)$, say $t_1$ and $t_2$, such that $t_1 \neq t_2$ and $p_A(t_1) = p_A(t_2)$. Take $P = \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22})$. We have $\phi(P) = tP + sI$ for some $t, s \in \mathbb{R}$ with $t \neq 0$. But then $\phi(P)$ commutes with $\phi(A)$, while $AP \neq PA$, a contradiction.

\[ \square \]
Proof of Theorem 1.3. Let $t$ be any real number. Clearly, $A(tI) = (tI)A$ for every $A \in H_n$, and consequently, $\phi$ is an injective continuous map from $H_n$ into $\phi(tI)'$. The commutant of $\phi(tI)$ is a real-linear subspace of $H_n$. By the invariance of domain theorem we have $\dim \phi(tI)' = \dim H_n = n^2$, or equivalently, $\phi(tI)' = H_n$ (here, as well as in the sequel, $\dim$ stands for the dimension of real spaces). It follows that $\phi(tI)$ is a scalar matrix. Thus, $\phi(\mathbb{R}I) \subset \mathbb{R}I$.

Let us next observe that if $\dim A' \geq n^2 - 2n + 2$, then $A = tP + sl$ for some real numbers $t$, $s$ and some rank one projection $P$ (equivalently, $A \in H_{n-1}^n \cup H_n^n$). Indeed, it is clear that if $A$ has exactly two eigenvalues, one of multiplicity $k$, and the other one of multiplicity $n-k$, then $\dim A' = k^2 + (n-k)^2$. Obviously, $k^2 + (n-k)^2 < n^2 - 2n + 2$ unless $k = 1$ or $k = n-1$. Observe further that for every $B \in H_n$ with at least three eigenvalues there exists $A \in H_n$ with exactly two distinct eigenvalues such that $B' \subset A'$ and $B' \neq A'$. It follows that $\dim A' \geq n^2 - 2n + 2$ implies that $A$ is a linear combination of a rank one projection and the identity.

As a consequence, every $A \in H_n$, which is a linear combination of a rank one projection and the identity, is mapped into a matrix of the same type. Otherwise, $\phi$ would map $A'$, which is of dimension at least $n^2 - 2n + 2$, continuously and injectively into $\phi(A)'$, whose dimension is strictly smaller than $n^2 - 2n + 2$. This is impossible by the invariance of domain theorem.

Let $P$ be a projection of rank one. We will show that there exists a nonzero real $t$, such that $\phi(tP) \notin \mathbb{R}I$. Assume on the contrary that $\phi(tP) = g(t)I$, $t \in \mathbb{R}$, for some continuous function $g : \mathbb{R} \to \mathbb{R}$. We know that $\phi(I) = k(t)I$, $t \in \mathbb{R}$, for some injective continuous function $k : \mathbb{R} \to \mathbb{R}$. The range of $k$ is an open (bounded or unbounded) interval. By continuity, $\lim_{t \to 0} g(t) = g(0) = k(0)$. Thus, there exist nonzero real $u$ and $v$ such that $g(u) = k(v)$, contradicting the injectivity of $\phi$.

So, for every projection $P$ of rank one there exist a nonzero real number $t$, a rank one projection $Q$, and $t_1, s_1 \in \mathbb{R}$ such that $t_1 \neq 0$ and $\phi(tP) = t_1Q + s_1I$. We will show that $\phi(\mathbb{R}P + \mathbb{R}I) \subset \mathbb{R}Q + \mathbb{R}I$. Indeed, if there are real numbers $t_2, s_2$ with $t_2 \neq 0$, such that $\phi(t_2P + s_2I) \notin \mathbb{R}Q + \mathbb{R}I$, then $\phi$ would map $P' = (t_2P + s_2I)'$ injectively and continuously into $Q' \cap \phi(t_2P + s_2I)'$, which would be a proper subspace of $Q'$, contradicting the invariance of domain theorem.

It remains to show that if for two different rank one projections $P_1$ and $P_2$ we have $\phi(t_1P_1) = r_1Q_1 + s_1I$ and $\phi(t_2P_2) = r_2Q_2 + s_2I$ with $t_j, r_j \neq 0$, $j = 1, 2$, then $Q_1 \neq Q_2$. Indeed, once we show this the proof can be completed using Lemma 3.4. Assume on the contrary that $Q_1 = Q_2 = Q$. Then, by the invariance of domain theorem, $\phi(P_1')$ is an open subset of $Q'$ containing $\phi(0) = rI$. Take a matrix $R \in P_2'$ such that $R \notin P_1'$. As $\phi(tR)$ tends to $rI$ as $t$ tends to zero we can find a small enough nonzero real number $t$ such that $\phi(tR) \in \phi(P_1')$, contradicting the injectivity of $\phi$.

$\square$
References


