Minimal locally linearly dependent spaces of operators

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Abstract
Let $S$ be an $n$-dimensional minimal locally linearly dependent space of operators acting between vector spaces $U$ and $V$. We obtain the sharp lower bound and the sharp upper bound for $\dim SU$ and give a complete description of those minimal locally linearly dependent spaces at which the upper bound is attained.

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1 Introduction, examples, and statement of the main result

Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$. We denote by $\mathcal{L}(U, V)$ the linear space of all linear transformations from $U$ into $V$. Linear operators $T_1, \ldots, T_n : U \to V$ are locally linearly dependent if $T_1 u, \ldots, T_n u$ are linearly dependent for every $u \in U$. When studying $n$-tuples of locally linearly dependent operators

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we can always assume that we have a nontrivial case, that is, $T_1, \ldots, T_n$ are linearly independent. It is then more natural to study the linear span of these operators $S = \text{span} \{T_1, \ldots, T_n\} \subset L(U, V)$ instead of the $n$-tuple $T_1, \ldots, T_n$.

The assumption of local linear dependence of $T_1, \ldots, T_n$ is then equivalent to the condition that for every $u \in U$ there exists a nonzero $S \in S$ such that $Su = 0$. Hence, we will say that an $n$-dimensional linear subspace of operators $S \subset L(U, V)$ is locally linearly dependent if $\dim Su = \dim \{Su : S \in S\} \leq n - 1$ for every $u \in U$.

When studying the structure of locally linearly dependent spaces of operators one may first observe that if $S_1 \subset S_2 \subset L(U, V)$ are linear spaces of operators and if $S_1$ is locally linearly dependent, then so is $S_2$. This trivial observation yields that it is enough to study only minimal locally linearly dependent spaces of operators $S \subset L(U, V)$, that is, if $T$ is a locally linearly dependent space of operators and $\{0\} \neq T \subset S$, then $T = S$.

The structure of $n$-tuples of locally linearly dependent operators and later locally linearly dependent spaces of operators has been studied not only because this is an interesting question by itself, but also because of applications in ring theory, problems concerning derivations and reflexivity of operator spaces (see, for example [1]-[4] and [6]-[8]). The basic theorem [3, 7] states that if $S \subset L(U, V)$ is an $n$-dimensional locally linearly dependent space, then there exists a nonzero $S \in S$ such that $\text{rank} S \leq n - 1$. If we assume that $\mathbb{F}$ has at least $n + 2$ elements then we can say even more: either there exists a nonzero $S \in S$ such that $\text{rank} S \leq n - 2$, or $\text{rank} S = n - 1$ for every nonzero $S \in S$ [7]. If we restrict our attention to minimal locally linearly dependent spaces $S$, then it was proved in [8] that the rank is bounded above on $S$ and the following (non-sharp) estimate was obtained:

$$\text{rank } S \leq \frac{(n - 1)(n - 2)}{2} + 1$$

for every $S \in S$. As $S$ is a finite-dimensional space of operators this yields that $\dim Su = \dim \text{span} \{Su : S \in S, u \in U\} < \infty$. It should be noted here that for an arbitrary linear subspace $T \subset L(U, V)$ the set $\{Tu : T \in T, u \in U\} \subset V$ need not be a linear subspace. The symbol $TU$ will always denote the linear span of this set.

So far, the upper bounds for the minimal and the maximal rank of operators belonging to an $n$-dimensional minimal locally linearly dependent space of operators were studied. A sharp estimate has been given only for the minimal rank. In this paper, instead of being interested in ranks of elements of $S$ we will be interested in the “rank of the whole space $S$”, that is, we will be interested in $\dim Su$. We will give the sharp upper bound and the sharp lower bound for this quantity. We will also describe the structure of $S$ in the extremal cases.

Let $n \geq 2$ be an integer and $U, V$ vector spaces over an arbitrary field $\mathbb{F}$. Assume that $\epsilon_1, \ldots, \epsilon_n \in U$ and $f_1, \ldots, f_{\frac{n(n-1)}{2}} \in V$ are linearly independent sets.
Moreover, let $W \subset U$ be a linear subspace such that $U = \text{span}\{e_1, \ldots, e_n\} \oplus W$ and let $T_1, \ldots, T_n : U \to V$ be linear operators defined by

\[
T_j \left( \sum_{k=1}^{n} \lambda_k e_k \right) = 0, \quad j = 1, \ldots, n, \quad w \in W;
\]

\[
T_1 \left( \sum_{k=1}^{n} \lambda_k e_k \right) = \sum_{k=2}^{n} \lambda_k f_{k-1},
\]

\[
T_2 \left( \sum_{k=1}^{n} \lambda_k e_k \right) = -\lambda_1 f_1 + \sum_{k=3}^{n} \lambda_k f_{[(n-1)+k-2]},
\]

\[
T_3 \left( \sum_{k=1}^{n} \lambda_k e_k \right) = -\lambda_1 f_2 - \lambda_2 f_{[(n-1)+1]} + \sum_{k=4}^{n} \lambda_k f_{[(n-1)+(n-2)+k-3]},
\]

\[
T_4 \left( \sum_{k=1}^{n} \lambda_k e_k \right) = -\lambda_1 f_3 - \lambda_2 f_{[(n-1)+2]} - \lambda_3 f_{[(n-1)+(n-2)+1]}
\]

\[
+ \sum_{k=5}^{n} \lambda_k f_{[(n-1)+(n-2)+(n-3)+k-4]},
\]

\[
\vdots
\]

\[
T_n \left( \sum_{k=1}^{n} \lambda_k e_k \right) = -\lambda_1 f_{n-1} - \lambda_2 f_{[(n-1)+(n-2)]} - \lambda_3 f_{[(n-1)+(n-2)+(n-3)]}
\]

\[
- \ldots - \lambda_{n-1} f_{[(n-1)+(n-2)+\ldots+1]},
\]

where $\lambda_1, \ldots, \lambda_n$ are any scalars. Note that according to this definition $T_1(e_1) = T_2(e_2) = \ldots = T_n(e_n) = 0$.

Let $u \in U$ be an arbitrary vector, $u = \lambda_1 e_1 + \ldots + \lambda_n e_n + w$, $\lambda_j \in \mathbb{F}$, $w \in W$. Then

\[
\lambda_1 T_1 u + \lambda_2 T_2 u + \ldots + \lambda_n T_n u = 0. \tag{1}
\]

Hence, the space $S = \text{span}\{T_1, \ldots, T_n\} \subset \mathcal{L}(U, V)$ is locally linearly dependent. It is easy to check that each nonzero $S \in \mathcal{S}$ has rank $\geq n - 1$. Indeed, all we have to do is to show that the null space of the restriction of $\mu_1 T_1 + \ldots + \mu_n T_n$ to span $\{e_1, \ldots, e_n\}$ is at most one-dimensional; here not all $\mu$'s are zero. It follows that $S$ is a minimal locally linearly dependent space of operators (otherwise $S$ would contain a locally linearly dependent subspace $T$ of dimension $k < n$, and by the basic theorem, $T$ would contain a nonzero operator of rank $\leq k - 1 < n - 1$, a contradiction). Clearly,

\[
\dim SU = \frac{n(n-1)}{2}.
\]

**Definition 1.1** Let $S \subset \mathcal{L}(U, V)$ be an $n$-dimensional linear space of operators, $n \geq 2$. If there exist vectors $e_1, \ldots, e_n \in U$, $f_1, \ldots, f_{\frac{n(n-1)}{2}} \in V$, a
two parts): we need to know is how the operators dependent space of operators. S = span \{T_1, \ldots, T_n\}, then S is called a standard n-dimensional locally linearly dependent space of operators.

Another way of representing such a space of operators is the following. All we need to know is how the operators $T_1, \ldots, T_n$ behave on the linear span of $e_1, \ldots, e_n$. For each $T_j$, $j = 1, \ldots, n$, we will write the coordinates of $T_j u = T_j(\lambda_1, \ldots, \lambda_n) = T_j(\sum_{k=1}^n \lambda_k e_k)$ with respect to the basis $f_1, \ldots, f_{n(n-1)}$ as the $j$-th row of an $n \times \frac{n(n-1)}{2}$ matrix (as this is a large matrix we will divide it into two parts):

\[
\begin{bmatrix}
\lambda_2 & \lambda_3 & \lambda_4 & \ldots & \lambda_n & 0 & 0 & 0 & \ldots & 0 \\
-\lambda_1 & 0 & 0 & \ldots & 0 & \lambda_3 & \lambda_4 & \lambda_5 & \ldots & \lambda_n \\
0 & -\lambda_1 & 0 & \ldots & 0 & -\lambda_2 & 0 & 0 & \ldots & 0 \\
0 & 0 & -\lambda_1 & \ldots & 0 & 0 & -\lambda_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & -\lambda_3 & 0 & 0 & 0 & \ldots & -\lambda_2 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\lambda_4 & \lambda_5 & \ldots & \lambda_n & 0 & 0 & \ldots & 0 & \ldots & 0 \\
-\lambda_3 & 0 & \ldots & 0 & \lambda_5 & \lambda_6 & \ldots & \lambda_n & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & -\lambda_3 & 0 & 0 & \ldots & -\lambda_4 & 0 & \ldots & -\lambda_{n-1}
\end{bmatrix}
\]

Obviously, the rows are designed in such a way that (1) holds.

Yet another way of representing such a space is to introduce different indices of the $f$'s, namely to index these vectors as $f_{ij}$, $1 \leq i < j \leq n$, and then define

$$T_j e_k = f_{jk}, \quad 1 \leq i, j \leq n,$$

where $f_{11} = \ldots = f_{nn} = 0$ and $f_{ij} = -f_{ji}$, $1 \leq j < i \leq n$.

It is clear that if $S \subseteq L(U, V)$ is an $n$-dimensional minimal locally linearly dependent linear space of operators, then $\dim S U \geq n - 1$. For if $\dim S U \leq n - 2$, then every $(n - 1)$-dimensional subspace of $S$ is locally linearly dependent as well. There are many ways to construct minimal $n$-dimensional locally linearly dependent spaces of operators satisfying $\dim S U = n - 1$. Let us give a few examples in the case $n = 3$ (these examples can be easily extended to higher dimensional cases). The easiest way is to take a 6-dimensional space $U = \text{span} \{e_1, \ldots, e_6\}$, a two-dimensional space $V = \text{span} \{f_1, f_2\}$, linear operators

$$T_j : U \rightarrow V, \quad j = 1, 2, 3,$$

defined by

$$T_1 e_1 = f_1, \quad T_1 e_2 = f_2, \quad T_1 e_3 = T_1 e_4 = T_1 e_5 = T_1 e_6 = 0,$$
$$T_2 e_3 = f_1, \quad T_2 e_4 = f_2, \quad T_2 e_5 = T_2 e_6 = 0,$$
$$T_3 e_5 = f_1, \quad T_3 e_6 = f_2, \quad T_3 e_1 = T_3 e_2 = T_3 e_3 = T_3 e_4 = 0.$$
and then define $S$ to be the linear span of $T_1, T_2, T_3$. To get another example we can modify the above example by replacing $U$ by a 4-dimensional space span $\{e_1, \ldots, e_4\}$ and operators $T_1, T_2, T_3$ by

\[
T_1 e_1 = f_1, \quad T_1 e_2 = f_2, \quad T_1 e_3 = T_1 e_4 = 0,
\]

\[
T_2 e_2 = f_1, \quad T_2 e_3 = f_2, \quad T_2 e_1 = T_2 e_4 = 0,
\]

\[
T_3 e_3 = f_1, \quad T_3 e_4 = f_2, \quad T_3 e_1 = T_3 e_2 = 0.
\]

In both cases we get a 3-dimensional minimal locally linearly dependent space of operators whose all nonzero members are of rank 2. For our final example we take a 2-dimensional space $U = \text{span} \{e_1, e_2\}$ and operators $T_1, T_2, T_3 : U \to U$ defined by

\[
T_1 e_1 = e_1, \quad T_2 e_2 = e_2, \quad T_1 e_2 = T_2 e_1 = 0, \quad T_3 e_1 = e_1 + e_2, \quad T_3 e_2 = e_1 + e_2,
\]

to get a 3-dimensional minimal locally linearly dependent space of operators whose basis consists of rank one operators.

Our main result gives the sharp lower bound and the sharp upper bound for $\dim S_U$ and a complete description of those minimal locally linearly dependent spaces of operators at which the upper bound is attained.

**Theorem 1.2** Let $n \geq 2$ be an integer and let $\mathbb{F}$ be a field with at least $n + 2$ elements. Suppose that $U$ and $V$ are vector spaces over $\mathbb{F}$ and $S \subset \mathcal{L}(U, V)$ is an $n$-dimensional minimal locally linearly dependent space of operators. Then

\[
n - 1 \leq \dim S_U \leq \frac{n(n - 1)}{2}.
\]

Both estimates are sharp. If

\[
\dim S_U = \frac{n(n - 1)}{2}
\]

then $S$ is a standard $n$-dimensional locally linearly dependent space of operators.

**2 Proof**

Throughout this section we will assume that $U$ and $V$ are vector spaces over a field $\mathbb{F}$ with at least $n + 2$ elements, $n \geq 2$ is a fixed integer, and $S \subset \mathcal{L}(U, V)$ is an $n$-dimensional minimal locally linearly dependent space of operators. We will frequently use [3, Lemma 2.1] stating that if $W$ is a vector space over $\mathbb{F}$, $r$ a positive integer, $w_1, \ldots, w_r$ linearly independent vectors in $W$ and $z_1, \ldots, z_r$ arbitrary vectors in $W$, then there are at most $r$ nonzero scalars $\alpha \in \mathbb{F}$ such that $w_1 + \alpha z_1, \ldots, w_r + \alpha z_r$ are linearly dependent (in fact, the statement of Lemma 2.1 in [3] is slightly different, but exactly the same proof gives the above statement). We need some more notation. By $U^*$ we denote the dual of $U$, that is, the linear space of all linear functionals on $U$. If $v \in V$ is a nonzero vector and $\varphi \in U^*$ a nonzero linear functional, then $v \otimes \varphi$ stands for the rank one


operator from $U$ into $V$ defined by $(v \otimes \varphi)u = \varphi(u)v$, $u \in U$. Note that every rank one operator in $\mathcal{L}(U, V)$ can be written in this form.

We will need the next statement in the proof of our main theorem. We believe it is of independent interest as it gives some insight into the structure of standard locally linearly dependent spaces of operators.

Proposition 2.1 Let $S_1, \ldots, S_n$ be any basis of a standard $n$-dimensional locally linearly dependent space of operators $S$. Then there exist nonzero scalars $\lambda_1, \ldots, \lambda_n$, vectors $e_1, \ldots, e_n \in U$, $f_1, \ldots, f_{n(n-1)/2} \in V$, a subspace $W \subset U$, and linear operators $T_1, \ldots, T_n : U \to V$ as in Definition 1.1 such that $S_j = \lambda_j T_j$, $j = 1, \ldots, n$.

Proof. Since $S$ is a standard $n$-dimensional locally linearly dependent space of operators it has a basis as described in Definition 1.1. We have to show that any other basis is of the same type up to multiplicative factors. If $a_1, \ldots, a_n$ is a basis of an arbitrary vector space $Z$, then any of the $n$-tuples:

- $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$,
- $a_1 + \mu a_2, a_2, \ldots, a_n$,

is a basis of $Z$. Here $\sigma$ is a permutation on $n$ elements and $\mu$ is a nonzero scalar. Moreover, any basis of $Z$ can be obtained from the original basis $a_1, \ldots, a_n$ using a finite sequence of the above operations: permuting elements and adding a scalar multiple of the second element to the first one, and then as the final step multiplying the elements of the obtained basis by nonzero scalars.

Thus, all we have to do is to show that if we start with a basis of $S$ as described in Definition 1.1 and if we apply any of the two operations described above we arrive at the basis of the same type. If we interchange $T_i$ and $T_j$, $i \neq j$, then we get the basis of the same type as in Definition 1.1 (of course, after interchanging $e_i$ and $e_j$, and after permuting basis vectors $f_1, \ldots, f_{n(n-1)/2}$ and multiplying them by $\pm 1$ accordingly). And if $T_1, \ldots, T_n$ is a basis as in Definition 1.1 and $\mu$ is a nonzero scalar, then $T_1 + \mu T_2, T_2, \ldots, T_n$ is a basis of the same type corresponding to the vectors $e_1 + \mu e_2, e_2, \ldots, e_n$ and $f_1, f_2 + \mu f_n, f_3 + \mu f_{n+1}, \ldots, f_{n-1} + \mu f_{2n-3}, f_n, f_{n+1}, \ldots, f_{n(n-1)/2}$.

We continue by two simple lemmas.

Lemma 2.2 Let $R_1, R_2 \in \mathcal{L}(U, V)$ satisfy $\text{rank} (\lambda R_1 + \mu R_2) = 2$ for all $\lambda, \mu \in \mathbb{F}$ not both of them zero and $\dim (\text{Im} R_1 \cap \text{Im} R_2) = 1$. Then either there exist linearly independent vectors $e_1, e_2 \in U$, linearly independent vectors $f_1, f_2, f_3 \in V$, and a subspace $Z \subset U$ such that

$$U = \text{span} \{e_1, e_2\} \oplus Z,$$
or there exist linearly independent vectors $e_1, e_2, e_3 \in U$, linearly independent vectors $f_1, f_2, f_3 \in V$, and a subspace $Z \subset U$ such that

$$U = \text{span} \{ e_1, e_2, e_3 \} \oplus Z,$$

$$R_1 e_1 = R_2 e_1 = f_1, \quad R_1 e_2 = f_2, \quad R_2 e_3 = f_3,$$

Proof. Set $Z = \text{Ker} R_1 \cap \text{Ker} R_2$. Choose subspaces $W_1, W_2 \subset U$ such that $\text{Ker} R_1 = Z \oplus W_1$ and $\text{Ker} R_2 = Z \oplus W_2$. Then

$$U = Z \oplus W_1 \oplus W_2 \oplus W_3$$

for some subspace $W_3 \subset U$. As $\text{rank} R_1 = \text{rank} R_2 = 2$ we have three possibilities:

- $\dim W_1 = \dim W_2 = 2$ and $W_3 = \{0\}$,
- $W_1 = W_2 = \{0\}$ and $\dim W_3 = 2$,
- $\dim W_1 = \dim W_2 = \dim W_3 = 1$.

We will show that the first possibility cannot occur. Assume on the contrary that we have the first possibility. Then $\text{Im} R_1 = \text{Im} (R_1|_{W_1}) = \text{Im} (R_2|_{W_1}) = \text{Im} R_2$, since otherwise we would have $\text{rank} (R_1 + R_2) \geq 3$. This contradicts the fact that $\dim (\text{Im} R_1 \cap \text{Im} R_2) = 1$.

In the second case we choose a nonzero $f_1 \in \text{Im} R_1 \cap \text{Im} R_2 \subset V$. There are unique vectors $e_1, e_2 \in W_3$ such that $R_1 e_1 = f_1 = R_2 e_2$. If $e_1$ and $e_2$ are linearly dependent, then $e_1 = \lambda e_2$ for some nonzero $\lambda \in \mathbb{F}$, which yields that $(\lambda R_1 - R_2)e_2 = 0$ and since $(\lambda R_1 - R_2)z = 0$ for every $z \in Z$ we conclude that $\text{rank} (\lambda R_1 - R_2) \leq 1$, a contradiction. Hence, $e_1$ and $e_2$ are linearly independent. Set $f_2 = R_1 e_2$ and $f_3 = R_2 e_1$. As $\text{rank} R_1 = 2$, the vectors $f_1$ and $f_2$ are linearly independent. Similarly, $f_1$ and $f_3$ are linearly independent and because $\dim (\text{Im} R_1 \cap \text{Im} R_2) = 1$ we have $f_3 \notin \text{span} \{ f_1, f_2 \}$. So, we are done in this case.

It remains to consider the last case. Let $e_1 \in W_2$ and $e_2 \in W_1$ be nonzero vectors. If $R_1 e_1$ and $R_2 e_2$ are linearly dependent, then after replacing $e_2$ by $\mu e_2$ for an appropriate nonzero $\mu \in \mathbb{F}$ we may, and will assume that $R_1 e_1 = R_2 e_2 = f_1$. Let $e_3 \in W_3$ be a nonzero vector and denote $R_1 e_3 = f_2$ and $R_2 e_3 = f_3$. It is easy to check that $f_1, f_2, f_3$ are linearly independent. So, we are done also in this case and it only remains to show that the case when $R_1 e_1 = f$ and $R_2 e_2 = g$ are linearly independent cannot occur. Indeed, denote $R_1 e_3 = h$ and $R_2 e_3 = k$. We will show that $h \in \text{span} \{ f, g \}$. If this was not the case, then the image of
\[ \lambda R_1 + R_2 \] would contain vectors \( \lambda f = (\lambda R_1 + R_2)e_1 \), \( g = (\lambda R_1 + R_2)e_2 \), and \( \lambda h + k = (\lambda R_1 + R_2)e_3 \), which are linearly independent for at least one nonzero \( \lambda \), contradicting the fact that \( \text{rank}(\lambda R_1 + R_2) = 2 \). Similarly, \( k \in \text{span} \{f, g\} \). It follows that \( \text{Im} \ R_1 = \text{Im} \ R_2 \), a contradiction.

\[ \square \]

The next lemma is likely known, but we do not have a handy reference, so we provide a short proof.

**Lemma 2.3** Let \( \xi_1, \ldots, \xi_{n-1}, \sigma_1, \sigma_2 \in U^* \) be functionals such that \( \xi_1 \) and \( \xi_j \) are linearly independent, \( j = 2, \ldots, n-1 \). Assume that for every \( x \in U \) satisfying \( \xi_1(x) = 0 \) and \( \xi_2(x) \neq 0, \ldots, \xi_{n-1}(x) \neq 0 \) we have \( \sigma_1(x) = 0 \) or \( \sigma_2(x) = 0 \). Then either \( \sigma_1 = c\xi_1 \) for some scalar \( c \), or \( \sigma_2 = c\xi_1 \) for some scalar \( c \).

**Proof.** If Ker \( \xi_1 \subset \text{Ker} \sigma_1 \), then clearly, \( \sigma_1 = c\xi_1 \) for some scalar \( c \). So, all we have to do is to show that the restriction of \( \sigma_1 \) to Ker \( \xi_1 \) is the zero functional or the restriction of \( \sigma_2 \) to Ker \( \xi_1 \) is the zero functional. We further know that the restrictions of \( \xi_2, \ldots, \xi_{n-1} \) to Ker \( \xi_1 \) are all nonzero functionals and that \( \sigma_1(x)\sigma_2(x)\xi_2(x) \cdots \xi_{n-1}(x) = 0 \) for every \( x \in \text{Ker} \xi_1 \). Thus, we have to show that if we have \( n \) nonzero linear functionals on some subspace \( W \), then there exists \( w \in W \) such that all these functionals are nonzero at \( w \). In other words, we have to see that the union of \( n \) proper subspaces of \( W \) cannot be the whole space which is trivially true as the cardinality of the underlying field is at least \( n+2 \).

\[ \square \]

**Proof of Theorem 1.2.** In order to prove the first part of Theorem 1.2 we only need to show that

\[ \dim SU \leq \frac{n(n-1)}{2}, \]

because the lower bound and the sharpness of both the upper and the lower bound have already been proved in the first section. This part of the proof is a modification of the proof of [8, Theorem 2.2].

By the basic theorem on locally linearly dependent operators there exists a nonzero \( S_1 \in S \) such that \( \text{rank} \ S_1 = k_1 \leq n-1 \). Denote by \( V_1 \) the image of \( S_1 \), \( V_1 = \text{Im} \ S_1 \subset V \). Then \( \dim V_1 = k_1 \). Set \( S_1 = \{ S \in S : \text{Im} S \subset V_1 \} \subset S \). Then \( S_1 \) is a linear subspace of \( S \) of dimension \( p_1 \geq 1 \). If \( p_1 = n \), then \( \dim SU = k_1 \leq n-1 \) and we are done. We choose a direct summand \( T_1 \) of \( S_1 \) in \( S \) and an idempotent operator \( P_1 \in L(V) = L(V, V) \) whose kernel is \( V_1 \).

In the next step we will show that the linear space of operators \( P_1 T_1 = \{ P_1 S : S \in T_1 \} \subset L(U, V) \) is locally linearly dependent space of dimension \( n - p_1 \). Obviously, \( P_1 T_1 \) is a linear subspace with \( \dim P_1 T_1 = \dim T_1 = n - p_1 \).
Assume that this subspace is not locally linearly dependent. Then we can find $u \in U$ and $S_{p_1+1}, \ldots, S_n \in T_i$ such that

$$P_1S_{p_1+1}u, \ldots, P_1S_nu$$

are linearly independent. Using minimality of $S$ we can find $y \in U$ and $S_1, \ldots, S_{p_1} \in S_i$ such that

$$S_1y, \ldots, S_{p_1}y$$

are linearly independent. There are at most $(n - p_1)$ nonzero scalars $\alpha$ such that $P_1S_{p_1+1}(u + \alpha y), \ldots, P_1S_n(u + \alpha y)$ are linearly dependent and at most $p_1$ nonzero scalars $\alpha$ such that $S_1(\alpha^{-1}u + y), \ldots, S_n(\alpha^{-1}u + y)$ are linearly dependent. Hence, there is a nonzero $\beta \in F$ such that both sets of vectors $P_1S_{p_1+1}(u + \beta y), \ldots, P_1S_n(u + \beta y)$ and $S_1(u + \beta y), \ldots, S_{p_1}(u + \beta y)$ are linearly independent. The first set of vectors belong to the image of the idempotent operator $P_1$, while the second set belong to the null space of $P_1$. It is then not difficult to see that the set of vectors

$$S_{p_1+1}(u + \beta y), \ldots, S_n(u + \beta y), S_1(u + \beta y), \ldots, S_{p_1}(u + \beta y)$$

is linearly independent, contradicting the fact that $S$ is locally linearly dependent.

As $P_1T_i$ is locally linearly dependent space of dimension $n - p_1$, there exists a nonzero $S_2 \in T_i$ such that $0 \neq \text{rank } P_1S_2 = k_2 \leq n - p_1 - 1$. Set $V_2 = V_1 \oplus \text{Im } P_1S_2$. Then $\text{dim } V_2 = k_1 + k_2$. We have $\text{Im } S_1 = V_1 \subset V_2$ and $\text{Im } S_2 = \text{Im } ((I - P_1)S_2 + P_1S_2) \subset V_1 + \text{Im } P_1S_2 = V_2$. Set $S_2 = \{ S \in S : \text{Im } S \subset V_2 \} \subset S$. Then $S_2$ is a linear subspace of $S$ of dimension $p_2 \geq 2$. If $p_2 = n$, then $\text{dim } SU \leq k_1 + k_2 \leq n - 1 + (n - p_1 - 1) \leq (n - 1) + (n - 2)$ and we are done. We choose a direct summand $T_2$ of $S_2$ in $S$ and an idempotent operator $P_2 \in \mathcal{L}(V) = \mathcal{L}(V, V)$ whose kernel is $V_2$. In the next step we prove in exactly the same way as before that $P_2T_2$ is a locally linearly dependent space of operators of dimension $n - p_2$. We continue by repeating the same procedure. We stop after $m$ steps when $S_m = S$. The subspace $S_k$ has dimension at least $k$ and since $P_kT_k$ is a locally linearly dependent space of operators and therefore $\text{dim } P_kT_k = \text{dim } T_k \geq 2$, we necessarily have $m \leq n - 1$. It follows that

$$\text{dim } SU \leq k_1 + k_2 + \ldots + k_m \leq n - 1 + (n - p_1 - 1) + \ldots + (n - p_{m-1} - 1)$$

$$\leq (n - 1) + (n - 2) + \ldots + (n - m)$$

$$\leq (n - 1) + (n - 2) + \ldots + 1 = \frac{n(n - 1)}{2}.$$ 

Note that we can achieve the extremal value $(1/2)n(n - 1)$ only if $k_1 = n - 1, p_1 = 1, k_2 = n - 2, p_2 = 2, \ldots$

Now we will deal with the extremal case. So, assume that $S$ is an $n$-dimensional minimal locally linearly dependent space of operators with $\text{dim } SU = \frac{n(n - 1)}{2}$.}

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We claim that then each nonzero $S \in \mathcal{S}$ has rank $n - 1$. Indeed, let $S_1 \in \mathcal{S}$ be a nonzero operator with the minimal rank. Denote this rank by $k_1$ and proceed like in the first part of the proof. As we have at the end the extremal value for the dimension of $SU$, we have necessarily $k_1 = n - 1$. But then, by [7, Theorem 2.4], $\text{rank } S = n - 1$ for every nonzero $S \in \mathcal{S}$.

Our next goal is to show that if $S_1, S_2 \in \mathcal{S}$ are linearly independent operators, then $\dim(\text{Im } S_1 \cap \text{Im } S_2) = 1$. Once again start the same procedure as in the first part of the proof with a chosen $S_1$. We know that $p_1 = 1$ and we can then choose $T_1$ in such a way that $S_2 \in T_1$. We have $\text{Im } S_2 = (V_1 \cap \text{Im } S_2) \oplus M$ for some subspace $M \subset V$. We further choose $P_1$ in such a way that $P_1 u = u$, $u \in M$. Then $\text{Im } S_2 = (V_1 \cap \text{Im } S_2) \oplus P_1 \text{Im } S_2 = (\text{Im } S_1 \cap \text{Im } S_2) \oplus P_1 \text{Im } S_2$. We know that $P_1 T_1$ is a locally linearly dependent space of dimension $n - 1$ and that the minimal rank of nonzero operators in this subspace is $n - 2$. Again, by [7, Theorem 2.4], $\text{rank } P_1 S = n - 2$ for every nonzero $S \in T_1$. In particular, rank $P_1 S_2 = n - 2$. It follows then from $\text{Im } S_2 = (\text{Im } S_1 \cap \text{Im } S_2) \oplus P_1 \text{Im } S_2$ that $\dim(\text{Im } S_1 \cap \text{Im } S_2) = 1$, as desired.

We will prove the second part of our theorem by induction on $n$. In the case $n = 2$ we have $\dim SU = 1$. Thus, $\mathcal{S} = \text{span } \{v \otimes f, v \otimes g\}$ for some nonzero $v \in V$ and linearly independent functionals $f, g \in U^*$. As $f$ and $g$ are linearly independent there exist $e_1, e_2 \in U$ such that

$$f(e_2) = 1 = -g(e_1) \quad \text{and} \quad f(e_1) = g(e_2) = 0.$$ 

Set $W = \text{Ker } f \cap \text{Ker } g$ and observe that $U = \text{span } \{e_1, e_2\} \oplus W$. Finally, denote $v = f_1$. It is now clear that $\mathcal{S}$ is a standard 2-dimensional locally linearly dependent space of operators.

So, assume now that our theorem holds true for some $n - 1 \geq 2$ and we want to prove it for $n$. We will begin the induction step by proving that $\text{Im } R_1 \cap \text{Im } R_2 \cap \text{Im } R_3 = \{0\}$ for all linearly independent $R_1, R_2, R_3 \in \mathcal{S}$. Assume that this is not true. We distinguish two cases. We first treat the case when $n = 3$. By Lemma 2.2 we have two possibilities for $R_1, R_2$. Let us first consider the case when there exist linearly independent vectors $e_1, e_2 \in U$, linearly independent vectors $f_1, f_2, f_3 \in V$, and a subspace $Z \subset U$ such that $U = \text{span } \{e_1, e_2\} \oplus Z$, $R_1 z = R_2 z = 0$, $z \in Z$, and $R_1 e_1 = R_2 e_2 = f_1$, $R_1 e_2 = f_2$, $R_2 e_1 = f_3$. Then, clearly, $\text{Im } R_1 \cap \text{Im } R_2 \cap \text{Im } R_3 = \text{span } \{f_1\}$. Since $R_1 e_1, R_2 e_1, R_3 e_1$ are linearly dependent we have $R_3 e_1 = \lambda_1 f_1 + \lambda_3 f_3$ for some $\lambda_1, \lambda_3 \in \mathbb{F}$. Similarly, $R_3 e_2 = \mu_1 f_1 + \mu_2 f_2$ for some $\mu_1, \mu_2 \in \mathbb{F}$. As $f_1, \lambda_1 f_1 + \lambda_3 f_3, \mu_1 f_1 + \mu_2 f_2 \in \text{Im } R_3$ and rank $R_3 = 2$, we have $\lambda_3 = 0$ or $\mu_2 = 0$. Let us consider just the first possibility. If $\mu_2 \neq 0$ then $\{f_1, f_2\} \subset \text{Im } R_3 \cap \text{Im } R_2$, a contradiction. Using once more the fact that rank $R_3 = 2$ we see that there exists $z \in Z$ such that $R_3 z \notin \text{span } \{f_1\}$. Then $(\lambda R_1 + R_3) e_1 = (\lambda + \lambda_1) f_1$, $(\lambda R_1 + R_3) e_2 = \lambda f_2 + \mu_1 f_1$, and $(\lambda R_1 + R_3) z = R_3 z$ and because rank $(\lambda R_1 + R_3) = 2$ for each $\lambda \in \mathbb{F}$ we have $R_3 z \in \text{span } \{f_1, f_2\}$. The same argument with $R_2$ instead of $R_1$ yields that $R_3 z \in \text{span } \{f_1, f_3\}$. Consequently, $R_3 z \in \text{span } \{f_1\}$, a contradiction.
Similar elementary arguments yield the contradiction when we have the second possibility from Lemma 2.2.

So, assume now that \( n > 3 \) and let \( z \in V \) be a nonzero vector such that \( z \in \text{Im} R_1 \cap \text{Im} R_2 \cap \text{Im} R_3 \). Then we claim that \( z \in \text{Im} R \) for every nonzero \( R \in S \).

Suppose that there exists \( R \in S, R \not\in \text{span} \{R_1, R_2, R_3\} \), such that \( z \not\in \text{Im} R \). Take an idempotent operator \( P \in \mathcal{L}(V) \) such that \( \text{Ker} P = \text{Im} R = V_1 \) and \( Pz = z \). Choose an \((n-1)\)-dimensional subspace \( T \subset S \) containing \( R_1, R_2, R_3 \) such that \( S = \text{span} \{R\} \oplus T \). We know that \( PT = S_2 \) is a locally linearly dependent space of operators of dimension \( n-1 \). As all nonzero members of this space have rank \( n-2 \), this space must be minimal. Moreover,

\[
SU \subset V_1 + S_2 U,
\]

and because \( \dim SU = (1/2)n(n-1) \), \( \dim V_1 = n-1 \), and \( \dim S_2 U \leq (1/2)(n-1)(n-2) \), we have necessarily that

\[
\dim S_2 U = \frac{(n-1)(n-2)}{2}.
\]

So, we can apply the induction hypothesis which yields together with Proposition 2.1 that \( \text{Im} PR_1 \cap \text{Im} PR_2 \cap \text{Im} PR_3 = \{0\} \). But \( z \in \text{Im} PR_1 \cap \text{Im} PR_2 \cap \text{Im} PR_3 = \{0\} \), a contradiction.

Suppose that \( R = \lambda R_1 + \mu R_2 + \delta R_3 \neq 0 \). Then at least one of \( \lambda, \mu, \delta \) is nonzero, say \( \lambda \neq 0 \). We know that there is an \( R_4 \) linearly independent of \( R_1, R_2, R_3 \) such that \( z \in \text{Im} R_4 \). Repeating the same arguments as above with \( R, R_2, R_3, \) and \( R_4 \) instead of \( R_1 \), we come to the conclusion that \( z \in \text{Im} R \) in this case as well.

Thus, \( z \in \text{Im} R \) for every \( R \in S \). It follows that

\[
\text{Im} T \cap \text{Im} S = \text{span} \{z\}
\]

for every pair of linearly independent \( T, S \in S \). Take any linearly independent set \( T_1, \ldots, T_n \in S \). By minimality of \( S \) there exists \( u \in U \) such that \( T_2u, \ldots, T_nu \) are linearly independent. Assume first that \( T_1u \not\in \text{span} \{z\} \). There exist uniquely determined scalars \( \lambda_2, \ldots, \lambda_n, \), not all zero, such that \( T_1u = \lambda_2 T_2 u + \ldots + \lambda_n T_n u \). Set \( S = \lambda_2 T_2 + \ldots + \lambda_n T_n \). Clearly, \( T_1 \) and \( S \) are linearly independent and \( T_1u \in \text{Im} T_1 \cap \text{Im} S \). This contradicts (2).

If \( T_1u \in \text{span} \{z\} \), then we can find \( w \in U \) such that \( T_1w \not\in \text{span} \{z\} \). It follows that \( T_1(u + \lambda w) \not\in \text{span} \{z\} \) for every nonzero scalar \( \lambda \). As we can find a nonzero \( \lambda \in \mathbb{F} \) such that \( T_2(u + \lambda w), \ldots, T_n(u + \lambda w) \) are linearly independent, we can get a contradiction in the same way as above. This completes the proof of the fact that \( \text{Im} R_1 \cap \text{Im} R_2 \cap \text{Im} R_3 = \{0\} \) for all linearly independent \( R_1, R_2, R_3 \in S \).

We will next show that if \( S_1, \ldots, S_n \in S \) are linearly independent operators, then for every \( j \in \{1, \ldots, n\} \) we have

\[
\text{Im} S_j = \oplus_{k \neq j}(\text{Im} S_j \cap \text{Im} S_k).
\]
Since $\text{Im} S_1 \cap \text{Im} S_k \subset \text{Im} S_j$, $k = 1, \ldots, n$, $k \neq j$, $	ext{dim} \text{Im} S_1 = n - 1$, and $	ext{dim}(\text{Im} S_j \cap \text{Im} S_k) = 1$, $k = 1, \ldots, n$, $k \neq j$, it is enough to show that one-dimensional subspaces $\text{Im} S_j \cap \text{Im} S_k$, $k = 1, \ldots, n$, $k \neq j$, are linearly independent (each one-dimensional linear subspace is a linear span of some nonzero vector and we say that these subspaces are linearly independent if the spanning vectors are linearly independent). We will show that, say

$$\text{Im} S_2 \cap \text{Im} S_1, \text{Im} S_2 \cap \text{Im} S_3, \ldots, \text{Im} S_2 \cap \text{Im} S_n$$

are linearly independent. We have $\text{Im} S_2 = (\text{Im} S_2 \cap \text{Im} S_1) \oplus Z$ for some linear subspace $Z \subset V$. Denote by $V_1$ the image of $S_1$. We choose an idempotent operator $P_1 \in \mathcal{L}(V)$ whose kernel is $V_1$ and whose image contains $Z$. We prove as above that $P_1 \text{span} \{S_2, \ldots, S_n\} = P_1 T_2 = S_2$ is a minimal locally linearly dependent space of operators of dimension $n - 1$ with

$$\text{dim} S_2 U = \frac{(n - 1)(n - 2)}{2}.$$

So, we can apply the induction hypothesis which together with Proposition 2.1 implies that

$$\text{Im} S_2 = (\text{Im} S_2 \cap \text{Im} S_1) \oplus Z = (\text{Im} S_2 \cap \text{Im} S_1) \oplus \text{Im} P_1 S_2$$

where $v_j$ is a nonzero vector belonging to the one-dimensional subspace $\text{Im} P_1 S_2 \cap \text{Im} P_1 S_j$, $j = 3, \ldots, n$. For each $j = 1, 3, \ldots, n$ choose a nonzero vector $u_j \in \text{Im} S_2 \cap \text{Im} S_j$. Then $u_j = S_2 z_j = S_j w_j$, $j = 3, \ldots, n$, for some $z_j, w_j \in U$. Hence, $P_1 S_2 z_j = P_1 S_j w_j$. This is a nonzero vector since otherwise $u_j \in \text{Im} S_2 \cap \text{Im} S_j \cap \text{Im} S_1$, a contradiction by what we have proved at the beginning of the induction step. Hence, $P_1 S_2 z_j = P_1 S_j w_j = \mu_j v_j$ for some nonzero scalar $\mu_j$, $j = 3, \ldots, n$. Thus,

$$u_j = (I - P_1) u_j + P_1 u_j = (I - P_1) u_j + \mu_j v_j, \quad j = 3, \ldots, n.$$

We have to show that $u_1, u_3, \ldots, u_n$ are linearly independent. Suppose that

$$\alpha_1 u_1 + \alpha_3 u_3 + \ldots + \alpha_n u_n = 0$$

for some scalars $\alpha_1, \alpha_3, \ldots, \alpha_n$. Then

$$P_1 (\alpha_1 v_1 + \alpha_3 ((I - P_1) u_3 + \mu_3 v_3) + \ldots + \alpha_n ((I - P_1) u_n + \mu_n v_n)) = 0,$$

and consequently,

$$\alpha_3 \mu_3 v_3 + \ldots + \alpha_n \mu_n v_n = 0.$$

It follows that $\alpha_3 = \ldots = \alpha_n = 0$, which further yields that also $\alpha_1 = 0$. This proves (3).
Now,

\[ SU = \sum_{j=1}^{n} \text{Im} S_j, \]

and because of (3) we have

\[ SU = \sum_{j<k} (\text{Im} S_j \cap \text{Im} S_k). \]

Since the subspaces appearing on the right-hand side of this equality are all one-dimensional and because \( \dim SU = (1/2)(n(n-1)) \), we have actually

\[ SU = \bigoplus_{j<k} (\text{Im} S_j \cap \text{Im} S_k). \]

Choose an idempotent operator \( P \in \mathcal{L}(V) \) such that \( \text{Ker} P = \text{Im} S_n \) and \( Pz = z \) for every \( z \in \text{Im} S_j \cap \text{Im} S_k, \ j < k < n \). We apply the induction hypothesis to conclude that after multiplying \( S_1, \ldots, S_{n-1} \) by appropriate nonzero scalars there exist linearly independent vectors \( e_1', \ldots, e_{n-1}' \in U \), linearly independent vectors \( f_1', \ldots, f_{(n-1)/2}', f_{(n-1)/2} + 1', \ldots, f_{(n+1)/2}', f_{(n+1)/2} + 1 \) and a linear subspace \( Y \subset U \) such that

\[
U = \text{span} \{ e_1', \ldots, e_{n-1}' \} \oplus Y,
\]

\[
PS_jy = 0, \quad j = 1, \ldots, n-1, \ y \in Y,
\]

\[
PS_1 \left( \sum_{k=1}^{n-1} \lambda_k e_k' \right) = \sum_{k=2}^{n-1} \lambda_k f_k' - 1,
\]

\[
PS_2 \left( \sum_{k=1}^{n-1} \lambda_k e_k' \right) = -\lambda_1 f_1' + \sum_{k=3}^{n-1} \lambda_k f_{(n-1)+k-2}',
\]

\[
\vdots
\]

\[
PS_{n-1} \left( \sum_{k=1}^{n-1} \lambda_k e_k' \right) = -\lambda_1 f_{n-2}' - \lambda_2 f_{(n-1)+(n-2)-1}' - \lambda_3 f_{(n-1)+(n-2)+(n-3)-1}' - \cdots - \lambda_{n-2} f_{(n-1)+(n-2)+\ldots+3+1}'
\]

where \( \lambda_1, \ldots, \lambda_{n-1} \) are any scalars.

Choose nonzero vectors

\[
f_1 \in \text{Im} S_1 \cap \text{Im} S_2, \ f_2 \in \text{Im} S_1 \cap \text{Im} S_3, \ldots, \ f_{n-1} \in \text{Im} S_1 \cap \text{Im} S_n,
\]

\[
f_{(n-1)+1} \in \text{Im} S_2 \cap \text{Im} S_3, \ldots, f_{(n-1)+(n-2)} \in \text{Im} S_2 \cap \text{Im} S_n,
\]

\[
f_{(n-1)+(n-2)+1} \in \text{Im} S_3 \cap \text{Im} S_4, \ldots, f_{(n-1)+(n-2)+\ldots+1} \in \text{Im} S_{n-1} \cap \text{Im} S_n.
\]
Then there exist uniquely determined linear functionals \( \tau_1, \ldots, \tau_{n-1} \in U^* \) such that

\[
\begin{align*}
S_1 &= f_1 \otimes \tau_1 + f_2 \otimes \tau_2 + \cdots + f_{n-1} \otimes \tau_{n-1}, \\
S_2 &= -f_1 \otimes \tau_1 + f_{[(n-1)+1]} \otimes \tau_{n+1} + \cdots + f_{[(n-1)+(n-2)]} \otimes \tau_{2(n-1)}, \\
S_3 &= -f_2 \otimes \tau_{2n-1} - f_{[(n-1)+1]} \otimes \tau_{2n} + f_{[(n-1)+(n-2)+1]} \otimes \tau_{2n+1} + \cdots \\
&\quad + f_{[(n-1)+(n-2)+(n-3)]} \otimes \tau_{3(n-1)}, \\
\vdots \\
S_{n-1} &= -f_{n-2} \otimes \tau_{[(n-2)(n-1)+1]} - f_{[(n-1)+(n-2)-1]} \otimes \tau_{[(n-2)(n-1)+2]} - \cdots \\
&\quad - f_{[(n-1)+(n-2)+3+1]} \otimes \tau_{[(n-1)^2-1]} \\
&\quad + f_{[(n-1)+(n-2)+3+2+1]} \otimes \tau_{[(n-1)^2]}, \\
S_n &= -f_{n-1} \otimes \tau_{[(n-1)^2+1]} - f_{[(n-1)+(n-2)]} \otimes \tau_{[(n-1)^2+2]} - \cdots \\
&\quad - f_{[(n-1)+(n-2)+3+1]} \otimes \tau_{n(n-1)}.
\end{align*}
\]

Obviously, \( f'_k \) and \( f_k \) are linearly dependent for all \( k = 1, \ldots, \frac{n(n-1)}{2}, k \notin \{n-1, (n-1)+(n-2), \ldots, (n-1)+(n-2)+\cdots+1\} \). Absorbing the constants in the tensor products in the above expressions for \( S_1, \ldots, S_{n-1} \), we may, and will assume that \( f'_k = f_k \) for all these integers \( k \). Now, we calculate \( PS_1, \ldots, PS_{n-1} \) from the above formulas (we delete the term \( f_{n-1} \otimes \tau_{n-1} \) in the above expression of \( S_1 \), we delete the term \( f_{[(n-1)+(n-2)]} \otimes \tau_{2(n-1)} \) in the above expression of \( S_2, \ldots \)) and compare these equations with (4), (5), ..., (6).

We conclude that \( \tau_1 \) is a linear functional defined by \( \tau_1(e'_k) = 1, \tau_k(y) = 0 \) for all \( y \in Y \). The same is true for \( \tau_2, \ldots, \tau_{(n-2)(n-1)+2} \), and thus, \( \tau_1 = \tau_2 = \cdots = \tau_{(n-2)(n-1)+2} \). Similarly, \( \tau_n = \tau_{2n-1} = \cdots = \tau_{[(n-2)(n-1)+1]} \) is a linear functional defined by \( \tau_0(e'_1) = 1, \tau_k(e'_k) = 0 \) for \( k = 2, 3, \ldots, n-1 \), and \( \tau_n(y) = 0 \) for all \( y \in Y \). We define functionals \( \varphi_k \in U^*, k = 1, \ldots, n-1 \), by

\[
\varphi_k(y) = 0, \quad y \in Y,
\]

and

\[
\varphi_k(e'_j) = \delta_{kj},
\]

where \( \delta_{kj} \) is the Kronecker symbol \( 1 \leq k, j \leq n-1 \). So, we have

\[
\begin{align*}
S_1 &= [f_1 \otimes \varphi_2 + f_2 \otimes \varphi_3 + \cdots + f_{n-2} \otimes \varphi_{n-1}] + f_{n-1} \otimes \tau_{n-1}, \\
S_2 &= [-f_1 \otimes \varphi_1 + f_{[(n-1)+1]} \otimes \varphi_3 + \cdots + f_{[(n-1)+(n-2)-1]} \otimes \varphi_{n-1}] \\
&\quad + f_{[(n-1)+(n-2)]} \otimes \tau_{2(n-1)}, \\
S_3 &= [-f_2 \otimes \varphi_1 - f_{[(n-1)+1]} \otimes \varphi_2 + f_{[(n-1)+(n-2)+1]} \otimes \varphi_4 + \cdots \\
&\quad + f_{[(n-1)+(n-2)+(n-3)-1]} \otimes \varphi_{n-1}] + f_{[(n-1)+(n-2)+(n-3)]} \otimes \tau_{3(n-1)}, \\
\vdots \\
\end{align*}
\]
\[
S_{n-1} = [-f_{n-2} \otimes \varphi_1 - f_{(n-1)+(n-2)-1} \otimes \varphi_2 - \ldots
- f_{(n-1)+(n-2)+3+1} \otimes \varphi_{n-2}] + f_{(n-1)+(n-2)+3+1} \otimes \tau_{[(n-1)^2]}.
\]
\[
S_n = -f_{n-1} \otimes \tau_{[(n-1)^2+1]} - f_{(n-1)+(n-2)} \otimes \tau_{[(n-1)^2+2]} - \ldots
- f_{(n-1)+(n-2)+3+1} \otimes \tau_n(n-1).
\]

We know that \( \varphi_1, \ldots, \varphi_{n-1} \) are linearly independent, and consequently, there exists \( u \in U \) such that \( \varphi_1(u) = 0 \), while \( \varphi_2(u) \neq 0 \), \( \ldots \), \( \varphi_{n-1}(u) \neq 0 \). There exist scalars \( \alpha_1, \ldots, \alpha_n \), not all of them zero, such that

\[
\alpha_1S_1u + \ldots + \alpha_nS_nu = 0. \tag{7}
\]

If we write down this equation using the above formulas we get a linear combination of vectors \( f_1, \ldots, f_{2(n-1)} \) and in this linear combination each of \( f_j \)'s appears at most two times. In particular, comparing the coefficients at \( f_j \), \( j \neq 1, 2, \ldots, n-1 \), \( (n-1)+(n-2), (n-1)+(n-2)+(n-3), \ldots, (n-1)+\ldots+1 \), we arrive at

\[
\alpha_j \varphi_k(u) = \alpha_k \varphi_j(u), \quad 2 \leq j, k \leq n-1.
\]

It follows that either \( \alpha_2 = \ldots = \alpha_{n-1} = 0 \), or all these scalars are nonzero. Moreover, since \( \varphi_1(u) = 0 \), there is only one term with \( f_1 \), that is \( \alpha_1 \varphi_2(u)f_1 \), and consequently, \( \alpha_1 = 0 \). It follows that in this linear combination we have only one term with \( f_{n-1} \), that is \( -\alpha_n \tau_{[(n-1)^2+1]}(u)f_{n-1} \). Thus, we have two possibilities; either \( \alpha_n = 0 \), or \( \tau_{[(n-1)^2+1]}(u) = 0 \).

Assume first that \( \alpha_n = 0 \). Because both \( \alpha_1 \) and \( \alpha_n \) are zero, we have \( \alpha_2 \neq 0, \ldots, \alpha_{n-1} \neq 0 \). Considering the coefficients at \( f_j \), \( j = (n-1)+(n-2), (n-1)+(n-2)+(n-3), \ldots, (n-1)+\ldots+1 \), we arrive at \( \tau_{k(n-1)}(u) = 0, k = 2, \ldots, n-1 \).

Hence, for every \( u \in U \) such that \( \varphi_1(u) = 0 \) and \( \varphi_2(u) \neq 0 \), \( \ldots \), \( \varphi_{n-1}(u) \neq 0 \) we have \( \tau_{[(n-1)^2+1]}(u) = 0 \). By Lemma 2.3 we conclude that \( \tau_{2(n-1)}(u) \) is a scalar multiple of \( \varphi_1 \) or \( \tau_{[(n-1)^2+1]}(u) \) is a scalar multiple of \( \varphi_1 \). The first possibility contradicts the fact that \( S_2 \) is an operator of rank \( n-1 \).

Thus, we have the second possibility, that is, \( \tau_{[(n-1)^2+1]} = c_1 \varphi_1 \) for some nonzero scalar \( c_1 \). In the same way we show that there are nonzero scalars \( c_1, \ldots, c_{n-1} \) such that

\[
\tau_{(n-1)^2+k} = c_k \varphi_k, \quad k = 1, \ldots, n-1.
\]

After replacing \( S_n \) by \( c_1^{-1}S_n \) we have \( c_1 = 1 \).

In the next step we will show that \( \tau_k(n-1) \) and \( \tau_{2(n-1)} \) are linearly dependent for every pair \( j,k, 1 \leq j,k \leq n-1 \). Assume on the contrary that this is not true, say, \( \tau_{n-1} \) and \( \tau_{2(n-1)} \) are linearly independent. Then there exists \( u \in U \) such that \( \tau_{n-1}(u) = 0 \), and \( \tau_{2(n-1)}(u) \neq 0 \). For every such \( u \) we can find scalars \( \alpha_1, \ldots, \alpha_n \), not all of them zero, such that (7) holds. There is only one term with \( f_{n-1} \) in this linear combination. Hence, \( \alpha_n \varphi_1(u) = 0 \). We will prove that \( \varphi_1(u) = 0 \). Otherwise, we would have \( \alpha_n = 0 \), and then we would get
by considering the term with $f_{(n-1)+(n-2)}$ that $\alpha_2 = 0$, which would further yield that the coefficient at $f_1$ must be zero. Hence, we would have either $\varphi_2(u) = 0$ or $\alpha_1 = 0$. In the second case we would get from $\varphi_1(u) \neq 0$ that $\alpha_1 = \alpha_3 = \ldots = \alpha_{n-1} = 0$, a contradiction. Thus, we have $\varphi_1(u) = 0$ or $\varphi_2(u) = 0$ for every $u \in U$ such that $\tau_{n-1}(u) = 0$ and $\tau_{2(n-1)}(u) \neq 0$. By Lemma 2.3 we have two possibilities. In the first case the functional $\varphi_2$ would be a scalar multiple of $\tau_{n-1}$, contradicting rank $S_1 = n - 1$. Hence, $\tau_{n-1}$ and $\varphi_1$ are linearly dependent. But then there exists a non-trivial linear combination of $S_1$ and $S_n$ of rank $\leq n - 2$, a contradiction.

Set $\tau_{n-1} = \varphi_n$. Then $\tau_{k(n-1)} = b_k \varphi_n$ for some nonzero scalars $b_2, \ldots, b_{n-1}$. Absorbing the constant in the tensor product we may, and will assume that $b_2 = \ldots = b_{n-1} = 1$. We will show that $\varphi_1, \ldots, \varphi_{n-1}, \varphi_n$ are linearly independent. Assume on the contrary that they are linearly dependent. As $\varphi_1, \ldots, \varphi_{n-1}$ are linearly independent we have $\varphi_n = \beta_1 \varphi_1 + \ldots + \beta_{n-1} \varphi_{n-1}$ for some scalars $\beta_1, \ldots, \beta_{n-1}$. Moreover, all the $\beta$'s are nonzero, since otherwise one of the operators $S_1, \ldots, S_{n-1}$ would be of rank $< n-1$. But then the operator $S_1 + \beta_1 S_n$ is of rank $< n-1$, a contradiction.

Hence, $\varphi_1, \ldots, \varphi_{n-1}, \varphi_n$ are linearly independent and therefore we can choose vectors $e_1, \ldots, e_n \in U$ such that $\varphi_k(e_j) = \delta_{kj}$, $1 \leq k, j \leq n$. It is now straightforward to check that also $e_1 = \ldots = e_{n-1} = 1$.

Finally, we set

$$W = \cap_{k=1}^n \ker \varphi_k$$

in order to see that $S$ satisfies all conditions of Definition 1.1.

\[ \square \]

3 Final remarks

Because of certain applications it is important to understand completely the structure of $n$-tuples of locally linearly dependent operators for small values of $n$. In particular, this problem has been solved for $n = 2$ and $n = 3$ in [3, Theorems 2.3 and 2.4]. The case $n = 2$ is rather trivial. Two operators $T_1, T_2 : U \to V$ are locally linearly dependent if and only if they are linearly dependent or they are both of rank one with the same one-dimensional image. This follows easily from our results but it is also easy to give a direct short proof.

The case $n = 3$ is much more difficult and has been resolved in [3] using some structural results for matrix spaces with zero determinant [5]. We will show here that it is easy to describe the general form of locally linearly dependent operators $T_1, T_2, T_3 : U \to V$ using our results. Here, $U$ and $V$ are vector spaces over a field with at least 5 elements.

The first trivial possibility is that $T_1, T_2, T_3$ are linearly dependent. If this is not the case, then we denote by $S$ the linear span of these three operators.
Again we have two possibilities. The first one is that $\mathcal{S}$ is not a minimal locally linearly dependent space of operators. Then there exists a two-dimensional locally linearly dependent subspace. In other words, $\mathcal{S}$ contains two linearly independent rank one operators with the same image. The second possibility is that $\mathcal{S}$ is minimal. Then, by our main result, we have

$$2 \leq \dim SU \leq 3.$$ 

If $\dim SU = 3$, then $\mathcal{S}$ is standard. Thus, we have the following result.

**Theorem 3.1.** Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$ with at least 5 elements, and let $T_1, T_2, T_3 : U \to V$ be linear operators. Then the following are equivalent.

- $T_1, T_2, T_3$ are locally linearly dependent.
- Either $T_1, T_2, T_3$ are linearly dependent, or $\mathcal{S}$ is a standard 3-dimensional locally linearly dependent space of operators, or there exist a one-dimensional subspace $W \subset V$ and a two-dimensional subspace $T \subset \mathcal{S}$ such that $TU = W$, or there exists a two-dimensional subspace $Z \subset V$ such that $SU = Z$.

**References**


