The Compact Hyperspace Monad, 
a Constructive Approach\(^1\)

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1. Monads

Definition
Let $\mathcal{C}$ be a category. A monad consists of an endofunctor $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta: \text{Id}_\mathcal{C} \to \mathcal{F}$ and $\mu: \mathcal{F}^2 \to \mathcal{F}$ so that for any $X \in \mathcal{C}$,

\[
\mu_X \circ \mathcal{F}(\mu_X) = \mu_X \circ \mu_{\mathcal{F}(X)} \quad \text{and} \quad \mu_X \circ \eta_{\mathcal{F}(X)} = \mu_X \circ \mathcal{F}(\eta_X) = \text{Id}_{\mathcal{F}(X)}.
\]

Example
Let $\mathcal{P}: \text{SET} \to \text{SET}$ be the powerset functor. Define

\[
\eta_X(x) := \{x\} \quad \text{and} \quad \mu_X(\mathbb{X}) = \bigcup \mathbb{X},
\]

where $X$ is a set and $\mathbb{X} \in \mathcal{P}(\mathcal{P}(X))$. Then $(\mathcal{P}, \eta, \mu)$ is a monad.
Theorem (E. Michael (1951))

Let \((X, \tau)\) be a compact Hausdorff space and \(\mathcal{K}(X)\) be the set of all non-empty compact subsets of \(X\). Then the following two statements hold:

1. \(\mathcal{K}(X)\) is compact Hausdorff again with respect to the Vietoris topology \(\tau_V\), which is generated by the sets

\[
[U; V_1, \ldots, V_n] := \{ A \in \mathcal{K}(X) \mid A \subseteq U \wedge (\forall 1 \leq i \leq n) A \cap V_i \neq \emptyset \},
\]

where \(U, V_1, \ldots, V_n\) are open subsets of \(X\).

2. If \(\mathbb{K}\) is a compact subset of \(\mathcal{K}(X)\), then \(\bigcup \mathbb{K}\) is a compact subset of \(X\).
Let $\mathbf{CH}$ be the category of all compact Hausdorff spaces with continuous maps as morphisms. For morphisms $f : X \to Y$ let

$$\mathcal{K}(f)(A) := f[A],$$

that is, $\mathcal{K}(f)(A)$ the direct image of $A$ under $f$. Then it follows for $\eta$ and $\mu$ as above that $(\mathcal{K}, \eta, \mu)$ is a monad.

Our aim is now to derive an analogous result in a framework that allows to compute with the elements of the spaces under consideration. We will follow the line of research of U. Berger et al. in which one works in an intuitionistic logic extended by inductive and co-inductive definitions.

Spaces under consideration will have co-inductive characterisation from which by a suitable realisability interpretation trees representing the elements of the spaces can be extracted with which one computes.
2. Iterated function systems

Let $X$ be a compact Hausdorff space and $D$ be a finite set of continuous functions $d : X \rightarrow X$. Then $(X, D)$ is called iterated function system (IFS).

**Definition**

An IFS $(X, D)$ is said to be covering if

$$X = \bigcup \{ \text{range}(d) \mid d \in D \}.$$

The covering condition allows to characterise $X$ co-inductively.

**Definition**

Define $C_X$ co-inductively to be the largest subset of $X$ such that for all $x \in X$,

$$x \in C_X \rightarrow (\exists d \in D)(\exists y \in C_X) x = d(y).$$
Lemma

Let \((X, D)\) be a covering IFS. Then

\[ X = \mathbb{C}_X. \]

Proof.
By definition, \(\mathbb{C}_X \subseteq X\). The converse inclusion follows by co-induction. Because \((X, D)\) is covering, the defining implication of \(\mathbb{C}_X\) remains correct if \(\mathbb{C}_X\) is replaced by \(X\). \qed
The existential quantifiers in the definition of $C_X$ need to be interpreted constructively. Then by applying the definition of $C_X$ again and again, one obtains a sequence $d_0, d_1, \ldots$ of maps in $D$ so that

$$x \in \bigcap_n \text{range}(d_0 \circ \cdots \circ d_{n-1}).$$

Note that since $X$ is compact,

$$\bigcap_n \text{range}(d_0 \circ \cdots \circ d_{n-1}) \neq \emptyset.$$

In order to insure that the sequence $\alpha := d_0, d_1, \ldots$ uniquely determines $x$, a further condition is needed.

**Definition**

An IFS $(X, D)$ is **weakly hyperbolic** if for all $\alpha \in D^\omega$

$$\|\bigcap_n \text{range}(\alpha_0 \circ \cdots \circ \alpha_{n-1})\| \leq 1.$$
3. The case $\mathcal{K}(X)$

In case of points $x$, we characterised $x$ by determining a $d \in D$ with $x \in \text{range}(d)$. In the case of compact sets $K$, we will characterise $K$ by a finite subset $D'$ of $D$ such that $d \in D'$ exactly if $K$ hits $\text{range}(d)$.

For $d_1, \ldots, d_r \in D$ define

$$[d_1, \ldots, d_r](K_1, \ldots, K_r) := \bigcup_{i=1}^{r} d_i[K_i] = \bigcup_{i=1}^{r} \mathcal{K}(d_i)(K_i).$$

Note that for any continuous $f : X \to X$, the map $K \in \mathcal{K}(X) \mapsto f[K]$ is continuous with respect to Vietoris topology, similarly for the binary operation ‘union’.

Hence $[d_1, \ldots, d_r] : \mathcal{K}(X)^r \to \mathcal{K}(X)$ is continuous. Let

$$D^+ := \{ [d_1, \ldots, d_r] \mid d_1, \ldots, d_r \in D \text{ pairwise distinct} \}.$$

Then $D^+$ is finite and $(\mathcal{K}(X), D^+)$ is a generalised IFS.
Lemma
If $(X, D)$ is covering, so is $(\mathcal{K}(X), D^+)$.

Definition
Define $\mathbb{C}_{\mathcal{K}(X)}$ co-inductively to be the largest subset of $\mathcal{K}(X)$ such that for all $K \in \mathcal{K}(X),$

$$K \in \mathbb{C}_{\mathcal{K}(X)} \rightarrow (\exists [d_1, \ldots, d_r] \in D^+)$$

$$(\exists K_1, \ldots, K_r \in \mathbb{C}_{\mathcal{K}(X)}) K = [d_1, \ldots, d_r](K_1, \ldots, K_r).$$

Lemma
Let $(X, D)$ be a covering IFS. Then

$$\mathcal{K}(X) = \mathbb{C}_{\mathcal{K}(X)}.$$
4. Weak hyperbolicity
It is open whether weakly hyperbolicity is inherited form \((X, D)\) to \((\mathcal{K}(X), D^+)\).

**Lemma**

*Let* \(X\) *be a compact metric space and assume all* \(d \in D\) *are contracting. Then the following statements hold:*

1. \(\mathcal{K}(X)\) *with the Hausdorff metric is a compact metric space.*
2. *The metric topology is equivalent to the Vietoris topology.*
3. *All maps in* \(D^+\) *are contracting.*

In what follows let \(X\) *be a compact metric space.*
5. The case $\mathcal{K}(\mathcal{K}(X))$

In order to obtain a finite set of covering maps it seems natural to iterate the above construction. Consider

$$[[d_1^{(1)}, \ldots, d_{r_1}^{(1)}], \ldots, [d_1^{(n)}, \ldots, d_{r_n}^{(n)}]](\mathbb{K}_1, \ldots, \mathbb{K}_n)$$

$$:= \bigcup_{i=1}^{n} \mathcal{K}([d_1^{(i)}, \ldots, d_{r_i}^{(i)}])(\mathbb{K}_i).$$

A set $\mathbb{K} \in \mathcal{K}^2(X)$ is covered by the range of this map, just if its elements $K$ are such that for some $1 \leq i \leq n$, $K$ hits exactly the ranges of the maps $d_1^{(i)}, \ldots, d_{r_i}^{(i)}$.

The map $[[d_1^{(1)}, \ldots, d_{r_1}^{(1)}], \ldots, [d_1^{(n)}, \ldots, d_{r_n}^{(n)}]]$ has type

$$\mathcal{K}(\mathcal{K}(X)^{r_1}) \times \cdots \times \mathcal{K}(\mathcal{K}(X)^{r_n}) \to \mathcal{K}^2(X).$$

Thus $(\mathcal{K}^2(X), D^{++})$ is not a generalised IFS.
The central operation on $\mathcal{K}^2(X)$ in a monad is $\bigcup$. Here, we have

$$
\bigcup[[d_1^{(1)}, \ldots, d_{r_1}^{(1)}], \ldots, [d_1^{(n)}, \ldots, d_{r_n}^{(n)}]](\mathbb{K}_1, \ldots, \mathbb{K}_n)
= \bigcup[[d_1^{(1)}, \ldots, d_{r_1}^{(1)}], \ldots, [d_1^{(n)}, \ldots, d_{r_n}^{(n)}]](\prod_{j_1=1}^{r_1} \text{pr}_{j_1}^{(r_1)}(\mathbb{K}_1), \ldots, \prod_{j_n=1}^{r_n} \text{pr}_{j_n}^{(r_n)}(\mathbb{K}_n))
= \bigcup[[d_1^{(1)}, \ldots, d_{r_n}^{(n)}]](\text{pr}_{1}^{(r_1)}(\mathbb{K}_1), \ldots, \text{pr}_{r_n}^{(r_n)}(\mathbb{K}_n))
= \bigcup[\mathcal{K}(d_1^{(1)}), \ldots, \mathcal{K}(d_{r_n}^{(n)})](\text{pr}_{1}^{(r_1)}(\mathbb{K}_1), \ldots, \text{pr}_{r_n}^{(r_n)}(\mathbb{K}_n))
$$

$[\mathcal{K}(d_1^{(1)}), \ldots, \mathcal{K}(d_{r_n}^{(n)})]$ is of type $\mathcal{K}^2(X)^{r_1+\cdots+r_n} \to \mathcal{K}^2(X)$.

Therefore, instead of $[[d_1^{(1)}, \ldots, d_{r_1}^{(1)}], \ldots, [d_1^{(n)}, \ldots, d_{r_n}^{(n)}]]$ we use $[\mathcal{K}(d_1^{(1)}), \ldots, \mathcal{K}(d_{r_n}^{(n)})]$ and redefine $\mathcal{K}$ correspondingly.
6. The objects of the category CMIFS

For every compact metric space $X$ and finite set $D$ of contracting endofunctions so that $(X, D)$ is covering we inductively define the higher-order compact generalised IFS as follows:

$$
D^{(0)} := D; \quad D^{(n+1)} := \{ K^n(d) \mid d \in D^{(n)} \}^+ \\
X^{(0)} := X; \quad X^{(n+1)} := \bigcup \{ \text{range}(d) \mid d \in D^{(n+1)} \}.
$$

The objects of the category are then the sets

$$
\mathbb{C}_X^{(n)}
$$

for IFS $(X, D)$ as above and $n \geq 0$. For $n \geq 0$, define $\hat{K}$ by

$$
\hat{K}(\mathbb{C}_X^{(n)}) := \mathbb{C}_X^{(n+1)}.
$$
7. Morphisms

Here, we use a generalisation of U. Berger’s co-inductive-inductive characterisation of the uniformly continuous functions on the unit interval.

For \( f : X^n \to X \), \( m \in \mathbb{N} \) and \( 1 \leq i \leq m \), define \( f^{(i,m)} : X^{n+m-1} \to X^m \) by

\[
f^{(i,m)}(x_1, \ldots, x_{i-1}, y_1, \ldots, y_n, x_{i+1}, \ldots, x_m) := (x_1, \ldots, x_{i-1}, f(y_1, \ldots, y_n), x_{i+1}, \ldots, x_m).
\]

Let

\[
\mathbb{F}(X, Y) := \{ f : X^m \to X \mid m \geq 0 \}
\]

and for \( m \in \mathbb{N} \) and \( F \subseteq \mathbb{F}(X, Y) \), let \( F^{(m)} \) be the set of maps in \( \mathbb{F}(X, Y) \) of arity \( m \).
Now, let the operator

\[ \Phi^{X,Y}: \mathcal{P}(\mathbb{F}(X, Y)) \rightarrow (\mathcal{P}(\mathbb{F}(X, Y)) \rightarrow \mathcal{P}(\mathbb{F}(X, Y))) \]

be given by

\[ \Phi^{X,Y}(F)(G) := \{ f \mid (\exists e \in E)(\exists f_1, \ldots, f_{\text{ar}(e)} \in F) f = e \circ (f_1 \times \cdots \times f_{\text{ar}(e)}) \vee (\exists 1 \leq i \leq \text{ar}(f)) (\forall d \in D) f \circ d^{(i,\text{ar}(f))} \in G \}. \]

Obviously, \( \Phi^{X,Y}(F) \) is monotone in \( G \), for all \( F \subseteq \mathbb{F}(X, Y) \). Thus,

\[ \mathcal{J}^{X,Y}(F) := \mu \Phi^{X,Y}(F) \]

exists. It follows that \( \mathcal{J}^{X,Y}(F) \) is the smallest subset \( G \) of \( \mathbb{F}(X, Y) \) such that

(W) If \( e \in E \) and \( f_1, \ldots, f_{\text{ar}(e)} \in F \), then \( e \circ (f_1 \times \cdots \times f_{\text{ar}(e)}) \in G \).

(R) If \( h \in \mathbb{F}(X, Y) \) and \( 1 \leq i \leq \text{ar}(h) \) so that \( h \circ d^{(i,\text{ar}(h))} \in G \), for all \( d \in D \), then \( h \in G \).
Since $J^{X,Y}$ is monotone as well, also

$$C_F(X,Y) := \nu J^{X,Y}$$

exists.

**Proposition**

Let $(X,D)$, $(Y, E)$ and $(Z, C)$ be (generalised) IFS. If $f \in C_F(Y,Z)$ and $g_\kappa \in C_F(X,Y)$, for $1 \leq \kappa \leq \text{ar}(f)$, then

$$f \circ (g_1 \times \cdots \times g_{\text{ar}(f)}) \in C_F(X,Z).$$

**Lemma**

Let $(X,D)$ be a (generalised) IFS and $\text{id}_X : X \to X$ be the identity on $X$. Then $\text{id}_X \in C_F(X,X)$. 
Proposition

Let \((X, D)\) and \((Y, E)\) be (generalised) IFS. Moreover, for \(m > 0\), let \(ev: F(m)(X, Y) \times X^m \to Y\) with

\[
ev(f, x) := f(x)
\]

be the evaluation map. Then

\[
ev[C_{F(X, Y)}^{(m)} \times C_X^m] \subseteq C_Y.
\]

For any two objects \(C_X\) and \(C_Y\) of the category \(\text{CMIFS}\), the corresponding hom-set is the set

\[
C_{F(C_X, C_Y)}.
\]

Theorem

\((\hat{\mathcal{K}}, \eta, \cup)\) is a monad.
8. Computable metric spaces

**Definition**
A digit space \((X, D)\) is a compact covering (generalised) IFS such that \(X\) is a metric space and all maps in \(D\) are contracting. They are called digit maps.

**Definition**
Let \((X, \rho, Q), (X', \rho', Q')\) be metric spaces with countable dense subspaces \(Q\) and \(Q'\), respectively. A uniformly continuous map \(h: X^i \rightarrow X'\) is computable if it has a computable modulus of continuity and there is a procedure \(G_h\), which given \(\vec{u} \in Q^i\) and \(n \in \mathbb{N}\) computes a basic element \(v \in Q'\) with \(\rho'(h(\vec{u}), v) < 2^{-n}\).

**Definition**
Let \((X, D)\) be a digit space such that the underlying metric space \((X, \rho)\) has a countable dense subset \(Q\) with respect to which it is computable. \((X, D, Q)\) is said to be a computable digit space if, in addition, all digit maps \(d \in D\) are computable.
Definition
A (generalised) IFS \((X, D)\) is well-covering if every element of \(X\) is contained in the interior \(\text{int} (\text{range}(d))\) of \(\text{range}(d)\), for some \(d \in D\).

Proposition
Let \((X, D)\) be a compact well-covering IFS. Then \((\mathcal{K}(X), D^+)\) is a compact well-covering (generalised) IFS.

Definition
Let \((X, D, Q)\) be a computable digit space. We call \((X, D, Q)\) decidable if for \(u \in Q\), \(\theta \in \mathbb{Q}_+\) and \(d \in D\) it can be decided whether

\[ B_\rho(u, \theta) \subseteq \text{range}(d). \]
Proposition

Let \((X, D, Q)\) be a computable digit space. Then \((\mathcal{K}(X), D^+, Q)\) is a computable digit space as well. Here, \(Q\) is the set of all finite subsets of \(Q\).

Proposition

Let \((X, D, Q)\) be a computable digit space. If \((X, D, Q)\) is decidable, so is \((\mathcal{K}(X), D^+, Q)\).

Proposition

Let \((X, D, Q)\) be a well-covering and decidable computable digit space. If the digit maps in \(D\) have a computable right inverse, then also the digit maps in \(D^+\) have a computable right inverse.
Let

\[ C(X, Y) := \{ f \in \mathcal{F}(X, Y) \mid f \text{ uniformly continuous with a} \]
\[
\text{computable modulus of continuity } \zeta : \mathbb{Q}_+ \rightarrow \mathbb{Q}_+ \}. \]

**Theorem**

Let \((X, D, Q_X)\) and \((Y, E, Q_Y)\) be decidable and well-covering computable digit spaces so that the digit maps in \(D\) and \(E\) have a computable right inverse. Then

\[ C_{\mathcal{F}}(C_X, C_Y) = C(C_X, C_Y). \]
9. Representing trees

Definition
Let $D$ be a finite non-empty set such that an arity $\text{ar}(d) \geq 1$ is assigned to each $d \in D$. Let $\mathcal{T}_D^\omega$ be the largest subset of $\mathcal{P}(D^\omega)$ such that for every $T \in \mathcal{T}_D^\omega$,

$$T \in \mathcal{T}_D^\omega \rightarrow (\exists d \in D)(\exists T_1, \ldots, T_{\text{ar}(d)} \in \mathcal{T}_D^\omega) \ T = [d; T_1, \ldots, T_{\text{ar}(d)}].$$

The elements of $\mathcal{T}_D^\omega$ are called infinite $D$-trees.

Obviously, every infinite $D$-tree is a finitely branching tree with only infinite paths. Note that $d$ and $T_1, \ldots, T_{\text{ar}(d)}$ are uniquely determined.

Each infinite $D$-tree $T = [d; T_1, \ldots, T_{\text{ar}(d)}]$ is uniquely determined by its finite initial segments $T^{(n)}$ recursively defined by

$$T^{(0)} := \{d\},$$
$$T^{(n+1)} := [d; T_1^{(n)}, \ldots, T_{\text{ar}(d)}^{(n)}].$$
Now, assume that \((X, D)\) is a generalised IFS. Then each finite initial segment \(\mathcal{T}^{(n)}\) defines a continuous map \(f_{\mathcal{T}^{(n)}} : X^{\text{ar}(f_{\mathcal{T}^{(n)}})} \to X:\)

- If \(\mathcal{T}^{(0)} = \{d\}\), for some \(d \in D\), then \(f_{\mathcal{T}} = d\).
- If \(\mathcal{T} = [d; T_1, \ldots, T_{\text{ar}(d)}]\), then \(f_{\mathcal{T}} = d \circ (f_{T_1} \times \cdots \times f_{T_{\text{ar}(d)}})\).

**Proposition**

Let \((X, D)\) be compact and weakly hyperbolic. Then for all \(\mathcal{T} \in \mathcal{T}_D\),

\[
\| \bigcap_{n \in \mathbb{N}} \text{range}(f_{\mathcal{T}^{(n)}}) \| = 1.
\]

The uniquely determined element in \(\bigcap_{n \in \mathbb{N}} \text{range}(f_{\mathcal{T}^{(n)}})\) is denoted by \([T]\).
Theorem
Let \((X, D)\) be a compact, covering and weakly hyperbolic generalised IFS. Then the realisers of a statement \(x \in \mathbb{C}_X\) are exactly the infinite \(D\)-trees \(T \in T_D^\omega\) representing \(x\), that is

\[ T \mathsf{r} (x \in \mathbb{C}_X) \iff \llbracket T \rrbracket = x. \]

In particular, from a constructive proof of \(x \in \mathbb{C}_X\) one can extract an infinite \(D\)-tree representation of \(x\).
For $1 \leq i \leq m$ let $R_i$ be a letter with $\text{ar}(R_i) := \|D\|$. Then a realiser for a map $h \in \mathcal{C}^{(m)}_{\text{IF}(X,Y)}$ extracted from the definition is an infinite $(E \cup \{ R_i \mid 1 \leq i \leq m \})$-tree such that

- each node is either a
  - writing node labelled with a digit in $E$ and $\text{ar}(e)$ immediate subtrees, or a
  - reading node labelled with $R_i$ and $\|D\|$ immediate subtrees;
- each path has infinitely many writing nodes.
The interpretation of such a tree as a tree transformer is easy. Given $m$ trees $T_1, \ldots, T_m \in \mathcal{T}_D^\omega$ as inputs, run through the tree and output a tree in $\mathcal{T}_E^\omega$ as follows:

1. At a writing node $[e; S_1, \ldots, S_{\text{ar}(e)}]$ output $e$ and continue with the subtrees $S_1, \ldots, S_{\text{ar}(e)}$.

2. At a reading node $[R_i; (S'_d)_{d \in D}]$ continue with $S'_d$, where $d$ is the root of $T_i$, and replace $T_i$ by its $\text{ar}(d)$ immediate subtrees.