Brooks Theorem for Generalized Dart Graphs

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Abstract

The well-known Brooks’ Theorem says that each graph $G$ of maximum degree $k \geq 3$ is $k$-colorable unless $G = K_{k+1}$. We generalize this theorem by allowing higher degree vertices with prescribed types of neighborhood.

1 Introduction

A $k$-coloring of a graph is a mapping from the set of vertices to $\{1, \ldots, k\}$ such that any two adjacent vertices have different colors. The decision problem whether a given graph $G$ has a $k$-coloring is a classical NP-complete problem for every fixed $k \geq 3$ (see [3, 4]).

By Brooks’ Theorem [1], every graph with maximum vertex degree at most $k \geq 3$ and without a component isomorphic to $K_{k+1}$ (a complete graph on $k + 1$ vertices) has a $k$-coloring. Furthermore, as follows from [2, 6, 7, 8, 9], there exists a linear-time algorithm that finds a $k$-coloring for such a graph.

Kochol, Lozin, and Randerath [6, Theorem 4.3] proved that if $\mathcal{D}$ is a class of graphs in which the neighborhood of each 4-degree vertex induces a graph isomorphic to a disjoint union of an isolated vertex and a path of length 2, then every graph from $\mathcal{D}$ is either 3-colorable or has a component isomorphic to $K_4$. Furthermore, there exists

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a linear-time algorithm that finds either a 3-coloring or a component isomorphic to \( K_4 \) for each graph from \( D \). This generalizes the Brooks’ Theorem for the case \( k = 3 \).

The aim of this paper is to generalize the Brooks’ Theorem and the result from [6, Theorem 4.3]. We consider classes of graphs where each vertex of degree at least \( k + 2 \) has a strictly prescribed neighborhood, so called “\((k, s)\)-dart graphs”, defined in the following section. Our main result, Theorem 1, is that if \( G \) is a \((k, s)\)-dart graph, \( k \geq \max\{3, s\} \), and \( s \geq 2 \), then \( G \) is \((k+1)\)-colorable if and only if it has no component isomorphic to \( K_{k+2} \). Furthermore, if \( G \) is \((k+1)\)-colorable, then a \((k+1)\)-coloring of \( G \) can be constructed in a linear time. We also show that if \( s > k \geq 3 \), then it is an NP-complete problem to decide whether a \((k, s)\)-dart graph is \((k+1)\)-colorable (see Theorem 2).

2 Definitions

In this paper we consider simple graphs, i.e., without multiple edges and loops. If \( G \) is a graph, then \( V(G) \) and \( E(G) \) denote the vertex and the edge sets of \( G \), respectively.

Let \( G \) be a graph and \( x, y \) two vertices of \( G \). Then \( G + xy \) denotes the graph constructed from \( G \) by adding an edge \( xy \). Since we consider simple graphs, \( G + xy = G \) if \( x, y \) are adjacent in \( G \). For a vertex \( v \) of \( G \), let \( d_G(v) \) denote the degree of \( v \) in \( G \). Let \( H, G \) be two graphs such that no subgraph of \( G \) is isomorphic with \( H \). Then we say that \( G \) is a \( H \)-free graph.

A \((k, s)\)-diamond is a join of a clique of size \( k \geq 1 \) and an independent set of size \( s \geq 1 \). Notice that these graphs are edge-maximal split graphs. In a \((k, s)\)-diamond \( D \), vertices that belong to the independent set are called pick vertices, and the remaining (i.e. those in the \( k \)-clique) are called central vertices. Denote by \( C(D) \) and \( P(D) \) the sets of central vertices and pick vertices of \( D \), respectively. An example of a \((4, 3)\)-diamond \( D \) with \( C(D) = \{c_1, \ldots, c_4\} \) and \( P(D) = \{p_1, p_2, p_3\} \) is in Figure 1.

![Figure 1: A (4,3)-diamond.](image)

Note that a \((k, 1)\)-diamond is isomorphic to \( K_{k+1} \); in this case the unique pick vertex does not distinguish from the central vertices. This is irrelevant for us, because in this paper we deal only with \((k, i)\)-diamonds where \( i \geq 2 \).

**Definition 1** A graph \( G \) is a \((k, s)\)-dart if each vertex of \( G \) of degree \( \geq k+2 \) is a central vertex of some \((k, i)\)-diamond \( D \) as an induced subgraph of \( G \) with \( 2 \leq i \leq s \), for which
(a) \( d_D(x) \geq d_G(x) - 1 \) for each \( x \in V(D) \);

(b) no two vertices of \( C(D) \) have a common neighbor in \( G - D \).

The following remarks related to Definition 1 are straightforward:

(1) Inequality \( i \geq 2 \) can be removed in Definition 1, because it follows from the fact that \( D \) contains a vertex of degree \( \geq k + 2 \).

(2) Every graph of maximum degree \( \leq k + 1 \) is a \((k, 1)\)-dart graph since in Definition 1, we only prescribe the structure on the neighborhood of vertices of higher degree.

(3) Every \((k, s_1)\)-dart graph is a \((k, s_2)\)-dart if \( s_1 \leq s_2 \).

Notice that \((2, 2)\)-diamonds and \((2, 2)\)-dart graphs are called diamonds and dart graphs, respectively, in [6]. By a generalized dart graph and generalized diamond we mean any \((k, s)\)-dart graphs and any \((k, s)\)-diamond, \( k, s \geq 2 \), respectively. In this paper we usually omit the word generalized, if it is clear from the context which term we have in mind.

In a \((k, s)\)-dart graph \( G \), every vertex of degree at least \( k + 2 \) belongs to an induced \((k, i)\)-diamond with \( 2 \leq i \leq s \). Denote by \( D(G) \) the set of all induced maximal \((k, i)\)-diamonds of \( G \) with \( i \geq 2 \).

We say that a vertex of a \((k, s)\)-dart graph \( G \) is central, if it is a central vertex of a \((k, i)\)-diamond of \( D(G) \), \( i \leq s \). Similarly define a pick vertex of \( G \). Denote the sets of central vertices and pick vertices by \( C(G) \) and \( P(G) \), respectively.

Let \( G \) be a \((k, s)\)-dart graph and \( D \in D(G) \). Then, each central vertex \( x \in C(D) \) is adjacent to at most one vertex \( v' \) from \( G - D \). In this case, \( v' \) is called an isolated neighbor of \( v \). The set of all isolated neighbors of the central vertices of \( D \) is denoted by \( I(D) \). Notice that the possibility that \( I(D) = \emptyset \) is not excluded.

We remark that the following observations for a \((k, s)\)-dart graph \( G \) hold:

(4) A central vertex \( v \) of a \((k, s)\)-dart graph \( G \) is not necessarily of degree at least \( k + 2 \). This happens only if \( v \) is a central vertex of a \((k, 2)\)-diamond \( D \in D(G) \) and it has no neighbor in \( G - D \). Then, \( v \) is of degree \( k + 1 \). The possibility that all central vertices of \( D \) are of degree \( k + 1 \) is not excluded.

(5) If \( K_{k+2} \) is a subgraph of a \((k, s)\)-dart graph \( G \), then it must be a component of \( G \). Thus a copy of \( K_{k+2} \) in \( G \) is disjoint from diamonds of \( D(G) \).

(6) No two pick vertices of the same diamond from \( D(G) \) are adjacent.

### 3 Properties of dart graphs

The next lemma assures that diamonds in a dart graph are vertex disjoint.

**Lemma 1** Let \( G \) be a \((k, s)\)-dart graph with \( k \geq 3 \). Then
(a) \( V(D_1) \cap V(D_2) = \emptyset \), for every two distinct diamonds \( D_1, D_2 \in D(G) \).

(b) \( C(G) \cap P(G) = \emptyset \); in particular each pick vertex is of degree \( k \) or \( k+1 \).

**Proof.** We prove (a). Suppose that \( v \) is a vertex of two distinct diamonds \( D_1, D_2 \in D(G) \).

Assume that \( v \in C(D_1) \cap C(D_2) \). If \( C(D_1) = C(D_2) \), then by Definition 1(b) we obtain that \( P(D_1) = P(D_2) \), whence \( D_1 = D_2 \). Thus \( C(D_1) \neq C(D_2) \).

Suppose first \( |C(D_1) \cap C(D_2)| = 1 \), i.e., \( C(D_1) \cap C(D_2) = \{v\} \). Then by Definition 1, either \( k-2 \) or \( k-1 \) vertices of \( C(D_2) \) (resp. \( C(D_1) \)) are pick vertices of \( D_1 \) (resp. \( D_2 \)). But then for \( k \geq 4 \), we obtain also two adjacent pick vertices of \( D_1 \) (resp. \( D_2 \)), a contradiction to (6). So we may assume that \( k = 3 \), \( C(D_1) = \{u_1, w_1, v\} \), \( C(D_2) = \{u_2, w_2, v\} \), and \( u_1 \) (resp. \( u_2 \)) are pick vertices of \( D_1 \) (resp. \( D_1 \)). By (6), \( w_1 \) (resp. \( w_2 \)) is not a pick vertex of \( D_2 \) (resp. \( D_1 \)). Then \( w_1 \in I(D_2) \) (resp. \( w_2 \in I(D_1) \)) is a common neighbor of \( v, u_2 \in C(D_2) \) (resp. \( v, u_1 \in C(D_1) \)), a contradiction with Definition 1(b).

Suppose now \( |C(D_1) \cap C(D_2)| \geq 2 \). Then each vertex \( u \in C(D_1) \setminus C(D_2) \) is a neighbor of at least two vertices from \( C(D_2) \), whence by Definition 1(b), \( u \in P(D_2) \) and thus \( C(D_1) \setminus C(D_2) \subseteq P(D_2) \). Similarly \( C(D_2) \setminus C(D_1) \subseteq P(D_1) \). Thus the subgraph of \( G \) induced by \( C(D_1) \cup C(D_2) \) is a clique, whence \( |C(D_1) \cup C(D_2)| = k+1 \), and so \( |C(D_1) \cap C(D_2)| = k-1 \). By assumption, \( D_1 \) is a \((k, s_1)\)-diamond, \( s \geq s_1 \geq 2 \). Thus there exists \( x_1 \in P(D_1) \setminus C(D_2) \). By (6), we infer that \( x_1 \in I(D_2) \), but then it is a common neighbor of at least two vertices from \( C(D_2) \), a contradiction with Definition 1(b).

By the above two paragraphs, we can assume that \( C(D_1) \cap C(D_2) = \emptyset \). If \( v \in V(D_1) \cap P(D_2) \), then \( d_{D_2}(v) + 1 < d_G(v) \), a contradiction with Definition 1(a). Similarly if \( v \in V(D_2) \cap P(D_1) \). This proves (a). Claim (b) is an easy consequence of (a). □

In the next few lemmas, we study properties of a graph \( G' \) obtained from \( G \) by applying some local modifications.

**Lemma 2** Let \( G \) be a \( K_{k+2} \)-free \((k, s)\)-dart graph with \( k \geq 3 \) and let \( D \in D(G) \). Suppose that \( a_1, a_2 \) are two central vertices of \( D \) and let \( x_1, x_2 \) be their isolated neighbors, respectively. Then the graph \( G' = G - x_1a_1 - x_2a_2 + x_1x_2 \) is a \( K_{k+2} \)-free graph unless there exists \( D' \in D(G) \) such that \( x_1, x_2 \) are pick vertices of \( D' \).

**Proof.** Suppose that \( G' \) contains a copy \( H \) of \( K_{k+2} \). Then, \( x_1, x_2 \) are vertices of \( H \), thus cannot be adjacent in \( G \) and there is a set \( S \) of \( k \) common neighbors of \( x_1 \) and \( x_2 \) in \( G \), which induce a clique. Notice that \( |S| = k \) and \( d_G(x_1), d_G(x_2) \geq k+1 \).

Suppose that \( d_G(x_1) \geq k+2 \). Then, \( x_1 \) is a central vertex of some diamond \( D' \in D(G) \), whence by Definition 1(b), \( S \subseteq V(D') \) and clearly, \( |S \cap C(D')| \geq k-1 \geq 2 \). Then \( x_2 \) has at least 2 neighbors in \( C(D') \), whence \( x_2 \) belongs to \( D' \), and so it is adjacent to \( x_1 \) in \( G \), a contradiction.

Thus, by previous paragraph, we may assume that \( d(x_1) = k + 1 \), and analogously \( d(x_2) = k + 1 \). Then \( x_1, x_2 \) and \( S \) belong to a diamond \( D' \in D(G) \) in which \( x_1, x_2 \in P(D') \) and \( S = C(D') \). □
Lemma 3 Let $G$ be a $(k, s)$-dart graph and let $D \in \mathcal{D}(G)$. Suppose that $a_1, a_2$ are two central vertices of $D$ and let $x_1, x_2$ be their isolated neighbors, respectively. Then the graph $G' = G - x_1(a_1 - a_2x_2 + x_1x_2)$ is a $(k, s)$-dart graph unless one of the following conditions occurs:

(7) there exists $D' \in \mathcal{D}(G)$ such that $x_1, x_2$ are pick vertices of $D'$;

(8) there exists $D' \in \mathcal{D}(G)$ and $i \in \{1, 2\}$ such that $x_i \in C(D')$ and $x_{3-i}$ is an isolated neighbor of a central vertex from $D'$, which is distinct from $x_i$.

Proof. Suppose that $G'$ is not a $(k, s)$-dart graph. Each vertex preserve its degree from $G$ except $a_1, a_2$, which belong to $D$. Notice that $D$ is a diamond in $G'$ as well. If there is some $D' \in \mathcal{D}(G)$ that is not induced diamond of $G'$, then $x_1$ and $x_2$ must be pick vertices of $D'$, which gives case (7).

Thus each diamond $D' \in \mathcal{D}(G)$ is an induced diamond of $G'$. Clearly $D'$ satisfies Definition 1(a) in $G'$. If $D'$ does not satisfy Definition 1(b) in $G'$, then there are two central vertices $u$ and $v$ of $D'$ with a common neighbor $w$ outside $D'$. Notice that $x_1x_2$ is one of the edges $uw$ or $vw$. Then without loss of generality, we may assume that $x_1$ is a central vertex in $D'$ and $x_2$ is an isolated neighbor of a central vertex of $D'$ distinct from $x_1$, which gives case (8). $\square$

Notice that in the exceptional case (7) of the above lemma, $G'$ may still be a dart graph, when $x_1, x_2$ are pick vertices of a $(k, 2)$-diamond $D'$ with no isolated vertices. Then, $D'$ becomes a copy of $K_{k+2}$ in $G'$.

4 An extension of Brooks theorem

For a diamond $D \in \mathcal{D}(G)$, a vertex of $I(D)$ could be a central or pick vertex of another diamond of $\mathcal{D}(G)$. Denote by $I_c(D)$ and $I_p(D)$ the subset of all such vertices of $I(D)$, respectively. By Lemma 1(b), sets $I_c(D)$ and $I_p(D)$ are disjoint. Finally, let $I_u(D)$ be the vertices of $I(D)$ that are neither in $I_c(D)$, nor in $I_p(D)$.

Lemma 4 Suppose that we have a $K_{k+2}$-free $(k, s)$-dart graph $G$, $k \geq \max\{3, s\}$, $s \geq 2$, together with the set $\mathcal{D}(G) \neq \emptyset$. Then we can find $D \in \mathcal{D}(G)$ and construct a $K_{k+2}$-free $(k, s)$-dart graph $G^*$ together with $\mathcal{D}(G^*)$ in $O(|E(D)|)$ time such that

(a) $|\mathcal{D}(G^*)| < |\mathcal{D}(G)|$;

(b) $|E(G^*)| \leq |E(G)| - |E(D)|$;

(c) From any $(k + 1)$-coloring $\lambda$ of $G^*$ one can construct a $(k + 1)$-coloring of $G$ in $O(|E(D)|)$ time.

Proof. Consider a $(k, i)$-diamond $D' \in \mathcal{D}(G)$, $2 \leq i \leq s$, and check three cases:

Case 1. $|I(D')| < k$. Thus there exists $v \in C(D')$ having no isolated neighbor. In this case we take $D := D'$ and $G^* := G - D'$. Suppose that $u'$ is an arbitrary vertex
of degree $\geq k + 2$ in $G^\ast$. Then, it is also of degree $\geq k + 2$ in $G$, and hence it belongs to a $(k, i)$-diamond $D'' \in \mathcal{D}(G)$ with $2 \leq i \leq s$. Diamonds $D$ and $D''$ are disjoint, by Lemma 1, and hence $D''$ is an induced $(k, s)$-diamond in $G^\ast$. Furthermore, Lemma 1 assures that $\mathcal{D}(G)$ consists of $D$ and $\mathcal{D}(G^\ast)$. Thus $G^\ast$ is a $(k, s)$-dart graph. Obviously, $G^\ast$ is a $K_{k+2}$-free graph and $|E(G^\ast)| \leq |E(G)| - |E(D)|$.

Let $\lambda^\ast$ be a $(k + 1)$-coloring of $G^\ast$. Since every pick vertex of $D$ has at most one neighbor outside $D$, and since $|P(D)| \leq k$, it follows that there exists a color $c$ that we can assign to all pick vertices of $D$. Denote by $u_1, \ldots, u_{k-1}$ the vertices from $C(D) \setminus \{v\}$ and take $u_k := v$. For $i = 1, \ldots, k$, take $L(u) = \{1, \ldots, k + 1\} \setminus \{c, \lambda^\ast(x_i)\}$ if $u_i$ has an isolated neighbor $x_i$, otherwise take $L(u) = \{1, \ldots, k + 1\} \setminus \{c\}$. Thus $k \geq |L(u_i)| \geq k - 1$ for $i < k$ and $|L(u_k)| = k$ (because $u_k = v$ has no isolated neighbor). For $i = 1, \ldots, k$ we assign $u_i$ a color from $L(u_i)$ and remove this color from all $L(u_j)$ where $j > i$. Clearly, each $L(u_i)$ is nonempty after $i - 1$ steps, thus this process gives a coloring $\lambda$ of $G$, and can be done in $O(|E(D)|)$ time.

Case 2. $|I(D')| = k$ and $I(D')$ does not consist of pick vertices of one diamond of $\mathcal{D}(G)$. Suppose that each pair $x_1, x_2 \in I(D')$ satisfies either (7), or (8). This implies immediately that $|I_s(D')| \leq 1$ and $|I_c(D')| \leq 1$. Thus $|I_p(D')| \geq 1$ (because $k \geq 3$).

Each $x_1 \in I_s(D') \cup I_c(D')$ and $x_2 \in I_p(D')$ satisfy neither (7), nor (8), whence $I_s(D') \cup I_c(D') = \emptyset$. Thus all vertices of $I(D')$ must be pick vertices of one diamond of $\mathcal{D}(G)$. This contradicts the assumption of Case 2.

Thus there exist two distinct vertices $x_1, x_2 \in I(D')$ satisfying neither (7), nor (8). To find them is an easy process. Take $x_1 \in I_s(D') \cup I_c(D')$ and $x_2 \in I_p(D')$ if possible. If $I_p(D') = \emptyset$, then either $|I_s(D')| \geq 2$, or $|I_c(D')| \geq 2$, and we can choose $x_1, x_2$ from one of them. If $I_s(D') \cup I_c(D') = \emptyset$, $I_p(D')$ has at least two vertices from different diamonds of $\mathcal{D}(G)$, and choose them.

After choosing $x_1, x_2$, take the graph $G' = G - x_1a_1 - x_2a_2 + x_1x_2$. By Lemmas 2 and 3, $G'$ is a $K_{k+2}$-free $(k, s)$-dart graph. Moreover, $|E(G')| < |E(G)|$ and $\mathcal{D}(G') = \mathcal{D}(G')$. $D' \in \mathcal{D}(G')$ but the number of isolated vertices of $D'$ in $G'$ is smaller then $k$. Thus we can apply the construction from Case 1 for $G'$ and $D'$; i.e., we take $D := D'$ and $G^\ast := G' - D'$. Analogously as in Case 1, $G^\ast$ is a $K_{k+2}$-free $(k, s)$-dart graph, $|E(G^\ast)| \leq |E(G)| - |E(D)|$ and $\mathcal{D}(G^\ast) = \mathcal{D}(G) \setminus \{D\}$.

Let $\lambda^\ast$ be a $(k + 1)$-coloring of $G^\ast$. Applying the process described in Case 1, we get a $(k + 1)$-coloring $\lambda$ of $G$. Clearly $\lambda(a_1) \neq \lambda(a_2)$ and $\lambda(x_1) \neq \lambda(x_2)$. By Definition 1, $a_1$ and $x_2$ are non-adjacent, and similarly $a_2$ and $x_1$ are non-adjacent. Notice that $\lambda$ is not a coloring of $G$ if and only if $\lambda(a_1) = \lambda(x_1)$ or $\lambda(a_2) = \lambda(x_2)$. But in that case, we can simply interchange the colors of $a_1$ and $a_2$, and obtain a proper $(k + 1)$-coloring $\lambda$ of $G$. Furthermore, $\lambda$ can be transformed to $\lambda$ in $O(|E(D)|)$ time.

Case 3. $|I(D')| = k$ and $I(D')$ consists of pick vertices of some $D'' \in \mathcal{D}(G)$. Now $D''$ is a $(k, k)$-diamond, because there exists a perfect matching between $C(D')$ and $P(D'')$. Thus $s = k$ and $|E(D')| \leq |E(D'')|$ (because $D'$ is a $(k, i)$-diamond where $i \leq k = s$). If Cases 1 or 2 are satisfied for $D''$, we set $D = D''$ and apply the constructions described in these cases for $D$ and obtain $G^\ast$ with required properties. Otherwise $|I(D'')| = k$ and $I(D'')$ consists of pick vertices of some $D''' \in \mathcal{D}(G)$. We consider two subcases:
Case 3.1. $D'' = D'$. Then vertices of $D'$ and $D''$ induce a component $G'$ of $G$. In this case we take $D := D'$ and $G^* := G - G'$. Notice that $G^*$ is a $(k, s)$-dart graph, $|E(G^*)| + 2|E(D')| = |E(G)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{D', D''\}$. Moreover, we can construct a $(k + 1)$-coloring of $G'$ in $O(k)$ time: just color all vertices of $P(D')$ and $P(D'')$ by the color $k + 1$, and assign colors $1, \ldots, k$ to the vertices of $C(D')$ and $C(D'')$.

Case 3.2. $D'' \neq D'$. In this case we take $D := D''$ and set $G^*$ to be the graph we obtain by removing the vertices of $D''$ and inserting a perfect matching between $C(D')$ and $P(D'')$. Then $G^*$ is a $(k, s)$-dart graph, $|E(G^*)| + |E(D)| = |E(G)|$ and $\mathcal{D}(G^*) = \mathcal{D}(G) \setminus \{D\}$. Let $\lambda^*$ be a $(k + 1)$-coloring of $G^*$. Then $\lambda^*$ assigns the same color $c$ to all vertices of $P(D'')$. Assign $c$ also to all vertices of $P(D'')$ and to each of the vertices of $C(D'')$ an unique color from $\{1, \ldots, k + 1\} \setminus \{c\}$. This gives a required coloring of $G$.

Clearly, we can check in $O(k)$ time whether $I(D')$ has cardinality $k$ or satisfies the conditions required in Cases 1, 2, 3.1, and 3.2. Thus all reductions from $G$ to $G^*$ and transformations of $k + 1$-colorings of $G^*$ to $k + 1$-colorings of $G$ can be done in $O(|E(D)|)$ time. This implies the statement. \hfill \square

Notice that $G^*$ from Lemma 4 also satisfy $|V(G^*)| \leq |V(G)| - |V(D)|$.

Now we are ready to prove the main result.

Theorem 1 Let $G$ be a $(k, s)$-dart graph where $s \geq 2$ and $k \geq \max\{3, s\}$ are arbitrary but fixed integers. Then $G$ is $(k + 1)$-colorable if and only if it has no component isomorphic to $K_{k+2}$. Furthermore, if $G$ is $(k + 1)$-colorable, then a $(k + 1)$-coloring of $G$ can be constructed in $O(|E(G)|)$ time.

Proof. The necessity of the first part of the theorem is trivial. We prove sufficiency and the second part of the theorem. Let $G$ be a $(k, s)$-dart graph. We can check in $O(|E(G)|)$ (linear) time whether $G$ is $K_{k+2}$-free. Analogously, we can find the set $\mathcal{D}(G)$ in linear time. Consequently, by means of Lemma 4 we can create in linear time a $K_{k+2}$-free graph $G'$ without vertices of degree more than $k + 1$ such that any $(k + 1)$-coloring of $G'$ can be transformed into a $(k + 1)$-coloring of $G$ in linear time. By [7] (see also [9, 6]), a $(k + 1)$-coloring of $G'$ can be found in linear time. \hfill \square

Notice that if $v$ is a vertex of a $(k, s)$-dart graph $G$ of degree at least $k + 2$ and $N(v)$ is the set of its incident vertices, then the graph induced by $N(v) \cup \{v\}$ is a $(k, i)$-diamond $(2 \leq i \leq s)$ with a possible pending edge. A similar property have central vertices of $G$ of degree $k + 1$. Thus the problem to find $\mathcal{D}(G)$ in $G$ is much easier then to find a maximal clique in a graph (a known NP-hard problem, see [3]). Also it is a trivial problem to determine in time $O(|E(G)|)$ whether a graph $G$ is a $(k, s)$-dart graph (where $k$ is arbitrary but fixed integer $\leq |V(G)|$).

5 NP-Completeness

In this section we show that Theorem 1 cannot be extended for $(k, s)$-dart graphs where $s > k \geq 2$ unless $P = NP$. 

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We need some more notation. Take \( n \) vertex disjoint copies of \((k, k+1)\)-diamonds \( D_1, \ldots, D_n \), \( k, n \geq 2 \). For \( i = 1, \ldots, n \), denote by \( v_{i,1}, \ldots, v_{i,k} \) and \( u_{i,1}, \ldots, u_{i,k+1} \) the central and pick vertices of \( D_i \), respectively. Add \( nk \) new edges \( v_{i,j}u_{i+1,j}, i = 1, \ldots, n, j = 1, \ldots, k \) (considering the sum \( i + 1 \mod n \)). Then the resulting graph is called a \((n, k+1)\)-bracelet and vertices \( u_{i,k+1}, \ldots, u_{n,k+1} \) are called its connectors. An example of a \((4, 3)\)-bracelet with connectors \( u_{1,3}, \ldots, u_{4,3} \) is in Figure 2.

![Figure 2: A (4, 3)-bracelet.](image)

**Lemma 5** Let \( G \) be a \((n, k+1)\)-bracelet, \( n, k \geq 2 \). Then in any \((k+1)\)-coloring of \( G \), all connectors of \( G \) have the same color.

**Proof.** By the above construction, \( G \) is composed from \( n \) vertex disjoint copies of \((k, k+1)\)-diamonds \( D_1, \ldots, D_n \). Consider a \((k+1)\)-coloring of \( G \). For every \( i \in \{1, \ldots, n\} \), the central vertices of \( D_i \) form a clique of order \( k \), whence must be colored by \( k \) different colors, and thus all pick vertices of \( D_i \) have the same color. Furthermore, each central vertex of \( D_i \) is adjacent with a pick vertex of \( D_{i+1} \). Therefore all vertices from \( P(D_1) \cup \ldots \cup P(D_n) \) have the same color, thus also the connectors of \( G \). \( \square \)

We study complexity of the following problem.

\((k, s)\)-DART-\((k+1)\)-COL

**Instance:** A \((k, s)\)-dart graph \( G \).

**Question:** Is \( G \) \((k+1)\)-colorable?

**Theorem 2** The problem \((k, s)\)-DART-\((k+1)\)-COL, \( k \geq 2 \), is

(a) NP-complete for \( s > k \),

(b) solvable in linear time for \( 2 \leq s \leq k \).

Claim (b) holds true by Theorem 1 for \( k \geq 3 \) and by [6, Theorem 4.3] for \( k = 2 \).

We prove (a). Let \( G \) be a graph. Replace each vertex \( v \) of \( G \) of degree \( \geq 2 \) by a \((d_G(v), k+1)\)-bracelet \( H_v \). Let \( H_v \) be an isolated vertex if \( d_G(v) = 1 \). Each edge \( uv \) of \( G \) replace by an edge joining a connector of \( H_v \) with a connector of \( H_u \) so that each connector is attached to at most one new edge. Denote the resulting graph by \( G' \). Clearly, \( G' \) is a \((k, k+1)\)-dart graph. From Lemma 5 it follows that by any \((k+1)\)-coloring \( G' \), all connectors of \( H_v, v \in V(G) \), must be colored by the same color. Hence
$G'$ is $(k+1)$-colorable if and only if $G$ is so. Thus the problem whether a $(k, k+1)$-dart graph is $k+1$-colorable can be polynomially reduced to the problem of $(k+1)$-coloring. This problem is NP-complete for every fixed $k \geq 2$ by Garey and Johnson [3, GT4]. This proves claim (a). □

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**References**


