On a conjecture about Wiener index in iterated line graphs of trees

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Abstract

Let $G$ be a graph. Denote by $L^i(G)$ its $i$-iterated line graph and denote by $W(G)$ its Wiener index. There is a conjecture which claims that there exists no nontrivial tree $T$ and $i \geq 3$, such that $W(L^i(T)) = W(T)$, see [5]. We prove this conjecture for trees which are not homeomorphic to the claw $K_{1,3}$ and $H$, where $H$ is a tree consisting of 6 vertices, 2 of which have degree 3.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. For any two of its vertices, say $u$ and $v$, by $d(u, v)$ we denote the distance from $u$ to $v$ in $G$. The Wiener index of $G$, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of $G$. Wiener index was introduced by Wiener in 1947, see [15]. In the next decades, it was intensively studied by chemists, as it is related to many physical properties of organical molecules, see [9]. Graph theorists reintroduced this parameter as the distance in 1970 and transmission in 1984, see [6] and [14], respectively. Recently, graph theoretic aspects of Wiener index are intensively studied, see e.g. [7] and [8], or surveys [3] and [4].

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By definition, if $G$ has a unique vertex, i.e., if $G = K_1$, then $W(G) = 0$. In this case we say that the graph $G$ is trivial. We set $W(G) = 0$ also when the set of vertices (and therefore also the set of edges) of $G$ is empty.

The line graph of $G$, $L(G)$, has vertex set identical with the set of edges of $G$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0 \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1], the following theorem was proved.

**Theorem 1.1** [1] If $T$ is a tree on $n$ vertices, then $W(L(T)) = W(T) - \binom{n}{2}$.

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees $T$ satisfying $W(L^2(T)) = W(T)$, see for example [2]. In [5] the following conjecture was posed (see also [3]).

**Conjecture 1.2** [5] Let $T$ be a nontrivial tree and $i \geq 3$. Then $W(L^i(T)) \neq W(T)$.

Denote by $P_n$ a path on $n$ vertices. If $n \geq 2$, then $W(P_n) > W(P_{n-1})$. Since $L(P_n) = P_{n-1}$ if $n \geq 2$, while $L(P_1)$ is an empty graph, it follows that $W(L^i(P_n)) < W(P_n)$ for every $i \geq 1$ provided that $n \geq 2$. Hence, Conjecture 1.2 is trivially true for paths of length at least 1.

In [11], we prove that for every graph $G$ the function $W(L^i(G))$ is convex in variable $i$. The following corollary is a straightforward consequence of this fact.

**Corollary 1.3** Let $T$ be a tree such that $W(L^3(T)) > W(T)$. Then for every $i \geq 3$ the inequality $W(L^i(T)) > W(T)$ holds.

Let $G$ be a graph. A pendant path (or a ray for short) $R'$ in $G$ is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in $G$. Observe that if $R'$ has length $t$, $t \geq 2$, then the edges of $R'$ correspond to vertices of a ray $R$ in $L(G)$ of length $t - 1$. In [11] we proved the following theorem.

**Theorem 1.4** [11] Let $T$ be a tree distinct from a path and the claw $K_{1,3}$ such that all of its rays have length 1. Then $W(L^3(T)) > W(T)$.

Here we extend this statement to trees with arbitrarily long rays. Denote by $H$ a tree on 6 vertices, two of which have degree 3 and four of which have degree 1. (That is, $H$ is the graph which "looks" like the letter H.) The main result of this paper is the following theorem.
**Theorem 1.5** Let $T$ be a tree not homeomorphic to a path, claw $K_{1,3}$ and $H$. Then $W(L^3(T)) > W(T)$.

Recall that graphs $G_1$ and $G_2$ are homeomorphic if and only if the graphs obtained from them by repeatedly removing a vertex of degree 2 (and making its two neighbours adjacent) are isomorphic. Combining Corollary 1.3 and Theorem 1.5 we obtain the following corollary, which proves Conjecture 1.2 for the trees $T$ satisfying the assumption in Theorem 1.5.

**Corollary 1.6** Let $T$ be a tree not homeomorphic to a path, claw $K_{1,3}$ and $H$. Then $W(L_i(T)) > W(T)$ for every $i \geq 3$.

We remark that trees homeomorphic to the claw $K_{1,3}$ and the graph $H$ are considered in forthcoming papers, see [12, 13].

For a tree $T$, let $D(T) = W(L^3(T)) - W(T)$. We prove $D(T) > 0$ by induction on the length of the longest ray in $T$. By Theorem 1.4, $D(T) > 0$ if the longest ray has length 1. Now we describe the induction step:

We suppose that $D(T) > 0$ for all trees rays of which have length at most $l + 1$. We would like to extend this statement to trees with rays of length at most $l + 2$. Let $a'$ be the last vertex of a ray of length $l + 1$ in $T$, $l \geq 0$. Since we extend only one ray in turn, namely the ray terminating at $a'$, we assume that all rays of $T$ have lengths at most $l + 2$. Add to $T$ one new vertex $b'$ and the edge $a'b'$, and denote the resulting tree by $T^*$. Denote by $a$ the edge of $T$ containing $a'$ and denote by $b$ the edge $a'b'$. Then $ab$ is an edge of $L(T^*)$ and the degree of $b$ is 1 in $L(T^*)$. Moreover, $a$ is an endvertex of a ray of length $l$ in $L(T)$ and $b$ is an endvertex of a ray of length $l + 1$ in $L(T^*)$; see Figure 1. By the assumption, all rays of $L(T)$ have lengths at most $l + 1$. Let

$$\Delta T = D(T^*) - D(T).$$

In the next section we present an exact formula for $\Delta T$. In Section 3 we prove $\Delta T \geq 0$ and this will establish Theorem 1.5 (for more detailed explanation see the proof of Theorem 1.5).

![Figure 1: Description of the induction step for $l = 2$.](image-url)
Now we introduce notation used throughout the paper. Since we work repeatedly with line graphs of trees, we simplify the notation and write $LG$ rather than $L(G)$. The degree of a vertex $z$ is denoted by $d_z$. If there are more graphs containing the vertex $z$, then $d_z$ denotes the degree of $z$ in $LT$ (whatever the meaning of $T$ at that point will be). Similarly, by $d(z, w)$ we denote the distance from $z$ to $w$, and this distance is preferably considered in $LT$ (rather than $T$). When no confusion is likely, any path starting at $u$ and terminating at $v$ will be denoted by $u - v$.

## 2 Preliminaries

Analogously as a vertex of $L(G)$ corresponds to an edge of $G$, a vertex of $L^2(G)$ corresponds to a path of length two in $G$. For $x \in V(L^2(G))$, we denote by $B_2(x)$ the corresponding path in $G$. For two subgraphs $S_1$ and $S_2$ of $G$, by $d(S_1, S_2)$ we denote the shortest distance in $G$ between a vertex of $S_1$ and a vertex of $S_2$. If $S_1$ and $S_2$ share $s$ edges, then we set $d(S_1, S_2) = -s$.

Let $x$ and $y$ be two vertices of $L^2(G)$, such that $u$ is the center of $B_2(x)$ and $v$ is the center of $B_2(y)$. Then $d_{L^2(G)}(x, y) = d(B_2(x), B_2(y)) + 2$; see [10, 11].

Let $u, v \in V(G)$, $u \neq v$. Denote by $\beta_i(u, v)$ the number of pairs $x, y \in V(L^2(G))$, with $u$ being the center of $B_2(x)$ and $v$ being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u, v) - 2 + i$. Since $d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v)$, we see that $\beta_i(u, v) = 0$ for all $i \notin \{0, 1, 2\}$. In [11] the following statement was proved:

**Proposition 2.1** Let $G$ be a connected graph. Then

\[
W(L^2(G)) = \sum_{u \neq v} \left[ \binom{d_u}{2} \binom{d_v}{2} d(u, v) + \beta_1(u, v) + 2\beta_2(u, v) \right] + \sum_u \left[ 3 \binom{d_u}{3} + 6 \binom{d_u}{4} \right],
\]

where the first sum runs through all unordered pairs of distinct vertices $u, v \in V(G)$ and the second one runs through all $u \in V(G)$.

We apply Proposition 2.1 to line graphs of trees. Let us recall the structure of these graphs. For any tree $F$, the graph $LF$ consists of cliques in the following sense: Denote by $C(LF)$ the set of maximal cliques of $LF$. Then every vertex of $LF$ belongs to at most two cliques from $C(LF)$; each pair of cliques from $C(LF)$ intersects in at most one vertex; and the cliques of $C(LF)$ have a “tree structure”, i.e., there are no cliques $C_0, C_1, \ldots, C_{t-1}, t \geq 3$, such that $C_i$ and $C_{i+1}$ have nonempty intersection, $0 \leq i \leq t - 1$, the addition being modulo $t$.  


We start with an exact formula for $\Delta T$. For $u \in V(LT) \setminus \{a\}$, let
\[
h_{LT}(u) = \left(\left(\frac{d_u}{2}\right)d_u - 1\right)d(u,a) + (d_u - 1)\left(d_u - \frac{1}{2}d_u\right) - 2 - \phi(u,a),
\]
where
\[
\phi(u,a) = \begin{cases} 
(d_u - 1)(d_u - 2) & \text{if } d(u,a) = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Proposition 2.2 For a nontrivial tree, the following equality holds:
\[
\Delta T = \sum_u h_{LT}(u) + \frac{1}{2}d_a(d_a - 1)(2d_a - 1) - 3,
\]
where the sum is taken over all vertices $u \in V(LT) \setminus \{a\}$.

**Proof** Let $F$ be a tree and let $u$ and $v$ be distinct vertices of $LF$. Consider vertices $x, y \in V(L^2(LF))$ such that $u$ is the center of $B_2(x)$ and $v$ is the center of $B_2(y)$. Due to the clique structure of $LF$, there is a unique shortest $u - v$ path in $LF$. Denote this path by $u = a_0, a_1, \ldots, a_t = v$. If $d(B_2(x), B_2(y)) = d(u,v) - 2$, then we must have $a_1 \in V(B_2(x))$ and $a_{t-1} \in V(B_2(y))$. There are $(d_a - 1)$ ways to choose the other endvertex of $B_2(x)$, and there are $(d_v - 1)$ ways to choose the other endvertex of $B_2(y)$. Hence, $\beta_0(u,v) = (d_u - 1)(d_v - 1)$.

Now we find $\beta_1(u,v)$. We distinguish two cases: $d(u,v) \geq 2$ and $d(u,v) = 1$.

Suppose first $d(u,v) \geq 2$. In this case $u$ and $v$ do not belong to a common clique from $C(LF)$. If $d(B_2(x), B_2(y)) = d(u,v) - 1$, then either $a_1 \in V(B_2(x))$ or $a_{t-1} \in V(B_2(y))$, but not both. In the first case we obtain $(d_u - 1)\binom{d_u - 1}{2}$ and in the second case $(d_u - 1)(d_v - 1) \binom{d_v - 1}{2}$ pairs $x, y$ and $y, x$. Thus
\[
\beta_1(u,v) = (d_u - 1)\binom{d_v - 1}{2} + (d_u - 1)\binom{d_u - 1}{2}.
\]

Suppose now that $d(u,v) = 1$. In this case, $u$ and $v$ belong to a common clique. All pairs $x, y$ mentioned in the previous case contribute to $\beta_1(u,v)$, but we have to add pairs $x, y$ such that $v \notin V(B_2(x))$, $u \notin V(B_2(y))$ and $d(B_2(x), B_2(y)) = d(u,v) - 1 = 0$. For these pairs, the paths $B_2(x)$ and $B_2(y)$ share at least one of their endvertices. Denote by $\alpha_{LF}(u,v)$ the number of these extra pairs. Then
\[
\beta_1(u,v) = (d_u - 1)\binom{d_v - 1}{2} + (d_u - 1)(d_v - 1) \alpha_{LF}(u,v).
\]

Since we do not need to evaluate $\alpha_{LF}(u,v)$ in general, we postpone this computation until later. To simplify the notation, we set $\alpha_{LF}(u,v) = 0$ for all pairs $u, v$ such that $d(u,v) \geq 2$. 

5
We have \(\binom{d_u}{2}\binom{d_v}{2}\) pairs \(x, y \in V(L^2(LF))\) such that \(u\) is the center of \(B_2(x)\) and \(v\) is the center of \(B_2(y)\). Since
\[
\binom{d_u}{2}\binom{d_v}{2} = (d_u-1)(d_v-1) + (d_u-1)\binom{d_v-1}{2} + \binom{d_u-1}{2}\binom{d_v-1}{2},
\]
we obtain \(\beta_2(u, v) = \binom{d_u-1}{2}\binom{d_v-1}{2} - \alpha_{LF}(u, v)\). By Proposition 2.1, it follows that
\[
W(L^2(LF)) = \sum \left[ \binom{d_u}{2}\binom{d_v}{2}d(u, v) + (d_u-1)\binom{d_v-1}{2} + \binom{d_u-1}{2}(d_v-1) + 2\binom{d_u-1}{2}\binom{d_v-1}{2} - \alpha_{LF}(u, v) \right] + \sum \left[ 3\binom{d_u}{3} + 6\binom{d_u}{4} \right].
\] (2)

Now we evaluate \(W(L^3(T^*)) - W(L^3(T)) = W(L^2(L^2(T^*))) - W(L^2(LT))\); see the notation following Corollary 1.6. The graph \(LT^*\) has one more vertex than \(LT\), namely the vertex \(b\) of degree 1, and the degree of \(a\) increased by 1 to \(d_a + 1\) in \(LT^*\). Therefore, all the terms of (2) for pairs \(u, v\) which do not contain neither \(a\) nor \(b\) cancel out in \(W(L^2(L^2(T^*))) - W(L^2(LT))\). However, we need to subtract the terms for pairs \(u, a\) in \(LT\) and to add the terms for pairs \(u, a\) in \(LT^*, \ u \in V(LT) \setminus \{a\}\). We can ignore the terms containing \(b\) in \(LT^*\), since the degree of \(b\) is 1, and thus \(b\) cannot be a center of \(B_2(y)\) for any \(y \in V(L^2(L^2(T^*)))\). (Observe that all terms of (2) are 0 if one of the vertices has degree 1.) As regards the second sum in (2), we have to subtract the term corresponding to \(a\) in \(LT\) and add the terms corresponding to \(a\) and \(b\) in \(LT^*\), the second one being 0 since the degree of \(b\) is 1 in \(LT^*\). Denote by \(\Delta \alpha(u, v)\) the difference \(\alpha_{LT^*}(u, v) - \alpha_{LT}(u, v)\) and denote by \(\Delta W L^2\) the difference \(W(L^2(L^2(T^*))) - W(L^2(LT))\). By (2), it follows that
\[
\Delta W L^2 = -\sum \left[ \binom{d_u}{2}\binom{d_a}{2}d(u, a) + (d_u-1)\binom{d_a-1}{2} + \binom{d_u-1}{2}(d_a-1) + 2\binom{d_u-1}{2}\binom{d_a-1}{2} - \alpha_{LT}(u, a) \right] + \sum \left[ \binom{d_u}{2}\binom{d_a+1}{2}d(u, a) + (d_u-1)\binom{d_a}{2} + \binom{d_u-1}{2}(d_a+1) + 2\binom{d_u-1}{2}\binom{d_a}{2} - \alpha_{LT^*}(u, a) \right] -3\binom{d_a}{3} - 6\binom{d_a}{4} + 3\binom{d_a+1}{3} + 6\binom{d_a+1}{4}.
\]
\[
\begin{align*}
\sum_u \left( \left( \frac{d_u}{2} \right) d_u d(u, a) + (d_u - 1)(d_u - 1) \\
+ \left( \frac{d_u - 1}{2} \right) + 2 \left( \frac{d_u - 1}{2} \right)(d_a - 1 - \Delta\alpha(u, v)) \right) \\
+ \frac{1}{4} d_a (d_a - 1) \left[ -2(d_a - 2) - (d_a - 2)(d_a - 3) \\
+ 2(d_a + 1) + (d_a + 1)(d_a - 2) \right] \\
= \sum_u \left( \left( \frac{d_u}{2} \right) d_u d(u, a) + (d_u - 1) \left( d_u d_a - d_a - \frac{1}{2} d_u \right) - \Delta\alpha(u, a) \right) \\
+ \frac{1}{2} d_a (d_a - 1)(2d_a - 1). \tag{3}
\end{align*}
\]

Now we determine \( \Delta\alpha(u, a) \). For \( u \in V(LT) \setminus \{a\} \), the distance from \( u \) to \( a \) in \( LT \) is the same as in \( LT^* \). Therefore \( \Delta\alpha(u, a) = \alpha_{LT^*}(u, a) - \alpha_{LT}(u, a) = 0 - 0 = 0 \) if \( d(u, a) \geq 2 \). If \( d(u, a) = 1 \), then in order to evaluate \( \alpha_{LT^*}(u, a) - \alpha_{LT}(u, a) \) we need to count pairs \( x, y \) such that \( b \in V(B_2(y)) \). Denote by \( C \) the clique of \( C(LT) \) containing both \( a \) and \( u \). The order of \( C \) is \( d_a + 1 \). We distinguish two cases.

- **Both endvertices of \( B_2(x) \) are in \( C \):** We have \( \binom{d_a - 1}{2} \) choices for \( B_2(x) \) in this case since \( a \notin V(B_2(x)) \). For each of these choices there are two choices for \( B_2(y) \) such that \( B_2(x) \) and \( B_2(y) \) share an endvertex and \( b \in V(B_2(y)) \). Therefore there are \( 2 \binom{d_a - 1}{2} \) pairs \( x, y \) contributing to \( \Delta\alpha(u, v) \) in this case.

- **Only one endvertex of \( B_2(x) \) is in \( C \):** For this vertex we have \( d_a - 1 \) choices, since \( a \notin V(B_2(x)) \), and for the other endvertex of \( B_2(x) \) we have \( d_u - d_a \) choices. In this case, for every \( x \) there is a unique \( y \) such that \( B_2(x) \) and \( B_2(y) \) share an endvertex and \( b \in V(B_2(y)) \). Thus there are \( (d_a - 1)(d_u - d_a) \) pairs \( x, y \) contributing to \( \Delta\alpha(u, v) \) in this case.

Hence,

\[
\Delta\alpha(u, v) = 2 \binom{d_a - 1}{2} + (d_a - 1)(d_u - d_a) = (d_a - 1)(d_u - 2) = \phi(u, a). \tag{4}
\]

Now we evaluate \( W(T^*) - W(T) \). If \( F \) is a tree with \( n_0 \) vertices, then \( W(LF) = W(F) - \binom{n_0}{2} \), by Theorem 1.1. Denote by \( n_1 \) the number of vertices of \( LF \). Since \( n_1 = n_0 - 1 \), we have \( W(F) = W(LF) + \binom{n_1 + 1}{2} \). Denote by \( n \) the number of vertices of \( LT \). Then

\[
W(T^*) - W(T) = W(LT^*) + \binom{n + 2}{2} - W(LT) - \binom{n + 1}{2} \\
= W(LT^*) - W(LT) + n + 1.
\]

\[7\]
In $W(LT^*) - W(LT)$, all terms for pairs $u, v$ which do not contain $b$ will cancel out. Therefore

$$W(T^*) - W(T) = \sum_u d(u, b) + d(a, b) + n + 1$$

$$= \sum_u (d(u, a) + 1) + 1 + \sum_u 1 + 2$$

$$= \sum_u (d(u, a) + 2) + 3.$$  \hspace{1cm} (5)

where the sum goes once again through $n - 1$ vertices $u \in V(LT) \setminus \{a\}$.

Since $\Delta T = D(T^*) - D(T) = W(L^3(T^*)) - W(T^*) - W(L^3(T)) + W(T) = \Delta W L^2 - (W(T^*) - W(T))$, combining (3), (4) and (5) we obtain the required result. □

3 Proof of Theorem 1.5

We prove that $\Delta T \geq 0$ for every tree $T$ which is not homeomorphic to a path, claw $K_{1,3}$ or the graph $H$. Let $l, a', b', a, b, T^*$ and $\Delta T$ be as in the discussion following Corollary 1.6. As explained there, we proceed by induction on $l$.

First we prove that $\Delta T \geq 0$ for the case $l = 0$. In this case, $a'$ is adjacent to a vertex of degree at least 3 in $T$, implying that in $LT$ we have $d_a \geq 2$.

Let $v$ be an endvertex of a ray (a pendant path) $R$ in $LT$, i.e., $d_v = 1$. By $\overline{v}$ we denote the first vertex of $R$, i.e., a vertex at shortest distance to $v$ whose degree is at least 3. Due to the clique structure of $LT$ described after Proposition 2.1, we have:

Observation 3.1 If $u$ and $v$ are distinct vertices of degree 1 in $LT$, then $\overline{u} \neq \overline{v}$.

We use Observation 3.1 repeatedly in the following proofs.

Lemma 3.2 Let $T$ be a tree different from a path in which all rays have length at most $l + 2$, and let $l = 0$. Then $\Delta T \geq 0$.

PROOF We find a lower bound for $\sum_u h_{LT}(u)$. Consider four cases:

- $d_a = 1$: Then $d(u, a) > 1$, and thus $h_{LT}(u) = -d(u, a) - 2$ by (1).

- $d_a = 2$: Since $(d_a - 1)(d_a - 2) = 0$, we see that $\phi(u, a) = 0$ in this case as well. By (1) we obtain

$$h_{LT}(u) = (d_a - 1)d(u, a) + d_a - 3 \geq d_a - 1 + d_a - 3 = 2d_a - 4 \geq 0$$

since $d_a \geq 2$.  

• $d_u \geq 3$ and $d(u, a) \geq 2$: By (1) it follows that

$$h_{LT}(u) = \left(\left(\frac{d_u}{2}\right)d_a - 1\right)d(u, a) + (d_a - 1)\left(d_u d_a - d_a - \frac{1}{2}d_u\right) - 2$$

$$\geq 5d(u, a) + (d_a - 1)\left[\frac{1}{2}(d_u - 2) + d_a(d_a - 1)\right] - 2$$

$$\geq 5d(u, a) + 5 - 2$$

$$\geq d(u, a) + 11$$

as $d_u \geq 3$, $d_a \geq 2$, and $d(u, a) \geq 2$.

• $d_u \geq 3$ and $d(u, a) = 1$: By (1) it follows that

$$h_{LT}(u) = \left(\left(\frac{d_u}{2}\right)d_a - 1\right)d(u, a) + (d_a - 1)\left(d_u d_a - d_a - \frac{1}{2}d_u\right) - 2 - (d_a - 1)(d_u - 2)$$

$$\geq 5d(u, a) + \frac{d_u}{2}d_a - \frac{1}{2}d_u^2 - 3d_u d_a + \frac{3}{2}d_a + 3d_a - \frac{3}{2} - \frac{5}{2}$$

$$= 5d(u, a) + \frac{1}{2}\left[(2d_a - 1)(d_u(d_u - 3) + 3) - 5\right]$$

$$\geq d(u, a) + 6$$

as $d_u \geq 3$, $d_a \geq 2$, and $d(u, a) = 1$.

Hence,

$$h_{LT}(u) \geq \begin{cases} -d(u, a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ d(u, a) + 6 & \text{if } d_u \geq 3. \end{cases} \quad (6)$$

Since $l = 0$, all rays of $T$ have length at most 2, implying that all rays of $LT$ have length at most 1. Hence, if $d_u = 1$, then $d(u, \overline{u}) = 1$ in $LT$. Thus

$$h_{LT}(u) + h_{LT}(\overline{u}) \geq -d(u, a) - 2 + d(\overline{u}, a) + 6 = -d(\overline{u}, a) - 3 + d(\overline{u}, a) + 6 \geq 0.$$

Denote by $V_1$ the set of vertices of degree 1 in $V(LT) \setminus \{a\}$. By Observation 3.1, $\overline{u} \neq \overline{v}$ whenever $u, v \in V_1$, $u \neq v$. Therefore, by (6) it follows that

$$\sum_u h_{LT}(u) \geq \sum_{u \in V_1} \left(h_{LT}(u) + h_{LT}(\overline{u})\right) \geq 0.$$

Since $d_a \geq 2$, we have $\frac{1}{2}d_a(d_a - 1)(2d_a - 1) \geq 3$, implying that

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2}d_a(d_a - 1)(2d_a - 1) - 3 \geq 0,$$
Now we prove that $\Delta T \geq 0$ for all $l \geq 1$, i.e., from now on we consider $l \geq 1$. In this case $\phi(u, a) = 0$ since $d_a = 1$, which simplifies the expression for $h_{LT}(u)$ in (1). The problem is that $h_{LT}(u) < 0$ even if $d_a = 2$, suggesting that we need sharper estimations. We prove $\Delta T \geq 0$ by induction on the number of vertices of degree at least 3 in $T$.

Let $G$ be a graph. A path of length at least one in $G$ is interior path if its endvertices have degrees both at least 3, its interior vertices (if any) have degree 2 in $G$, and its edges are bridges of $G$. In the next lemma we show that it suffices to prove $\Delta T \geq 0$ for trees whose interior paths have lengths at most 2, i.e., we reduce the class of trees for which we need to prove $\Delta T \geq 0$.

**Lemma 3.3** Let $T^*$ be obtained from $T$ by subdividing one edge of an interior path of length $t$, $t \geq 2$, and let $l \geq 1$. Then $\Delta T^* \geq \Delta T$.

**Proof** Denote by $P'$ the interior path of $T$, whose edge was subdivided to obtain $T^*$. Since $P'$ has length $t \geq 2$, the edges of $P'$ form an interior path $P$ of length $t - 1 \geq 1$ in $LT$. Obviously, $LT^*$ can be obtained from $LT$ by subdividing one edge of $P$. Denote by $e$ the endvertex of $P$, which has among the vertices of $P$ the greatest distance from $a$. Let $LT^*$ be obtained from $LT$ by subdividing that edge of $P$ which is incident to $e$. Denote the new vertex by $w$. Observe that for every vertex $u \in V(LT)$, the degree of $u$ in $LT$ is the same as its degree in $LT^*$.

Since the degree of $a$ is the same in $LT^*$ as in $LT$, namely 1, by Proposition 2.2 it suffices to show that $\sum_{u \in V(LT^*) \setminus \{a\}} h_{LT^*}(u) \geq \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u)$. We distinguish three cases.

- **$u$ is a vertex of $LT$ such that $e$ does not lie on $u - a$ path in $LT$:** Then $d_{LT^*}(u, a) = d_{LT}(u, a)$, implying that $h_{LT^*}(u) = h_{LT}(u)$ and $h_{LT^*}(u) - h_{LT}(u) = 0$, see (1).

- **$u$ is a vertex of $LT$, such that $e$ lays on $u - a$ path in $LT$:** Then $d_{LT^*}(u, a) = d_{LT}(u, a) + 1$, implying that $h_{LT^*}(u) - h_{LT}(u) = (d_a) - 1$ since $d_a = 1$; see (1). Thus $h_{LT^*}(u) - h_{LT}(u) = -1$ if $d_a = 1$, $h_{LT^*}(u) - h_{LT}(u) = 0$ if $d_a = 2$ and $h_{LT^*}(u) - h_{LT}(u) \geq 2$ if $d_a \geq 3$.

- **$u = w$:** Since the degree of $w$ is 2 in $LT^*$, we see that $h_{LT^*}(w) = -2$, by (1).

Every vertex $u$ of degree 1 in $LT$ is an endvertex of a ray starting at vertex $\overline{a}$ of degree at least 3. By Observation 3.1, if $u$ and $v$ are distinct vertices of degree 1 in $LT$, then $\overline{u} \neq \overline{v}$. Denote by $V_e$ the set of vertices $u$ of $LT$ such that $d_u = 1$ and $e$ lays on $u - a$ path. Observe that $e \neq \overline{a}$ for any $u \in V_e$. 

10
Denote \( \Delta h = \sum_{u \in V(LT) \setminus \{a\}} h_{LT^*}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u) \). By the analysis above only vertices of \( V_e \cup \{w\} \) contribute negative value to \( \Delta h \). Therefore

\[
\Delta h \geq \sum_{u \in V_e} \left( h_{LT^*}(u) - h_{LT}(u) \right) + \left( h_{LT^*}(\bar{a}) - h_{LT}(\bar{a}) \right) + \left( h_{LT^*}(c) - h_{LT}(c) \right) + h_{LT^*}(w)
\]

\[
\geq \sum_{u \in V_e} (-1 + 2) + 2 - 2 \geq 0.
\]

Hence \( \Delta T^* \geq \Delta T \).

Let \( F \) be a tree with a ray of length \( l + 1 \) terminating in the edge \( a \). Denote by \( S_{LF} \) the set of first edges of rays of \( F \). Then \( S_{LF} \) is also a set of vertices of \( LF \). These vertices have degree at least 3, with the exception when the corresponding edge is incident to vertices of degree 1 and 3 in \( F \). Let \( u \in S_{LF} \). If there is a ray in \( LF \) starting at \( u \), then denote by \( R_{LF}(u) \) the set of vertices (other than \( a \)) of this ray; otherwise set \( R_{LF}(u) = \{u\} \). Since \( l \geq 1 \), there is a ray in \( LF \) starting at \( \bar{a} \) and \( \bar{a} \neq s \). Observe also that \( R_{LF}(u) \cap R_{LF}(v) = \emptyset \) whenever \( u, v \in S_{LF} \), \( u \neq v \).

**Lemma 3.4** Let \( F \) be a tree ray of which have length at most \( l + 2 \), \( l \geq 1 \). Moreover, one ray of \( F \) has length exactly \( l + 1 \) and this ray terminates by the edge \( a \). Let \( c \in S_{LF} \) be a vertex of a clique from \( C(LF) \) of order \( r \geq 3 \). Then

\[
\sum_{u \in R_{LF}(c)} h_{LF}(u) \geq \begin{cases} \binom{r}{2} - 3 + \binom{r-1}{2} & \text{if } c = \bar{a} \\ \binom{r-1}{2} - 1 + \binom{r-2}{2} - 2 & \text{if } c \neq \bar{a} \text{ and } |R_{LF}(c)| = 1 \\ \binom{r}{2} - 2 - 3l + \binom{r-1}{2} - 5 & \text{if } c \neq \bar{a} \text{ and } |R_{LF}(c)| \geq 2. \end{cases}
\]

**Proof** We distinguish three cases.

- **\( c = \bar{a} \)**: Then \( R_{LF}(c) \) has one vertex of degree \( r \), namely \( c \) with \( d(c, a) = l \), and \( l - 1 \) vertices of degree 2. Since the degree of \( a \) is 1, by (1) we have

\[
\sum_{u \in R_{LF}(c)} h_{LF}(u) = \left( \binom{r}{2} - 1 \right) d(c, a) + \left( r - 1 \right) \left( \frac{r-2}{2} \right) - 2 + (l - 1)(-2)
\]

\[
= \left( \binom{r}{2} - 3 \right) l + \left( r - 1 \right). \]

- **\( c \neq \bar{a} \) and \( |R_{LF}(c)| = 1 \)**: Since the degree of \( c \) is \( r - 1 \), by (1) we obtain

\[
\sum_{u \in R_{LF}(c)} h_{LF}(u) = h_{LF}(c) = \left( \binom{r-1}{2} - 1 \right) d(c, a) + (r - 2) \left( \frac{r - 3}{2} \right) - 2.
\]

11
• \( c \neq \overline{a} \) and \( |R_{LF}(c)| \geq 2 \): Then \( R_{LF}(c) \) has one vertex of degree \( r \), namely \( c \), one vertex of degree 1 at distance at most \( d(c, a) + l + 1 \) from \( a \) and at most \( l \) vertices of degree 2 since all rays of \( LF \) have length at most \( l + 1 \). By (1) it follows that

\[
\sum_{u \in R_{LF}(c)} h_{LF}(u) \geq \left( \binom{r}{2} - 1 \right) d(c, a) + (r - 1) \left( \frac{r - 2}{2} \right) - 2
\]

\[
- (d(c, a) + l + 1) - 2 + l(-2)
\]

\[
= \left( \binom{r}{2} - 2 \right) d(c, a) - 3l + \left( \frac{r - 1}{2} \right) - 5.
\]

Before we state the lemmas necessary for the basis of induction, we give the proof of induction step. That is, we prove that if \( \Delta T^{gh} \geq 0 \) for every tree \( T^{gh} \) homeomorphic to \( T \) rays of which have lengths at most \( l + 2 \), then \( \Delta T^{gh} \geq 0 \) for all trees \( T^{gh} \) homeomorphic to \( T^{gh} \) rays of which have lengths at most \( l + 2 \), where \( T^{gh} \) is obtained from \( T \) by inserting a star at the end of one ray of \( T \) (of course, we cannot attach this star to \( a' \)).

Let \( R' \) be a ray of \( T \) which does not terminate at \( a' \). Remove \( R' \) from \( T \) and replace it by a path \( PR' \) of length \( i \), \( 1 \leq i \leq 2 \). Denote by \( c' \) the vertex of degree 1 in \( PR' \). Now attach to \( c' \) exactly \( j - 1 \) rays, each of length at most \( l + 2 \), and denote the resulting graph by \( T_{i,j} \), \( j \geq 3 \). In the next two lemmas we prove that \( \Delta T_{i,j} \geq 0 \).

**Lemma 3.5** Suppose that \( \Delta T^{gh} \geq 0 \) for all trees \( T^{gh} \) homeomorphic to \( T \) rays of which have lengths at most \( l + 2 \), \( l \geq 1 \), and in which the ray terminating in the edge \( a \) has length \( l + 1 \). Then \( \Delta T_{i,3} \geq 0 \) for all \( i \in \{1, 2\} \).

**Proof** Since \( \Delta T^{gh} \geq 0 \) for all trees homeomorphic to \( T \) rays of which have length at most \( l + 2 \), we may assume that the length of \( R' \) is exactly \( l + 2 \), \( l \geq 1 \). Then the
edges of $R'$ form a ray $R$ in $LT$ of length $l + 1$. Denote by $e$ the first vertex of $R$.
By (1) it follows that
\[
\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -2l - (d(e, a) + l + 1) - 2 = -d(e, a) - 3l - 3
\]
since $R_{LT}(e)$ has $l$ vertices of degree 2 and one vertex of degree 1 at distance $d(e, a) + l + 1$ from $a$. We distinguish two cases.

- **$i = 1$:** Then $PR'$ has length 1 and the unique edge of $PR'$ corresponds to the vertex $e$ in $LT_{1,3}$. In $LT_{1,3}$ the degree of $e$ is $d_e + 3 - 2 = d_e + 1$ since $e$ is in two cliques from $C(LT_{1,3})$, one of them has order $d_e$ and the other one has order 3. Denote by $c$ any one of the other two vertices of this clique of order 3. Since $d(c, a) \geq l + 2$, we see that $d(c, a) - 3l - 4 \geq -2l - 2$. Hence, by Lemma 3.4,
\[
\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq \begin{cases}
-2 & \text{if } |R_{LT_{1,3}}(c)| = 1 \\
-2l - 2 & \text{if } |R_{LT_{1,3}}(c)| \geq 2.
\end{cases}
\]
Since $-2l - 2 \leq -2$, we have $\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq -2l - 2$.
Denote
\[
\Delta h = \sum_{u \in V(LT_{1,3}) \setminus \{a\}} h_{LT_{1,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).
\]
In $\Delta h$ all terms cancel out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing $e$, the vertex $e$ itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that
\[
\Delta h \geq 2(-2l - 2) + \left(\left(\frac{d_e}{2} + 1\right) - 1\right)d(e, a) + d_e \left(\frac{d_e}{2} + 1\right) - 1 - 2
\]
\[
- \left(\left(\frac{d_e}{2} - 1\right) + d(e, a) - (d_e - 1)\left(\frac{d_e}{2} - 1\right) + 2 + (d(e, a) + 3l + 3)
\]
\[
\geq (d_e + 1)d(e, a) + (d_e - 1) - l - 1
\]
\[
\geq 4d(e, a) - l + 1 \geq 0
\]
since $d_e \geq 3$ and $d(e, a) \geq l + 1$. By Proposition 2.2, $\Delta T_{1,3} - \Delta T = \Delta h \geq 0$, implying that $\Delta T_{1,3} \geq \Delta T \geq 0$.

- **$i = 2$:** Then $PR'$ has length 2. One edge of $PR'$ corresponds to $e$, while the other corresponds to a vertex of degree 3, say $f$, in $LT_{2,3}$. Observe that the degree of $e$ is $d_e$ in $LT_{2,3}$ and the degree of $f$ is 3 in $LT_{2,3}$. Analogously as in the previous case, denote by $c$ any one of the two vertices of the triangle containing
Since $d(c, a) = d(e, a) + 2 \geq l + 3$, we see that $d(c, a) - 3l - 4 \geq -2l - 1$. Hence, by Lemma 3.4

$$\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{2,3}}(c)| = 1 \\ -2l - 1 & \text{if } |R_{LT_{2,3}}(c)| \geq 2. \end{cases}$$

Since $l \geq 1$ it follows that $-2l - 1 \leq -2$, implying that $\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq -2l - 1$. Denote

$$\Delta h = \sum_{u \in V(LT_{2,3}) \setminus \{a\}} h_{LT_{2,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In $\Delta h$ all terms cancel out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing $f$, the vertex $f$ itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that

$$\Delta h \geq 2(-2l - 1) + (2d(f, a) - 1) + (d(e, a) + 3l + 3) \geq 3d(e, a) - l + 2 \geq 0$$

since $d(f, a) = d(e, a) + 1$ and $d(e, a) \geq l + 1$. By Proposition 2.2, $\Delta T_{2,3} - \Delta T \geq \Delta h \geq 0$, implying that $\Delta T_{2,3} \geq \Delta T \geq 0$.

In both cases the inequality $\Delta T_{i,3} \geq 0$ holds, which completes the proof. \qed

Now we extend the previous lemma to trees $T_{i,j}$ with higher $j$.

**Lemma 3.6** Suppose that $\Delta T^h \geq 0$ for all trees $T^h$ homeomorphic to $T$ rays of which have lengths at most $l + 2$, $l \geq 1$, and in which the ray terminating in the edge $a$ has length $l + 1$. Then $\Delta T_{i,j} \geq 0$ for all $j \geq 4$ and $i \in \{1, 2\}$.

**Proof** We use the notation of the proof of Lemma 3.5. Analogously as in the proof of Lemma 3.5, assume that the length of $R'$ is $l + 2$, $l \geq 1$. Then again

$$\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -d(e, a) - 3l - 3.$$

Let $c$ be one of the $j - 1$ vertices of the clique of order $j$ obtained from the edges incident to $c'$, other than $e$ (in the case $i = 1$) or $f$ (in the case $i = 2$). By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq \begin{cases} \left(\binom{j - 1}{2} - 1\right)d(c, a) + \binom{j - 2}{2} - 2 & \text{if } |R_{LT_{i,j}}(c)| = 1 \\ \left(\binom{j}{2} - 2\right)d(c, a) - 3l + \binom{j - 1}{2} - 5 & \text{if } |R_{LT_{i,j}}(c)| \geq 2. \end{cases}$$
Since $j \geq 4$ and $d(c, a) \geq l + 2 \geq 3$, in any case we have $\sum_{u \in R_{LTi,j}(c)} h_{LTi,j}(u) \geq 0$.

Now if $i = 1$, then $h_{LTi,j}(e) - h_{LT}(e) \geq 0$ since the degree of $e$ is $d(e) + j - 2$ in $T_{i,j}$; see (1). On the other hand, if $i = 2$, then $h_{LTi,j}(e) = h_{LT}(e)$ while $h_{LTi,j}(f) \geq 0$, since the degree of $f$ is $j \geq 4$ in $T_{i,j}$. Hence

$$\Delta h = \sum_{u \in V(L_{Ti,j}) \setminus \{a\}} h_{LTi,j}(u) - \sum_{u \in V(L_{T}) \setminus \{a\}} h_{LT}(u) \geq (j - 1) \cdot 0 + 0 + d(e, a) + 3l + 3 \geq 0.$$ 

By Proposition 2.2, $\Delta T_{i,j} - \Delta T = \Delta h \geq 0$, showing that $\Delta T_{i,j} \geq \Delta T \geq 0$.

Now we prove $\Delta T \geq 0$ for the basis of induction. In all graphs in this basis, $a'$ is an endvertex of a ray of length $l + 1$ and $a$ is the edge incident with $a'$.

**Lemma 3.7** Let $T$ be a tree homeomorphic to a star $K_{1,k}$, $k \geq 4$, in which all rays have lengths at most $l + 2$, $l \geq 1$, and in which the ray terminating in the edge $a$ has length $l + 1$. Then $\Delta T \geq 0$.

**Proof** Here $|S_{LT}| = k$ and $\cup_{u \in S_{LT}} R_{LT}(u) = V(L_{T}) \setminus \{a\}$ where $R_{LT}(u) \cap R_{LT}(v) = \emptyset$ if $u \neq v$. Thus $\sum_{u} h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u))$. We prove that $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$.

Choose $c \in S_{LT}$. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \begin{cases} \binom{k}{2} - 3 \cdot l + \binom{k-1}{2} & \text{if } c = \overline{a} \\ \binom{k-1}{2} - 1 \cdot d(e, a) + \binom{k-2}{2} - 2 & \text{if } c \neq \overline{a} \text{ and } |R_{LT}(c)| = 1 \\ \binom{k}{2} - 2 \cdot d(e, a) - 3l + \binom{k-1}{2} - 5 & \text{if } c \neq \overline{a} \text{ and } |R_{LT}(c)| \geq 2. \end{cases}$$

Since $d(e, a) = l + 1$ in the last two cases, $k \geq 4$ and $l \geq 1$, in all three cases we conclude $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$.

We have $\sum_{u} h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u)) \geq k \cdot 1 \geq 4$. Since $d_{a} = 1$, we see that $\Delta T = \sum_{u} h_{LT}(u) - 3$ by Proposition 2.2, implying that $\Delta T \geq 0$.

Denote by $H_{i,j}$ a tree having $i + j$ vertices, $i, j \geq 3$. One of these vertices has degree $i$, another one has degree $j$ and the remaining $i + j - 2$ vertices have degrees 1. Obviously, the vertices of degrees $i$ and $j$ must be adjacent in $H_{i,j}$ and $H = H_{3,3}$.

**Lemma 3.8** Let $T$ be a tree homeomorphic to $H_{3,j}$, $j \geq 4$, in which all rays have lengths at most $l + 2$, $l \geq 1$, and in which the ray terminating in the edge $a$ has length $l + 1$. Suppose that the interior path of $H_{3,j}$ has length at most 2 and moreover suppose that the first vertex of the ray terminating in $a$ has degree 3. Then $\Delta T \geq 0$.
PROOF Denote \( e = \mathbf{1} \) and let \( P' \) be the unique interior path of \( T \). If \( P' \) has length 1, then the unique vertex of \( LP' \) (denote it by \( v \)) has degree \( 3 + j - 2 \geq 5 \), while if \( P' \) has length 2, then one of the vertices of \( LP' \) has degree 3 and the other (denote it by \( v \)) has degree \( j \geq 4 \). Since by (1), \( h_{LT}(u) \geq 0 \) if \( d_u \geq 3 \) and \( h_{LT}(u) \geq 5d(u, a) + 1 \) if \( d_u \geq 4 \), the vertices of \( LP' \) contribute to \( \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u) \) by at least \( 5d(v, a) + 1 \geq 5l + 6 \) since \( d(v, a) \geq l + 1 \).

Denote by \( c \) any one of the \( j - 1 \) vertices of the clique of order \( j \) from \( C(H_{3,j}) \), which is not in \( LP' \). By Lemma 3.4 it follows that
\[
\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \begin{cases} 
\left( \binom{j-1}{2} - 1 \right) d(c, a) + \left( \binom{j-2}{2} \right) - 2 & \text{if } |R_{LT}(c)| = 1 \\
\left( \binom{j-2}{2} \right) d(c, a) - 3l + \left( \binom{j-1}{2} \right) - 5 & \text{if } |R_{LT}(c)| \geq 2.
\end{cases}
\]

Since \( j \geq 4 \) and \( d(c, a) \geq l + 2 \geq 3 \), in any case we have \( \sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 0 \).

Now consider the rays attached to the clique of order 3 from \( C(H_{3,j}) \). By Lemma 3.4
\[
\sum_{u \in R_{LT}(e)} h_{LT}(u) = \left( \binom{3}{2} - 3 \right) l + \left( \binom{3-1}{2} \right) = 1.
\]

Denote by \( f \) that vertex of the clique of order 3 from \( C(H_{3,j}) \) which is different from \( e \) and which is not in \( LP' \). By Lemma 3.4 it follows that
\[
\sum_{u \in R_{LT}(f)} h_{LT}(u) \geq \begin{cases} 
-2 & \text{if } |R_{LT}(f)| = 1 \\
d(f, a) - 3l - 4 & \text{if } |R_{LT}(f)| \geq 2.
\end{cases}
\]

Since \( d(f, a) = l + 1 \) and \( l \geq 1 \), in any case we have \( \sum_{u \in R_{LT}(f)} h_{LT}(u) \geq -2l - 3 \).

Now summing up the inequalities above we obtain
\[
\sum_u h_{LT}(u) \geq (5l + 6) + (j - 1) \cdot 0 + 1 + (-2l - 3) = 3l + 4 \geq 3.
\]

Since \( d_a = 1 \), we have \( \Delta T = \sum_u h_{LT}(u) - 3 \) by Proposition 2.2, showing that \( \Delta T \geq 0 \).

Denote by \( Y_{i,j} \), \( 1 \leq i, j \leq 2 \), a tree having three vertices of degree 3, namely \( y_1' \), \( y_2' \) and \( y_3' \). All the other vertices of \( Y_{i,j} \) have degree at most 2. There are two interior paths in \( Y_{i,j} \), namely \( y_1' - y_2' \) and \( y_2' - y_3' \), and their lengths are \( i \) and \( j \), respectively. Moreover, there are five rays in \( Y_{i,j} \). Two such rays start at \( y_1' \), one starts at \( y_2' \) and two start at \( y_3' \). Of course, one of these rays has length exactly \( l + 1 \) and it terminates in \( a' \).

**Lemma 3.9** Let \( T \) be the tree \( Y_{i,j}, 1 \leq i, j \leq 2 \), in which all rays have lengths at most \( l + 2 \), \( l \geq 1 \). Then \( \Delta T \geq 0 \).

16
PROOF Denote by \( x_1, x_2, x_3, x_4 \) and \( x_5 \) the five vertices of \( S_{LT} \) corresponding to the first edges of rays starting at \( y_1', y_2', y_3' \), respectively. Since the degrees of \( y_1', y_2' \) and \( y_3' \) are 3 in \( T \), all \( x_1, x_2, \ldots, x_5 \) are vertices of cliques of order 3 in \( LT \). Let \( x_t = a, 1 \leq t \leq 5 \). By Lemma 3.4

\[
\sum_{u \in R_{LT}(x_t)} h_{LT}(u) = 1.
\]

For all other \( x_r, 1 \leq r \leq 5 \) and \( r \neq t \), by Lemma 3.4 we obtain

\[
\sum_{u \in R_{LT}(x_r)} h_{LT}(u) \geq \min\{-2, d(x_r, a) - 3l - 4\}.
\]

Since \( l \geq 1 \), this minimum equals \( d(x_r, a) - 3l - 4 \) if \( d(x_r, a) \leq l + 4 \). If \( d(x_r, a) = l + 5 \) then \( \sum_{u \in R_{LT}(x_r)} h_{LT}(u) \geq \min\{-2, -2l + 1\} \geq -2l \).

Now we consider vertices corresponding to the edges of interior paths. If such a path has length 1, then its unique edge corresponds to a vertex, say \( e \), the degree of which is 4 in \( LT \). By (1) it follows that

\[
h_{LT}(e) = 5d(e, a) + 1.
\]

On the other hand if such a path has length 2, then its edges correspond to two vertices, say \( e \) and \( f \), both of degree 3. Suppose that \( e \) is closer to \( a \) than \( f \). By (1) it follows that

\[
h_{LT}(e) + h_{LT}(f) = 2d(e, a) - 1 + 2d(f, a) - 1 = 4d(e, a).
\]

In what follows, we list contributions to \( \sum_u h_{LT}(u) \) first by vertices of rays starting at \( x_1, x_2, \ldots, x_5 \) and then by the vertices corresponding to edges of paths \( y_1' - y_2 ' \) and \( y_2' - y_3' \). By symmetry, there are two cases to consider. First, suppose that \( t = 1 \), i.e., \( \pi = x_1 \). We distinguish 4 subcases.

- \( i = j = 1 \): Then \( d(x_2, a) = l + 1, d(x_3, a) = l + 2 \) and \( d(x_4, a) = d(x_5, a) = l + 3 \). Since \( l \geq 1 \), we see that
  \[
  \sum_{u} h_{LT}(u) \geq 1+(-2l-3)+(-2l-2)+2(-2l-1)+(5l+6)+(5l+11) \geq 2l+11 \geq 3.
  \]

- \( i = 1 \) and \( j = 2 \): Analogously as above we obtain
  \[
  \sum_{u} h_{LT}(u) \geq 1+(-2l-3)+(-2l-2)+2(-2l)+(5l+6)+(4l+8) \geq l+10 \geq 3.
  \]

- \( i = 2 \) and \( j = 1 \): Then
  \[
  \sum_{u} h_{LT}(u) \geq 1+(-2l-3)+(-2l-1)+2(-2l)+(4l+4)+(5l+16) \geq l+17 \geq 3.
  \]
• $i = j = 2$: Here $d(x_4, a) = d(x_5, a) = l + 5$. Hence
\[\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (-2l - 1) + 2(-2l) + (4l + 4) + (4l + 12) \geq 13 \geq 3.\]

Now suppose that $t = 3$, i.e., $\overline{a} = x_3$. By symmetry, it suffices to consider 3 subcases.

• $i = j = 1$: Then $d(x_1, a) = d(x_2, a) = l + 2$ and also $d(x_4, a) = d(x_5, a) = l + 2$. Since $l \geq 1$, we see that
\[\sum_u h_{LT}(u) \geq 2(-2l - 2) + 1 + 2(-2l - 1) + (5l + 6) + (5l + 6) \geq 2l + 5 \geq 3.\]

• $i = 1$ and $j = 2$: Then the following holds
\[\sum_u h_{LT}(u) \geq 2(-2l - 2) + 1 + 2(-2l - 1) + (5l + 6) + (4l + 4) \geq l + 5 \geq 3.\]

• $i = j = 2$: Then
\[\sum_u h_{LT}(u) \geq 2(-2l - 1) + 1 + 2(-2l - 1) + (4l + 4) + (4l + 4) \geq 5 \geq 3.\]

Since $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, we conclude that $\Delta T \geq 0$. \qed

Now we prove $\Delta T \geq 0$ for the last graph of the basis of induction. Denote by $X_k$, $k \geq 4$, the tree having two vertices of degree 3, namely $y'_1$ and $y'_2$, and one vertex of degree $k$, namely $y'_3$. All other vertices of $X_k$ have degree at most 2. There are two interior paths in $X_k$, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, both of length at most 2. Moreover, there are $k + 2$ rays in $X_k$. Two such rays start at $y'_1$, one starts at $y'_2$ and the remaining $k - 1$ start at $y'_3$.

**Lemma 3.10** Let $T$ be the tree $X_k$, $k \geq 4$, in which all rays have lengths at most $l + 2$, $l \geq 1$. Suppose that the ray having length $l + 1$ and terminating at $a'$ starts at $y'_1$. Then $\Delta T \geq 0$.

**Proof** We use the notation of the proof of Lemma 3.9. Denote by $x_1, x_2, x_3, x_4, \ldots x_{k+2}$ the $k + 2$ vertices of $S_{LT}$ corresponding to the first edges of rays starting at $y'_1, y'_1, y'_2, y'_3, \ldots, y'_3$, respectively. The vertices $x_1, x_2$ and $x_3$ are in cliques of order 3, while $x_4, \ldots, x_{k+2}$ are in the clique of order $k$. Assume that $\overline{a} = x_1$. As shown in the proof of Lemma 3.9, we have $\sum_{u \in R_{LT}(x_1)} h_{LT}(u) = 1$. Further, $\sum_{u \in R_{LT}(x_3)} h_{LT}(u) \geq -2l - 3$ since $d(x_2, a) = l + 1$. The vertices corresponding to edges of $y'_1 - y'_2$ path
contribute to \( \sum_u h_{LT}(u) \) by at least \( \min\{5d(e, a) + 1, 4d(e, a)\} = 4d(e, a) = 4l + 4 \) as \( d(e, a) = l + 1 \). Finally, \( \sum_{u \in R_{LT}(x)} h_{LT}(u) \geq \min\{-2, d(x_3, a) - 3l - 4\} \geq -2l - 2 \) as \( d(x_3, a) \geq l + 2 \).

Since the vertices corresponding to edges of \( y_2' - y_3' \) path have degree \( k + 1 \) (in the case when the length of \( y_2' - y_3' \) is 1) or 3 and \( k \) (in the case when the length of \( y_2' - y_3' \) is 2), and since \( h_{LT}(u) \geq 0 \) if \( d_u \geq 3 \) by (1), the contribution of these vertices to \( \sum_u h_{LT}(u) \) is nonnegative.

Finally, consider \( \sum_{j \in R_{LT}(x)} h_{LT}(u) \) when \( i \geq 4 \). By Lemma 3.4 it follows that

\[
\sum_{u \in R_{LT}(x)} h_{LT}(u) \geq \begin{cases} 
\binom{k-1}{2} - 1 d(x_i, a) + \binom{k-2}{2} - 2 & \text{if } |R_{LT}(x)| = 1 \\
\binom{k-2}{2} - 2 d(x_i, a) - 3l + \binom{k-1}{2} - 5 & \text{if } |R_{LT}(x)| = 2 
\end{cases}
\]

Since \( d(x_i, a) \geq l + 3, k \geq 4 \) and \( l \geq 1 \), we obtain \( \sum_{u \in R_{LT}(x)} h_{LT}(u) \geq \min\{7, 11\} = 7 \).

Summing up these inequalities we obtain

\[
\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (4l + 4) + (-2l - 2) + 0 + (k - 1)7 = 7k - 7 \geq 3.
\]

Since \( d_u = 1 \), by Proposition 2.2 we conclude that \( \Delta T = \sum_u h_{LT}(u) - 3 \geq 0 \).

Now we summarize the proof of Theorem 1.5

PROOF OF THEOREM 1.5 Let \( T \) be a tree, not homeomorphic to a path, claw \( K_{1,3} \) and \( H \). We prove that \( D(T) = W(L^3(T)) - W(T) > 0 \). Denote by \( l + 2, l \geq -1 \), the length of a longest ray in \( T \). If \( l = -1 \) then \( D(T) = W(L^3(T)) - W(T) > 0 \) by Theorem 1.4.

Suppose that \( l \geq 0 \) and suppose that the statement of the theorem is true for all trees (not homeomorphic to a path, claw \( K_{1,3} \) and \( H \)) rays of which have lengths at most \( l + 1 \). Let \( R_t', R_{t+2}', \ldots, R_t' \) be rays of \( T \) having length \( l + 2 \). Further, denote by \( c'_1 \) the first vertex of \( R_1' \), denote by \( b'_l \) its last vertex and denote by \( a'_i \) the neighbour of \( b'_i \) in \( T \), \( 1 \leq i \leq t \). Finally, denote by \( T_i \) a tree obtained from \( T \) by removing the vertices \( b_i', b_{i+2}', \ldots, b_t' \) and edges \( a_i' b_{i+1}', a_{i+2}' b_{i+2}', \ldots, a_t' b_t' \), \( 0 \leq i \leq t \). Then \( T_t = T \) and \( T_0 \) is a tree rays of which have length at most \( l + 1 \). By induction we have \( D(T_0) = W(L^3(T_0)) - W(T_0) > 0 \). Let \( \Delta T_i = D(T_{i+1}) - D(T_i) \), \( 0 \leq i \leq t - 1 \).

Suppose that \( l = 0 \). All rays of \( T_i \) have length at most \( l + 2 \), and the ray \( R_{i+1}' \) terminating at \( a_{i+1}' \) has length \( l + 1 \). Moreover, \( T_{i+1} \) is obtained from \( T_i \) by adding the vertex \( b_{i+1}' \) and the edge \( a_i' b_{i+1}' \). Hence \( \Delta T_i \geq 0 \) by Lemma 3.2, \( 0 \leq i \leq t - 1 \), where the vertex \( a_{i+1}' \) and the tree \( T_i \) play the role of \( a \) and \( T \), respectively. Consequently
\[ \sum_{i=0}^{t-1} \Delta T_i \geq 0. \] Since

\[ 0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_i) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)], \]

we have \( W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0. \)

Now suppose that \( t \geq 1 \). In \( T_i \) shorten all interior paths of length at least 3 to paths of length 2, and denote the resulting graph by \( T^{-}_i \). Analogously as \( T_{i+1} \) is obtained from \( T_i \), the tree \( T^{-}_{i+1} \) is obtained from \( T^{-}_i \) by adding the vertex \( b'_i \) and the edge \( a'_i b'_i \). We prove that \( \Delta T^{-}_i = D(T^{-}_{i+1}) - D(T^{-}_i) \geq 0 \) by induction on the number of vertices of degree at least 3. Observe that \( T^{-}_i \), as well as \( T_i \), is a tree, rays of which have length at most \( l + 2 \) and the ray terminating at \( a'_i \) has length \( l + 1 \), \( 0 \leq i \leq t - 1 \).

Denote by \( V^3_i \) the set of vertices of degree at least 3 in \( T^{-}_i \). We distinguish four cases.

- \( |V^3_i| = 1 \): Then \( T^{-}_i \) is homeomorphic to \( K_{1,k} \). Since \( T \) is not homeomorphic to \( K_{1,3} \), it follows that \( k \geq 4 \). By Lemma 3.7 we have \( \Delta T^{-}_i \geq 0 \).

- \( |V^3_i| = 2 \): If the degree of \( c'_{i+1} \) is 3, then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.8, since \( T \) is not homeomorphic to \( H = H_{3,3} \). On the other hand if the degree of \( c'_{i+1} \) is \( k \geq 4 \), then denote by \( c'' \) the other vertex of \( V^3_i \). Remove the rays starting at \( c'' \) from \( T^{-}_i \), and denote the resulting graph by \( T'' \). Then \( T'' \) is a tree, rays of which have length at most \( l + 2 \), and \( T'' \) is homeomorphic to \( K_{1,k} \). By Lemma 3.7 we have \( \Delta T'' \geq 0 \). If the degree of \( c'' \) is 3, then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.5, while if the degree of \( c'' \) is at least 4 then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.6.

- \( |V^3_i| = 3 \): Denote by \( T^* \) a graph obtained from \( T^{-}_i \) by removing the edges of all rays. Since \( T^* \) is a tree, it has at least two vertices of degree 1. (We remark that in this case \( T^* \) is a path.) Denote by \( c'' \) a pendant vertex in \( T^* \), \( c'' \neq c'_{i+1} \), the degree of which is the smallest possible in \( T^{-}_i \). Finally, denote by \( T'' \) a tree obtained from \( T^{-}_i \) by removing all rays starting at \( c'' \). We distinguish two subcases.

  - \( T'' \) is homeomorphic to \( H \): If the degree of \( c'' \) is 3 in \( T \) then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.9. Suppose that the degree of \( c'' \) is \( k \geq 4 \). By the choice of \( c'' \), the vertex \( c'_{i+1} \) is a leaf of \( T^* \). Hence \( T \) is \( X_k \) and \( c'_{i+1} \) is the vertex \( y' \) in the notation of Lemma 3.10. Therefore \( \Delta T^{-}_i \geq 0 \) by Lemma 3.10.

  - \( T'' \) is homeomorphic to \( H_{1,j} \), \( i \leq j \) and \( j \geq 4 \): Since \( T'' \) is not homeomorphic to \( H \), it follows that \( \Delta T'' \geq 0 \) by the previous case (the case \( |V^3_i| = 2 \)). If the degree of \( c'' \) is 3, then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.5; while if the degree of \( c'' \) is at least 4, then \( \Delta T^{-}_i \geq 0 \) by Lemma 3.6.

20
Thus we proved $\Delta T_i^- \geq 0$ for every tree $T_i^-$ rays of which have length at most $l + 2$ and $|V_i^3| = 3$.

- $|V_i^3| \geq 4$: Analogously as in the previous case, denote by $T''$ a tree obtained from $T_i^-$ by removing all rays starting at a pendant vertex $c''$ of $T^*$, $c'' \neq c_{i+1}'$. By induction we assume that $\Delta T'' \geq 0$. If the degree of $c''$ is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5, while if the degree of $c''$ is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

Hence, in any case we have $\Delta T_i^- \geq 0$. If $T_i^- = T_i$, then it follows that also $\Delta T_i \geq 0$. Otherwise form a sequence $T_i^- = F_0, F_1, \ldots, F_r = T_i$ such that $F_{j+1}$ is obtained from $F_j$ by subdividing one edge of one interior path, $0 \leq j \leq r - 1$. By Lemma 3.3 we have $\Delta F_{j+1} - \Delta F_j \geq 0$. Hence $\sum_{j=0}^{r-1} (\Delta F_{j+1} - \Delta F_j) \geq 0$. Since

$$0 \leq \sum_{j=0}^{r-1} (\Delta F_{j+1} - \Delta F_j) = \Delta T_i - \Delta T_i^-,$$

we see that $\Delta T_i \geq \Delta T_i^- \geq 0$.

Thus we proved that $\Delta T_i \geq 0$ for every $i \in \{0, 1, \ldots, t-1\}$. Hence $\sum_{i=0}^{t-1} \Delta T_i \geq 0$. Since

$$0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_i) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we conclude that $W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0$.

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References


