Houston Journal of Mathematics

Volume 37, No. 4, 2011

SUBLINEAR HIGSON CORONA AND LIPSCHITZ EXTENSIONS

M. CENCELJ, J. DYDAK, J. SMREKAR, AND A. VAVPETIČ

Communicated by Charles Hagopian

Abstract. We show that the dimension of the sublinear Higson corona of a metric space X is the smallest non-negative integer m with the following property: Any norm-preserving asymptotically Lipschitz function from a closed subset A of X to the Euclidean space of dimension $m+1$ extends to a norm-preserving asymptotically Lipschitz function from X to the Euclidean space of dimension $m+1$. As an application we obtain another proof of the following result of Dranishnikov and Smith: Let X be a cocompact proper metric space, which is M-connected for some M, and has the asymptotic Assouad-Nagata dimension finite. Then this dimension equals the dimension of the sublinear Higson corona of X.

1. Introduction

In [6] Dranishnikov and Smith related the dimension $\dim(\nu_L(X))$ of the sublinear Higson corona $\nu_L(X)$ of X (see Section 4) to the asymptotic Assouad-Nagata dimension $\mathrm{asdim}_{AN}(X)$ of X as follows:

Theorem 1.1 (Dranishnikov-Smith)**.** *Suppose (X,d) is a connected cocompact proper metric space. If $\mathrm{asdim}_{AN}(X)$ is finite, then $\mathrm{asdim}_{AN}(X) = \dim(\nu_L(X))$.*

2000 *Mathematics Subject Classification.* Primary 54F45; Secondary 55M10, 54C65.

Key words and phrases. Assouad-Nagata dimension, Higson compactifications, Higson corona, Lipschitz functions, Lipschitz extensors.

Supported in part by the Slovenian Research Agency grants BIUS/07-08/029, P1-0292-0101 and J1-2057-0101.

The second-named author was partially supported by Grant No. 2004047 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

Since Assouad-Nagata dimension of X is related to the sphere S^m being a Lipschitz extensor of X (see [3] and [9]), a natural problem arises of expressing $\dim(\nu_L(X))$ in terms of Lipschitz extensions.

Dranishnikov [5] (p.1105) introduced dimension $\dim^c(X)$ for proper metric spaces X as the smallest integer $n \geq 0$ such that any proper asymptotically Lipschitz function $f\colon A \to \mathbf{R}^{n+1}$, A closed in X, extends to a proper asymptotically Lipschitz function $f'\colon X \to \mathbf{R}^{n+1}$. Thus, $\dim^c(X) \leq n$ if and only if $\mathbf{R}^{n+1}$ is an absolute extensor of X in the asymptotic category $\mathcal{A}$ of [5]. His basic result is that $\dim^c(X)$ equals the dimension of Higson corona $\nu(X)$ of X ([5], Theorem 6.6 on p.1111).

In this paper we consider the case of a special class of proper functions to $\mathbf{R}^{n+1}$: the norm-preserving functions. Our spaces are assumed to be equipped with a base-point x_0. In case of Euclidean space $\mathbf{R}^m$ the base-point is always 0. Now, the *norm* $|x|$ of x is defined to be $d(x, x_0)$. $f\colon X \to \mathbf{R}^m$ is *norm-preserving* if $|f(x)| = |x|$ for all $x \in X$.

In analogy to $\dim^c(X)$ of Dranishnikov we introduce the *sublinear asymptotic dimension* $\operatorname{asdim}_L(X)$ as the smallest integer $n \geq 0$ such that any norm-preserving asymptotically Lipschitz function $f\colon A \subset X \to \mathbf{R}^{n+1}$ extends to a norm-preserving asymptotically Lipschitz function $f'\colon X \to \mathbf{R}^{n+1}$. Our main result is that $\operatorname{asdim}_L(X)$ equals the dimension of sublinear Higson corona $\nu_L(X)$ of X if X is proper.

Another way of generalizing $\dim^c$ would be to look at the smallest integer $n \geq 0$ such that $\mathbf{R}^{n+1}$ is an absolute extensor of X in the category $\tilde{\mathcal{A}}$ of [5]. In Section 6 we show that definition to be equivalent to asdim_L.

As an application of our results we prove a slight generalization of 1.1 (see 5.3).

2. Norm-preserving Lipschitz functions

An ϵ-*net* in a metric space X is a subset X_1 of X such that $d(x, X_1) < \epsilon$ for all $x \in X$.

A metric space X is *discrete* if there is $\epsilon > 0$ such that $d(x, y) \geq \epsilon$ for all $x, y \in X$, $x \neq y$. In that case the term ϵ-*discrete* will be used.

A function $f\colon (X, d_X) \to (Y, d_Y)$ of metric spaces is *asymptotically Lipschitz* (or *large scale Lipschitz*) if there are numbers $\lambda, M \geq 0$ such that $d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y) + M$ for all $x, y \in X$. We will use the shortcut of (λ, M)-asymptotically Lipschitz in such case. If $M = 0$, then such maps are called *Lipschitz* or λ-*Lipschitz*. Also, the notation $\operatorname{Lip}(f) \leq \lambda$ is being used in that case. More precisely, $\operatorname{Lip}(f)$ is defined as $\sup\{\frac{d_Y(f(x), f(y))}{d_X(x,y)} \mid x \neq y \in X\}$.

Notice that if X is ϵ-discrete and $f\colon X \to Y$ is (λ, M)-asymptotically Lipschitz, then it is $(\lambda + \frac{M}{\epsilon})$-Lipschitz.

Let us show how to relate $\mathrm{asdim}_L(X)$ to existence of Lipschitz extensions.

Proposition 2.1. *Suppose (X, d) is a metric space and X_1 is a discrete ϵ-net in X for some $\epsilon > 0$. If $m \geq 0$, then the following conditions are equivalent:*

a. *Any asymptotically Lipschitz norm-preserving function $f'\colon A \subset X \to \mathbf{R}^{m+1}$ extends to an asymptotically Lipschitz norm-preserving function $F'\colon X \to \mathbf{R}^{m+1}$.*
b. *Any Lipschitz norm-preserving function $f'\colon A \subset X_1 \to \mathbf{R}^{m+1}$ extends to a Lipschitz norm-preserving function $F'\colon X_1 \to \mathbf{R}^{m+1}$.*

Proof. (a) $\Longrightarrow$ (b). Suppose $f'\colon A \subset X_1 \to \mathbf{R}^{m+1}$ is a norm-preserving Lipschitz function. Extend it to an asymptotically Lipschitz function $F'\colon X_1 \to \mathbf{R}^{m+1}$ and notice that it is Lipschitz as X_1 is discrete.

(b) $\Longrightarrow$ (a). Suppose $f'\colon A \subset X \to \mathbf{R}^{m+1}$ is a norm-preserving (λ, M)-asymptotically Lipschitz function. We may assume $x_0 \in A$ and f' is induced by $f\colon A \to S^m$ in the sense of $f'(x) = |x| \cdot f(x)$ for all $x \in A$. Let A_1 be the set of all points x in X_1 such that $d(x, A) < \epsilon$. For each $x \in A_1 \setminus A$ pick a point $a(x) \in A$ so that $d(x, a(x)) < \epsilon$. If $x \in A_1 \cap A$, put $a(x) = x$. Define $g\colon A_1 \to S^m$ by $g(x) = f(a(x))$. We need the induced function $g'\colon A_1 \to \mathbf{R}^{m+1}$ to be Lipschitz. Given $x, y \in A_1$, we have $g'(x) - g'(y) = |x| \cdot f(a(x)) - |y| \cdot f(a(y)) = f'(a(x)) - f'(a(y)) + (|x| - |a(x)|) \cdot f(a(x)) + (|y| - |a(y)|) \cdot f(a(y))$, so $|g'(x) - g'(y)| \leq \lambda d(a(x), a(y)) + M + \epsilon + \epsilon \leq \lambda \cdot (d(x, y) + 2\epsilon) + M + 2\epsilon$. Thus, g' is asymptotically Lipschitz, hence Lipschitz as A_1 is discrete. Extend g' over X_1 to a γ-Lipschitz norm-preserving function $G'\colon X_1 \to \mathbf{R}^{m+1}$. G' is induced by $G\colon X_1 \to S^m$, an extension of g. Given $x \in X \setminus (A \cup X_1)$ find a point $b(x) \in X_1$ such that $d(x, b(x)) < \epsilon$. For $x \in A \cup X_1$ put $b(x) = x$. Extend G to $F\colon X \to S^m$ by $F(x) = f(x)$ if $x \in A$ and $F(x) = G(b(x))$ if $x \notin A$. Let $G'(x) := |x| \cdot G(x)$. Estimation of $|G'(x) - G'(y)|$ is obvious if both x and y belong to A. In case $x, y \notin A$ follow the argument above, so the only case of interest is $x \in A$ and $y \notin A$. Pick $x_1 \in X_1$ satisfying $d(x, x_1) < \epsilon$ and notice $x_1 \in A_1$, so $g(x_1) = f(a(x_1))$. Now, $G'(x) - G'(y) = (G'(x) - G'(x_1)) + (G'(b(y)) - G'(y)) + (G'(x_1) - G'(b(y))$ and $|G'(x_1) - G'(b(y))| \leq \gamma \cdot d(x_1, b(y)) \leq \gamma \cdot (d(x, y) + 2\epsilon)$, $G'(b(y)) - G'(y) = |b(y)| \cdot g(b(y)) - |y| \cdot g(b(y))$ is of size at most ϵ, so the only remaining task is estimation of $G'(x) - G'(x_1)$. However, $G'(x) - G'(x_1) = f'(x) - f'(a(x_1)) + (|x_1 - |x|) \cdot f(x) + (|a(x_1)| - |x_1|) \cdot f(a(x_1))$ is of size at most $\lambda \cdot \epsilon + M + \epsilon + \epsilon$. □

In the remainder of the paper by $\mathrm{An}(X, r, s)$ we denote the annulus $\{x \in X \mid r \le |x| < s\}$, where $0 \le r \le s$.

Proposition 2.2. *Suppose (X, d) is a discrete metric space, $r > 0$ and $M > 1$. Put $X_k = \mathrm{An}(X, r\cdot M^{k-1}, r\cdot M^{k+1})$ and $Y_k = \mathrm{An}(X, r\cdot M^{k-1}, \infty)$. If $f\colon X \to S^m$ is a function and $m \ge 0$, then the following conditions are equivalent:*

a. *f induces a norm-preserving $f'\colon X \to \mathbf{R}^{m+1}$ that is Lipschitz.*
b. *The sequence $\{M^k \cdot \mathrm{Lip}(f|_{Y_k})\}_{k=1}^{\infty}$ is bounded.*
c. *The sequence $\{M^k \cdot \mathrm{Lip}(f|_{X_k})\}_{k=1}^{\infty}$ is bounded.*

Proof. (a) $\Longrightarrow$ (b). Suppose f' is λ-Lipschitz. Since X is discrete, each $\mathrm{Lip}(f_k|_{Y_k})$ is finite. If $x, y \in Y_k$ for some $k \ge 2$ and $|x| \le |y|$, then $\lambda \cdot d(x, y) \ge |f'(x) - f'(y)| = ||x|f(x) - |y|f(y)| = ||x|(f(x) - f(y)) + (|x| - |y|)f(y)| \ge |x||f(x) - f(y)| - d(x, y)$, so $M^k \frac{|f(x)-f(y)|}{d(x,y)} \le M^k \frac{\lambda+1}{|x|} \le M(\lambda + 1)/r$.

(c) $\Longrightarrow$ (a). Suppose $\{M^k \cdot \mathrm{Lip}(f|_{X_k})\}_{k=1}^{\infty}$ is bounded by C. Suppose $x, y \in X$ and $|x| \le |y|$. If $x, y \in X_k$ for some k, then $||x|f(x)-|y|f(y)| = ||x|(f(x)-f(y))+(|x| - |y|)f(y)| \le |x||f(x) - f(y)| + d(x, y) \le (|x| \cdot C/M^k + 1) \cdot d(x, y) \le (rMC + 1)\cdot d(x, y)$. If x and y do not belong to the same X_k, then there is $n \ge 0$ such that $|x| \le rM^n$ and $|y| \ge rM^{n+1}$. In that case $d(x, y) \ge |y| - |x| \ge r(M - 1) \cdot M^n \ge (M - 1)|x|$ and $|f'(x) - f'(y)| = ||x|f(x) - |y|f(y)| = ||x|(f(x) - f(y)) + (|x| - |y|)f(y)| \le |x||f(x)-f(y)|+d(x, y) \le 2|x|+d(x, y) \le (2/(M-1)+1)\cdot d(x, y)$. □

Corollary 2.3. *Suppose (X, d) is a discrete metric space. If $m \ge 0$, then the following conditions are equivalent:*

a. *Every Lipschitz norm-preserving function $f'\colon A \subset X \to \mathbf{R}^{m+1}$ extends to a Lipschitz norm-preserving function $F'\colon X \to \mathbf{R}^{m+1}$.*
b. *For any $r, s > 0$, any $M > 1$, and any sequence of functions $f_k\colon A_k \subset \mathrm{An}(X, r \cdot M^{2k}, r \cdot M^{2k+1}) \to S^m$, $k \ge 1$, satisfying $\mathrm{Lip}(f_k) \le \frac{s}{M^{2k}}$ there are $c > 0$ and functions $g_k\colon \mathrm{An}(X, r \cdot M^{2k}, r \cdot M^{2k+1}) \to S^m$ so that $\mathrm{Lip}(g_k) \le \frac{c}{M^{2k}}$ and $g_k|A_k = f_k$ for all $k \ge 1$.*
c. *For any $r, s > 0$, any $M > 1$, and any sequence of functions $f_k\colon A_k \subset \mathrm{An}(X, r \cdot M^k, r \cdot M^{k+1}) \to S^m$, $k \ge 1$, satisfying $\mathrm{Lip}(f_k) \le \frac{s}{M^k}$ there are $c > 0$ and functions $g_k\colon \mathrm{An}(X, r \cdot M^k, r \cdot M^{k+1}) \to S^m$ so that $\mathrm{Lip}(g_k) \le \frac{c}{M^k}$ and $g_k|A_k = f_k$ for all $k \ge 1$.*

Proof. (a) $\Longrightarrow$ (b). Let $A = \bigcup_{k=1}^{\infty} A_k$. Paste all f_k to obtain $f\colon A \to S^m$. By 2.2 the function $f'\colon A \to \mathbf{R}^{m+1}$ is Lipschitz, so it extends to a norm-preserving Lipschitz $g'\colon X \to \mathbf{R}^{m+1}$ induced by g. Consider $g_k = g|_{\mathrm{An}(X, r\cdot M^{2k}, r\cdot M^{2k+1})}$ and apply 2.2 again.

(b) $\Longrightarrow$ (c). It is a consequence of the fact that $\mathrm{An}(X, r\cdot M^{2k+1}, r\cdot M^{2k+2}) = \mathrm{An}(X, s\cdot M^{2k}, s\cdot M^{2k+1})$, where $s = r\cdot M$, so one can apply b) twice and get c).

(c) $\Longrightarrow$ (a). First observation is the following:

For any $s > 0$, any $M > 1$, and any sequence of functions $f_k\colon A_k \subset \mathrm{An}(X, M^k, M^{k+3}) \to S^m$ satisfying $\mathrm{Lip}(f_k) \le \frac{s}{M^k}$ there are $c > 0$ and functions $g_k\colon \mathrm{An}(X, M^k, M^{k+3}) \to S^m$ so that $\mathrm{Lip}(g_k) \le \frac{c}{M^k}$ and $g_k|A_k = f_k$ for all $k \ge 1$.

It follows from setting $M' = M^3$ and applying c) three times with values for r being respectively 1, M, and M^2.

Second observation is that given λ_i-Lipschitz functions $u_i\colon X_i \to Y$, $i = 1, 2$, from subsets of X to a bounded metric space Y such that $\mathrm{dist}(X_1 \setminus X_2, X_2 \setminus X_1) \ge \mu$ and $u_1|_{X_1\cap X_2} = u_2|_{X_1\cap X_2}$, then $u = u_1 \cup u_2\colon X_1 \cup X_2 \to Y$ is $\max(\lambda_1, \lambda_2, \frac{\mathrm{diam}(Y)}{\mu})$-Lipschitz.

Suppose $f'\colon A \to \mathbf{R}^{m+1}$ is a norm-preserving Lipschitz function and $A \subset X$. Define $f(a) = f'(a)/|a|$ for $a \in A \setminus \{x_0\}$ and define $f(x_0)$ arbitrarily. By 2.2 there is $s > 0$ such that $\mathrm{Lip}(f|_{\mathrm{An}(A,2^k,\infty)}) \le s/2^k$ for all $k \ge 0$. Find $t > 0$ and $f_k\colon \mathrm{An}(X, 2^{k-1}, 2^{k+2}) \to S^m$ for each $k \ge 2$ so that f_k extends $f|_{\mathrm{An}(A,2^{k-1},2^{k+2})}$ and $\{2^k \cdot \mathrm{Lip}(f_k)\}$ is bounded. Paste f_k and f_{k+2} to get g_k on $\mathrm{An}(X, 2^k, 2^{k+1}) \cup \mathrm{An}(A, 2^{k+1}, 2^{k+2}) \cup \mathrm{An}(X, 2^{k+2}, 2^{k+3})$ so that $\{2^k \cdot \mathrm{Lip}(g_k)\}$ is bounded. Get extensions $h_k\colon \mathrm{An}(X, 2^k, 2^{k+3}) \to S^m$ of g_k so that $\{2^k \cdot \mathrm{Lip}(h_k)\}$ is bounded. Splice the even-numbered h_k to get $h\colon X \to S^m$ so that h' is Lipschitz and h extends f. $\square$

Corollary 2.4. *Let (X, d) be a discrete metric space and $m \ge 0$. If $K \subset S^m$ is a Lipschitz extensor of X and $f\colon A \subset X \to K$ induces a Lipschitz function $f'\colon X \to \mathbf{R}^{m+1}$, then it extends to $g\colon X \to K$ inducing a Lipschitz function $g'\colon X \to \mathbf{R}^{m+1}$.*

Proof. K being a Lipschitz extensor of X means that there is $C > 0$ such that any λ-Lipschitz $f\colon A \subset X \to K$ extends over X to a $C\cdot\lambda$-Lipschitz function. Follow the same procedure as in the proof of 2.3 using the fact K is a Lipschitz extensor of X. $\square$

Corollary 2.5. *Suppose (X, d) is a discrete metric space. If S^m is a Lipschitz extensor of X, then any norm-preserving Lipschitz function $f\colon A \to \mathbf{R}^{m+1}$ extends to a norm-preserving Lipschitz function $f'\colon X \to \mathbf{R}^{m+1}$.*

In [6] the concept of a *cocompact* metric space X was introduced to mean that there is a compact subset K of X such that for any $x \in X$ there is an isometry γ of X with $x \in \gamma(K)$. In other words, the group of isometries of X acts on X

cocompactly. Analogously, one can introduce the concept of a *cobounded* space X to mean that $\mathrm{Isom}(X)$ acts on X coboundedly, i.e. there is a bounded subset B of X with $X = \bigcup_{\gamma \in Isom(X)} \gamma(B)$.

(X, d) is *M-scale connected* if any pair of points $x, y \in X$ can be connected by a chain $x_1 = x, \ldots, x_k = y$ such that $d(x_i, x_{i+1}) \leq M$ for all $i \leq k-1$.

Corollary 2.6. *Suppose (X, d) is a metric space and $\epsilon > 0$. If X is cobounded and M-scale connected for some $M > 0$, then the following conditions are equivalent:*

a. *Any asymptotically Lipschitz norm-preserving function $f' \colon A \subset X \to \mathbf{R}^{m+1}$ extends to an asymptotically Lipschitz norm-preserving function $F' \colon X \to \mathbf{R}^{m+1}$.*
b. *There is a function $c \colon \mathbf{R}_+ \to \mathbf{R}_+$ such that any $f \colon B \subset Y \to S^m$, Y an ϵ-discrete subset of X, satisfying $\mathrm{Lip}(f) \leq \frac{s}{\mathrm{diam}(Y)}$ for some $s > 0$, extends to $g \colon Y \to S^m$ so that $\mathrm{Lip}(g) \leq c(s) \cdot \mathrm{Lip}(f)$.*

Proof. (a) $\Longrightarrow$ (b). Suppose there is $s > 0$ such that for each $n \geq 1$ one can find $f_n \colon B_n \subset Y_n \to S^m$, Y_n an ϵ-discrete subset of X, satisfying $\mathrm{Lip}(f_n) \leq \frac{s}{\mathrm{diam}(Y_n)}$ but any extension $g \colon Y_n \to S^m$ of f_n satisfies $\mathrm{Lip}(g) > \alpha(n) \cdot \mathrm{Lip}(f_n)$, where $\alpha(n) \to \infty$ as $n \to \infty$. Notice that each f_n has an extension g such that $\mathrm{Lip}(g) \leq \frac{2}{\epsilon}$. That means $\mathrm{Lip}(f_n) \leq \frac{2}{\epsilon \cdot \alpha(n)}$ and $\mathrm{Lip}(f_n) \to 0$. By enlarging M if necessary we may assume that $M > 6$ and each $\mathrm{An}(X, 2M^k, \frac{1}{3}M^{k+1})$ contains a point x_k. For each $n \geq 1$ pick $k(n)$ so that $M^{k(n)} \leq \frac{2Ms}{\mathrm{Lip}(f_n)} < M^{k(n)+1}$ and by applying an isometry assume $d(x_{k(n)}, Y_n) < M$ for all $n \geq 1$. We may assume $k(n)$ is a strictly increasing function and $k(n) \geq 2$ for all $n \geq 1$ (by switching to a subsequence if necessary). Notice $\mathrm{diam}(Y_n) \leq \frac{s}{\mathrm{Lip}(f_n)} \leq M^{k(n)}/2$. Given $x \in Y_n$ one has $|x_{k(n)}| - M - \mathrm{diam}(Y_n) < |x| < |x_{k(n)}| + M + \mathrm{diam}(Y_n)$, so $2M^{k(n)} - M - M^{k(n)}/2 \leq |x| < \frac{1}{3}M^{k(n)+1} + M + M^{k(n)}/2$ and $M^{k(n)} \leq |x| < M^{k(n)+1}$ as $M > 6$. That means $Y_n \subset \mathrm{An}(X, M^{k(n)}, M^{k(n)+1})$.

Since $\mathrm{Lip}(f_n) \leq \frac{2Ms}{M^{k(n)}}$ for all $n \geq 1$, 2.3 implies existence of $c > 0$ and extensions $g_n \colon Y_n \to S^m$ such that $\mathrm{Lip}(g_n) \leq \frac{c}{M^{k(n)}}$. Thus, for all $n \geq 1$, $\frac{c}{M^{k(n)}} \geq \mathrm{Lip}(g_n) \geq \alpha(n) \cdot \mathrm{Lip}(f_n) \geq \alpha(n) \cdot \frac{2Ms}{M^{k(n)}}$ resulting in $\frac{c}{2Ms} \geq \alpha(n)$ for all n, a contradiction.

(b) $\Longrightarrow$ (a). Use 2.3 and 2.1. □

Corollary 2.7. *Suppose (X, d) is a metric space such that any asymptotically Lipschitz norm-preserving function $f' \colon A \subset X \to \mathbf{R}^{m+1}$ extends to an asymptotically Lipschitz norm-preserving function $F' \colon X \to \mathbf{R}^{m+1}$ for some $m \geq 0$. If X*

is cobounded, M-scale connected for some $M > 0$, and $\mathrm{asdim}_{AN}(X) \le m+1$, *then* $\mathrm{asdim}_{AN}(X) \le m$.

Proof. Suppose any asymptotically Lipschitz norm-preserving function $f'\colon A \subset X \to \mathbf{R}^{m+1}$ extends to an asymptotically Lipschitz norm-preserving function $F'\colon X \to \mathbf{R}^{m+1}$. Pick a 1-net X_1 in X. By 2.6 there is a function $c\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that any $f\colon B \subset Y \to S^m$, $Y \subset X_1$, satisfying $\mathrm{Lip}(f) \le \frac{s}{\mathrm{diam}(Y)}$ extends to $g\colon Y \to S^m$ so that $\mathrm{Lip}(g) \le c(s) \cdot \mathrm{Lip}(f)$. As Assouad-Nagata dimension of X_1 is at most $m+1$, there is a constant $C > 0$ such that for each $r > 0$ there is a cover $\mathcal{U}^r$ of X_1 that can be expressed as the union of $m+2$ families $\mathcal{U}^r_i$, $1 \le i \le m+2$, so that each $\mathcal{U}^r_i$ is r-disjoint and the diameter of elements of $\mathcal{U}^r$ is at most $C \cdot r$. Let A_r be all the points of X_1 contained in at most $m+1$ elements of $\mathcal{U}^r$. Consider the canonical map $\phi^r\colon X_1 \to \mathcal{N}(\mathcal{U}^r)$ to the nerve $\mathcal{N}(\mathcal{U}^r)$ of $\mathcal{U}^r$ for each $r > 0$. Notice $\mathrm{Lip}(\phi^r) \le \frac{\lambda}{r}$ for some constant $\lambda > 0$ (see [1] and [4]) and $\mathrm{diam}((\phi^r)^{-1})(\Delta)) \le 2Cr$ for each $r > 0$ and each $(m+1)$-simplex Δ of $\mathcal{N}(\mathcal{U}^r)$. Let $X^r_\Delta = (\phi^r)^{-1}(\Delta)$ and $\phi^r_\Delta = \phi^r|_{X^r_\Delta}\colon X^r_\Delta \to \Delta$ for any $(m+1)$-simplex Δ of $\mathcal{N}(\mathcal{U}^r)$. Thus $\mathrm{Lip}(\phi^r_\Delta) \le \frac{2\lambda}{\mathrm{diam}(X^r_\Delta)}$ and for each $r > 0$ and for each $(m+1)$-simplex Δ of $\mathcal{N}(\mathcal{U}^r)$ there is an extension $\psi^r_\Delta\colon X^r_\Delta \to \partial\Delta$ of $\phi^r_\Delta|(X^r_\Delta \cap A_r)$ satisfying $\mathrm{Lip}(\psi^r_\Delta) \le t \cdot \mathrm{Lip}(\phi^r_\Delta)$, where $t = c(2\lambda)$. Given a vertex U of $\mathcal{U}^r$ consider all Δ containing U as a vertex and let $s(U)$ be the union of all $(\psi^r_\Delta)^{-1}(st(U))$ and $U \cap A_r$. That means $s(U)$ shrinks U to points that are either contained in $A_r \cap U$ or their U-barycentric coordinate of ψ^r is positive. Thus the mesh of the cover $\{s(U)\}_{U \in \mathcal{U}^r}$ is at most $C \cdot r$, so it suffices to show that its Lebesgue number is at least $K \cdot r$ for some constant K (independent of r) and the multiplicity of that cover is at most $m+1$.

The reason multiplicity does not exceed $m+1$ is that each $x \in X$ is being mapped to the n-skeleton of the nerve by maps ψ^r_Δ.

Given $x \in X$ find all elements $U_1, \ldots, U_k$ of $\mathcal{U}^r$ containing it. If $k < m+2$, then the ball $B(x, r/2)$ intersects exactly one U_i and $s(U_i)$ must contain $B(x, r/2) \subset A_r \cap U_i$. If $k = m+2$, then for $\Delta = [U_1, \ldots, U_{m+2}]$ pick i such that the U_i-barycentric coordinate of $\psi^r_\Delta(x)$ is at least $\frac{1}{m+2}$. If $y \in B(x, \frac{r}{\lambda(m+2)t})$ and the U_i-barycentric coordinate of $\psi^r_\Delta(y)$ is 0, then $1/(m+2) \le |\psi^r_\Delta(x) - \psi^r_\Delta(y)| \le d(x,y) \cdot t\lambda/r < 1/(m+2)$, a contradiction. Thus $B(x, \frac{r}{\lambda(m+2)t}) \subset s(U_i)$ and the Lebesgue number of $\{s(U)\}_{U \in \mathcal{U}^r}$ is at least $K \cdot r$, where $K = \frac{1}{\lambda(m+2)t}$. That means $m \ge \dim_{AN}(X_1) = \mathrm{asdim}_{AN}(X_1) = \mathrm{asdim}_{AN}(X)$. □

3. Open cones and partitions of unity

Consider a compact subset K of $\mathbf{R}^m \setminus \{0\}$ such that each ray from 0 intersects at most one point of K. The union of rays from 0 via points of K is denoted by $\mathrm{OpenCone}(K)$. Thus, every point of $\mathrm{OpenCone}(K) \setminus \{0\}$ has unique representation as $t \cdot k$, $t > 0$ and $k \in K$. The most important examples are $K = S^{m-1}$ in which case $\mathrm{OpenCone}(K) = \mathbf{R}^m$ and $K \subset \Delta^{m-1}$, Δ^{m-1} being the standard simplex spanned by the basis e_i, $1 \leq i \leq m$, of $\mathbf{R}^m$.

We are interested in open cones over sets K such that any point $x \in K$ has a closed neighborhood A_x in K with the property that the distance between any 'tangent' vector $(v-w)/|v-w|$, $v, w \in A_x$, and $x/|x|$ is bigger than $\alpha(x)$, where $\alpha\colon K \to (0,\infty)$ is a function. For simplicity, let us call such sets *regular*. Notice that both S^{m-1} and Δ^{m-1} are regular.

The reason for next proposition is that simplexes offer better geometric properties (convexity) than subsets of spheres. Our main use for it is in replacing $\mathbf{R}^{m+1}$ by the open cone over Δ^m.

Proposition 3.1. *Suppose $f\colon K \to L$ is a Lipschitz map of compact subsets of Euclidean spaces. If K is regular, then*

$$f'\colon \mathrm{OpenCone}(K) \to \mathrm{OpenCone}(L)$$

defined by $f'(t \cdot k) = t \cdot f(k)$ is Lipschitz. In particular, if $f\colon K \to L$ is a bi-Lipschitz homeomorphism of regular sets, then f' is a bi-Lipschitz homeomorphism.

Proof. Suppose there is a sequence of points $t_n \cdot k_n, s_n \cdot l_n \in \mathrm{OpenCone}(K)$ such that $|t_n f(k_n) - s_n f(l_n)| > \beta(n)|t_n k_n - s_n l_n|$ and $\beta(n) \to \infty$ as $n \to \infty$. We may assume $s_n \neq 0$ (only one of t_n, s_n can be 0 for a given n). Replacing t_n by t_n/s_n and s_n by 1 (divide both sides of the inequality by s_n), we may assume $s_n = 1$ for all n. Notice 0 cannot be the limit of an infinite subsequence of $\{t_n\}$ (in that case $\beta(n) < 2\mathrm{diam}(L)/d(0,K)$ for infinitely many n, a contradiction). Similarly, no subsequence of $\{t_n\}$ can approach infinity (reduce to the previous case by switching to $1/t_n$ via dividing). Without loss of generality, we may assume $t_n \to t_0 > 0$ and $k_n \to k_0$, $l_n \to l_0$ (as K, L are compact). Notice $t_n = 1$ only for finitely many n (as f is Lipschitz), so assume $t_n \neq 1$ for all n. Similarly, we may assume $k_n \neq l_n$ for all n. Now $t_0 = 1$ (otherwise the minimum distance from K to $t_0 K$ is positive and one can get a bound on $\beta(n)$) which implies $k_0 = l_0$. It is here where we use regularity of K: assume all k_n, l_n lie in a closed neighborhood A of $k_0 = l_0$ so that there is $\alpha > 0$ with the property that $|(k-l)/|k-l| - tp| > \alpha$ for all $k, l, p \in A$ and all $t \in \mathbf{R}$ (the original assumption was satisfied for all

$t > 0$ but it implies such inequality for all t). Put $(t_n - 1)/|k_n - l_n| = r_n$ and $(k_n - l_n)/|k_n - l_n| = v_n$. The inequality $|t_n f(k_n) - s_n f(l_n)| > \beta(n)|t_n k_n - s_n l_n|$ can be easily transformed into $|r_n f(k_n) + (f(k_n) - f(l_n))/|k_n - l_n|| > \beta(n)|r_n k_n + v_n|$. As $|f(k_n) - f(l_n)|/|k_n - l_n|$ is bounded (f is Lipschitz), no subsequence of r_n can be unbounded, so we may assume $r_n \to r_0$. That means $r_n k_n + v_n \to 0$ and $r_0 = -1/|k_0|$. In that case $|r_n k_n + v_n| > \alpha/2$ for large n, hence $|r_n f(k_n) + (f(k_n) - f(l_n))/|k_n - l_n||$ is bounded resulting in $\beta(n)$ being bounded, a contradiction. □

The next result shows benefits in switching to open cones over simplexes. Namely, it provides a simple criterion for the canonical partition of unity subordinate to a cover to induce a Lipschitz 'norm-preserving' map.

Given an open cover $\mathcal{U} = \{U_1, \ldots, U_k\}$ of a metric space X by proper subsets, by the *canonical partition of unity $\phi_{\mathcal{U}}$ induced by $\mathcal{U}$* we mean functions $\phi_i(x) := \frac{d(x, X \setminus U_i)}{\sum_{j=1}^{k} d(x, X \setminus U_j)}$, $i = 1, \ldots, k$. Such a partition of unity gives the *canonical map* from X to the nerve $\mathcal{N}(\mathcal{U})$ of $\mathcal{U}$ defined by $\phi(x) = \sum_{i=1}^{k} \phi_i(x) \cdot v_i$, where v_i is the vertex corresponding to U_i.

Proposition 3.2. *Suppose $\mathcal{U} = \{U_1, \ldots, U_k\}$ is an open cover of a metric space X by proper subsets and $\phi_{\mathcal{U}}$ is the canonical partition of unity induced by $\mathcal{U}$. $\phi'_{\mathcal{U}} \colon X \to \mathrm{OpenCone}(\mathcal{N}(\mathcal{U}))$ is Lipschitz if and only if there is $\epsilon > 0$ such that one has*

$$\sum_{i=1}^{k} d(x, X \setminus U_i) \geq \epsilon |x|$$

for all x.

Proof. Fix $M > 1$. Choose $i \leq k$ and put $f(x) = d(x, X \setminus U_i)$, $g(x) = \sum_{i=j}^{k} d(x, X \setminus U_j) - f(x)$. Notice f and g are k-Lipschitz and $f(x) + g(x) \geq \epsilon|x|$. Let $h = \frac{f}{f+g}$. We want to show $|h(x) - h(y)| \leq \frac{3kd(x,y)}{\epsilon|y|}$ if $|x| \geq |y| > 0$ for some $x, y \in X$. Put $a = \frac{3kd(x,y)}{\epsilon|y|}$. If $h(x) - h(y) > a$, then $\frac{f(x)}{f(x)+g(x)} - \frac{f(x)-kd(x,y)}{f(x)+g(x)+2\cdot kd(x,y)} > a$ as well. Since $\frac{f(x)}{f(x)+g(x)} - \frac{f(x)-kd(x,y)}{f(x)+g(x)+2\cdot kd(x,y)} = \frac{f(x)\cdot 2kd(x,y)+kd(x,y)\cdot(f(x)+g(x))}{(f(x)+g(x))\cdot(f(x)+g(x)+2kd(x,y))} \leq \frac{3kd(x,y)}{f(x)+g(x)+2kd(x,y)} \leq \frac{3kd(x,y)}{f(x)+g(x)} \leq a$, we arrive at a contradiction.

Now, if $y = x_0$, then $||x|h(x) - |y|h(y)| = |x|h(x) \leq |x| = d(x, y)$. If $|x| \geq |y| > 0$, then $||x|h(x) - |y|h(y)| = ||y|(h(x) - h(y) + h(x)(|x| - |y|)| \leq 3kd(x, y)/\epsilon + d(x, y) = (3k/\epsilon + 1)d(x, y)$.

Suppose ϕ' is λ-Lipschitz. Let $S(x) = \sum_{i=1}^{k} d(x, X \setminus U_i)$. Suppose $x \in X$ and choose i such that $x \in U_i$. Find $y \in X \setminus U_i$ so that $d(x, X \setminus U_i) > d(x,y)/2$. Since $\lambda \cdot d(x,y)S(x) \geq |x|d(x, X \setminus U_i) - |y|d(y, X \setminus U_i)S(x)/S(y) = |x|d(x, X \setminus U_i) > |x|d(x,y)/2$, we get $S(x) > \frac{|x|}{2\lambda}$. □

Our last result on open cones shows that functions from X to Δ^m that induce Lipschitz maps from X to $\mathrm{OpenCone}(\Delta^m)$ form a 'convex' set.

Proposition 3.3. *Suppose (X,d) is a metric space and $\gamma\colon X \to \Delta^1$ induces a Lipschitz function $\gamma'\colon X \to \mathrm{OpenCone}(\Delta^1)$. If $f, g\colon X \to \Delta^m$ induce Lipschitz functions $f', g'\colon X \to \mathrm{OpenCone}(\Delta^m)$ for some $m \geq 1$, then $h = \alpha \cdot f + \beta \cdot g\colon X \to \Delta^m$, where α, β are barycentric coordinates of γ, induces a Lipschitz function $h'\colon X \to \mathrm{OpenCone}(\Delta^m)$.*

Proof. $h'(x) - h'(y) = |x|\alpha(x)f(x) + |x|\beta(x)g(x) - |y|\alpha(y)f(y) - |y|\beta(y)g(y) = \alpha(x)(f'(x) - f'(y)) + (\alpha(x) - \alpha(y))|y|f(y) + \beta(x)(g'(x) - g'(y)) + (\beta(x) - \beta(y))|y|g(y)$. Also, $|(\alpha(x) - \alpha(y)) \cdot |y|| = |(|x|\alpha(x) - |y|\alpha(y)) + \alpha(x) \cdot (|y| - |x|)| \leq |\gamma'(x) - \gamma'(y)| + d(x,y) \leq (1 + \mathrm{Lip}(\gamma')) \cdot d(x,y)$ (the same holds for β), so $|h'(x) - h'(y)| \leq \mathrm{Lip}(f') \cdot d(x,y) + \mathrm{Lip}(g') \cdot d(x,y) + (2\mathrm{Lip}(\gamma') + 2) \cdot d(x,y)$ and h' is Lipschitz. □

4. Sublinear coarse structure

The sublinear coarse structure was introduced in [6]. Our approach is a little bit different to allow for a better transition from previous sections of our paper.

Definition 4.1. A continuous function $s\colon \mathbf{R}_+ \to \mathbf{R}_+$ is called *asymptotically sublinear* if for each non-constant linear function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ there is M such that $s(r) \leq f(r) + M$. Equivalently, there is $r_0 > 0$ such that $s(r) \leq f(r)$ for $r \geq r_0$.

Proposition 4.2. *If $t_n, a_n \in \mathbf{R}_+$ and $t_n \to \infty$, $a_n \to 0$, then there is an asymptotically sublinear function $s\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that $s(t_k) = a_k \cdot t_k$ for infinitely many k.*

Proof. If for each n there is $k > n$ such that $a_k \cdot t_k \leq a_n \cdot t_n$, then such s is easy to construct as a piecewise linear function that is decreasing starting from some t. Without loss of generality assume $a_k \cdot t_k > a_n \cdot t_n$ if $k > n$. The idea is to pick a sequence $n(i)$ such that the slopes $\frac{a_{n(i+1)}t_{n(i+1)} - a_{n(i)}t_{n(i)}}{t_{n(i+1)} - t_{n(i)}}$ form a decreasing sequence tending to 0. Once it is done (easy exercise) create the graph of s in a piecewise linear fashion. □

Definition 4.3. A continuous function $f\colon (X,d_X)\to(Y,d_Y)$ is called *Higson sublinear* if for each asymptotically sublinear function $s\colon \mathbf{R}_+\to\mathbf{R}_+$ the conditions $x_n,y_n\to\infty$ and $d_X(x_n,y_n)\le s(|x_n|)$ for all $n\ge 1$ imply $d_Y(f(x_n),f(y_n))\to 0$.

Using asymptotically sublinear functions one can introduce the *sublinear coarse structure* $\mathcal{E}_L(X)$ on X either following [7] and declaring a family of subsets of X to be uniformly bounded if and only if it refines $\{B(x,s(|x|))\}_{x\in X}$ for some asymptotically sublinear s. Alternatively, one can use [10] and declare $E\in\mathcal{E}_L(X)$ if and only if $E\subset\bigcup\limits_{x\in X}B(x,s(|x|))\times B(x,s(|x|))$ for some asymptotically sublinear s. The corresponding *Higson sublinear compactification* $h_L(X)$ has the property that all bounded maps $f\colon h_L(X)\to\mathbf{R}$ induce Higson sublinear maps $f|_X$ and every Higson sublinear bounded $f\colon X\to\mathbf{R}$ extends over $h_L(X)$. If X is proper, then $\nu_L(X):=h_L(X)\setminus X$ is called the *sublinear Higson corona of* X.

Proposition 4.4. *Suppose $f\colon X\to S^m$ is a continuous function. If $f'\colon X\to\mathbf{R}^{m+1}$ defined by $f'(x)=|x|\cdot f(x)$ is asymptotically Lipschitz, then f is Higson sublinear.*

Proof. Suppose $s\colon\mathbf{R}_+\to\mathbf{R}_+$ is asymptotically sublinear and $x_n,y_n\to\infty$ so that $d(x_n,y_n)\le s(|x_n|)$ for all n. Let $\lambda,M>0$ satisfy $|f'(x)-f'(y)|\le\lambda\cdot d(x,y)+M$ for all $x,y\in X$ and suppose $\epsilon>0$. Pick $r_0>0$ such that $s(r)\le\frac{\epsilon}{\lambda+1}\cdot r$ for $r>r_0$. If n is sufficiently large so that $|x_n|>r$, then

$$\begin{aligned}&|f(x_n)-f(y_n)|\cdot|x_n| = ||x_n|\cdot f(x_n)-|x_n|\cdot f(y_n)|\\ &=|f'(x_n)-f'(y_n)-(|x_n|-|y_n|)\cdot f(y_n)|\le|f'(x_n)-f'(y_n)|+||x_n|-|y_n||\\ &\le\lambda\cdot d(x_n,y_n)+M+d(x_n,y_n)\le(\lambda+1)\cdot s(|x_n|)+M\le\epsilon\cdot|x_n|+M.\end{aligned}$$

Therefore $|f(x_n)-f(y_n)|\le 2\epsilon$ for large n. □

Remark 4.5. Consider $f\colon\mathbf{N}\to S^1$ defined by $f(2n)$ being at distance $\frac{1}{\sqrt{n}}$ from 1 and $f(2n+1)=1$ for all $n\in\mathbf{N}$. Notice that $f'(n)=n\cdot f(n)$ is not asymptotically Lipschitz but f is Higson sublinear.

Our next result generalizes part of Lemma 2.3 of [6] and our proof seems simpler.

Proposition 4.6. *Suppose (X,d) is a discrete metric space. If $\mathcal{U}=\{U_1,\dots,U_k\}$ is an open cover of $h_L(X)$, then either $X\subset U_i$ for some i or there is $\epsilon>0$ such that $\sum\limits_{i=1}^{k}d(x,X\setminus U_i)\ge\epsilon\cdot|x|$ for all $x\in X$.*

Proof. By contradiction. Using 4.2 find a sequence of points $x_n \in X$ tending to infinity and an asymptotically sublinear function s such that $\sum_{i=1}^{k} d(x_n, X \setminus U_i) < s(|x_n|)$ for all $n \geq 1$. Given that data, the sequence $\{x_n\}$ has a cluster point $x_0 \in \nu_L(X)$. As $x_0 \in U_i$ for some i, we may as well assume $x_n \in U_1$ for all $n \geq 0$. Since $d(x_n, X \setminus U_1) < s(|x_n|)$ for all $n \geq 1$, either $X \subset U_1$ or for each n there is $y_n \in X \setminus U_1$ such that $d(x_n, y_n) < s(|x_n|)$. Pick a continuous function $u\colon h_L(X) \to [0,1]$ such that $u(x_0) = 1$ and $u(h_L(X) \setminus U_1) = \{0\}$. $u|_X$ is Higson sublinear, so $1 = u(x_n) - u(y_n) \to 0$, a contradiction. □

Proposition 4.7. *Suppose (X, d) is a discrete metric space. A function $f\colon X \to S^m$ is Higson sublinear if and only if it is the uniform limit of functions $g\colon X \to S^m$ such that the functions $g'\colon X \to \mathbf{R}^{m+1}$, $g'(x) = |x| \cdot g(x)$, are Lipschitz.*

Proof. Suppose $f\colon X \to S^m$ is Higson sublinear. Replace S^m by $\partial\Delta^{m+1}$ using a bi-Lipschitz homeomorphism. Given $\epsilon > 0$ approximate f by a partition of unity on a cover of X that is a restriction of an open cover of $h_L(X)$. Such partitions of unity induce $g\colon X \to \partial\Delta^{m+1}$ for which $g'\colon X \to \mathrm{OpenCone}(\partial\Delta^{m+1})$ is Lipschitz (see 4.6 and 3.2). Switch back to S^m using 3.1. □

Proposition 4.8. *Suppose (X, d) is a proper metric space. If X' is a discrete ϵ-net in X for some $\epsilon > 0$, then the inclusion $i\colon X' \to X$ induces a homeomorphism of sublinear Higson coronas $\nu_L(X') \to \nu_L(X)$.*

Proof. Notice the image of $h_L(X')$ in $h_L(X)$ must contain $\nu_L(X)$. Otherwise there is a continuous $\alpha\colon h_L(X) \to [0,1]$ sending $i_*(h_L(X'))$ to 0 and some point $x_1 \in \nu_L(X)$ to 1. $\alpha|_X$ is Higson sublinear, so there is $r > 0$ such that $\alpha(X \setminus B(x_0, r)) \subset [0, 1/2]$ (otherwise there are points $x_n, y_n \in X$ tending to infinity with $x_n \in X'$, $d(x_n, y_n) < \epsilon$, and $\alpha(y_n) > 1/2$ contradicting $\alpha|_X$ being Higson sublinear). Since $X \setminus B(x_0, r)$ contains x_1 in its closure in $h_L(X)$, $\alpha(x_1) \leq 1/2$, a contradiction. Thus $\nu_L(X') \to \nu_L(X)$ is surjective.

Suppose $x_1 \neq x_2 \in \nu_L(X')$. Pick two closed neighborhoods A_1 of x_1 and A_2 of x_2 in $h_L(X')$ that are disjoint. Notice $X \cap i(A_j) = X' \cap A_j$, $j = 1, 2$, so call that set C_j. By 4.6 applied to the cover $\{h_L(X') \setminus A_1, h_L(X') \setminus A_2\}$ there is $\delta > 0$ such that $d(x, C_1) + d(x, C_2) \geq \delta \cdot |x|$ for all $x \in X'$. Let us show that implies existence of $\gamma > 0$ such that $S(x) := d(x, C_1) + d(x, C_2) \geq \gamma \cdot |x|$ for all $x \in X$.

Given $x \in X$ pick $x_1 \in X_1$ so that $d(x, x_1) < \epsilon$. Since S is 2-Lipschitz, $S(x) \geq S(x_1) - 2d(x, x_1) \geq \delta|x_1| - 2\epsilon$ and $S(x) \geq (\delta/2)|x|$ if $|x| \geq 2\epsilon(2+\delta)/\delta$. For x such that $0 < |x| \leq 2\epsilon(2+\delta)/\delta$ pick $a_1 \in C_1$ and $a_2 \in C_2$ so that $d(x, a_i) \leq \lambda|x|$ for some small $\lambda > 0$. Hence $d(a_1, a_2) \leq 2\lambda|x|$ is small but the minimal distance between C_1 and C_2 is at least the discreteness of X_1. By 3.2 and 4.4 the function $x \to \frac{d(x,C_1)}{d(x,C_1)+d(x,C_2)}$ is Higson sublinear on X and extends uniquely over $h_L(X)$. Its restriction to X' vanishes on C_1 and is 1 on C_2 implying $i(x_1) \neq i(x_2)$. $\square$

5. Dimension of sublinear Higson corona

Theorem 5.1. *Suppose (X,d) is a discrete metric space. The dimension of the sublinear Higson compactification $h_L(X)$ of X is the smallest integer $m \geq 0$ with the following property: Any norm-preserving Lipschitz function $f'\colon A \to \mathbf{R}^{m+1}$, $A \subset X$, extends to a norm-preserving Lipschitz function $g'\colon X \to \mathbf{R}^{m+1}$.*

Proof. Suppose $\dim(h_L(X)) \leq m$ and $f'\colon A \subset X \to \mathbf{R}^{m+1}$ is a Lipschitz function induced by $f\colon A \to S^m$. Replace S^m by $\partial\Delta^{m+1}$ using 3.1. Using 2.4 extend f to $f\colon X \to \Delta^{m+1}$ inducing Lipschitz f'. Find a Lipschitz $r\colon \Delta^{m+1} \to \Delta^{m+1}$ that induces a retraction $N \to \partial\Delta^{m+1}$ from a closed neighborhood N of $\partial\Delta^{m+1}$ in Δ^{m+1}. Let $g_1\colon A_1 = f^{-1}(N) \to \partial\Delta^{m+1}$ be defined by $g_1(x) = r(f(x))$. g_1 is Higson sublinear by 4.4. As $\dim(h_L(X)) \leq m$, it extends over $h_L(X)$ to a continuous function g_2. Using 4.7 choose $g_3\colon X \to \partial\Delta^{m+1}$ with the property that the segment joining $g_2(x)$ and $g_3(x)$ in Δ^{m+1} is always contained in N for all $x \in X$ and g_3 induces Lipschitz $g_3'\colon X \to \text{OpenCone}(\partial\Delta^{m+1})$.

Put $U_1 = f^{-1}(int(N))$ and $U_2 = f^{-1}(\Delta^{m+1} \setminus \partial\Delta^{m+1})$. By 4.6 and 3.2, the canonical partition of unity $\{\alpha, \beta\}$ of the cover $\{U_1 \cap X, U_2 \cap X\}$ gives a map from X to Δ^1 inducing Lipschitz $X \to \text{OpenCone}(\Delta^1)$. Since $\alpha \cdot (r \circ f) + \beta \cdot g_3$ induces a Lipschitz function to the open cone (see 3.3) and its values are in N, $g = r \circ (\alpha \cdot (r \circ f) + \beta \cdot g_3)$ is the required extension of f.

Suppose $\text{asdim}_L(X) \leq m$. Given a map $f\colon h_L(X) \to \Delta^{m+1}$ it induces an open cover $\mathcal{U} = \{U_1, \ldots, U_{m+2}\}$ of $h_L(X)$, where $U_i = f^{-1}(St(e_i))$. By 4.6, 3.2, and 3.1 there is $g\colon h_L(X) \to \partial\Delta^{m+1}$ with the property that $g(x) \notin St(e_i)$ if $f(x) \notin St(e_i)$. That is sufficient to conclude that S^m is an absolute extensor of $h_L(X)$, i.e. $\dim(h_L(X)) \leq m$. Indeed, if f is an extension of $f_A\colon A \subset h_L(X) \to \partial\Delta^{m+1}$, then $g|_A$ and f are homotopic (via straight segments) resulting in f_A being extendible over $h_L(X)$. $\square$

Corollary 5.2. *Suppose (X,d) is a proper metric space. The dimension of the sublinear Higson corona $\nu_L(X)$ of X is the smallest integer $m \geq 0$ with the following property: Any norm-preserving asymptotically Lipschitz function $f'\colon A \to \mathbf{R}^{m+1}$, $A \subset X$, extends to a norm-preserving asymptotically Lipschitz function $g'\colon X \to \mathbf{R}^{m+1}$.*

Proof. Pick a discrete 1-net X_1 in X and notice $h_L(X_1)$ is the closure of X_1 in $h_L(X)$ and $h_L(X_1) = X_1 \cup \nu_L(X)$ by 4.8. Use 2.1. □

Corollary 5.3. *Suppose (X, d) is a cocompact proper metric space that is M-scale connected for some M. If $\mathrm{asdim}_{AN}(X)$ is finite, then $\mathrm{asdim}_{AN}(X) = \dim(\nu_L(X))$.*

Proof. As in 5.2 consider a discrete 1-net X_1 in X while using 2.7 to get inequality $\mathrm{asdim}_{AN}(X) \leq \dim(\nu_L(X))$. The other inequality follows from 2.5 using the fact that S^m is a Lipschitz extensor if $\dim_{AN}(X) \leq m$ (see [9] or [3]). □

6. Final comments

In [5] Dranishnikov introduced the category $\tilde{\mathcal{A}}$ whose objects are pointed proper metric spaces X and morphisms are asymptotically Lipschitz functions $f\colon X \to Y$ such that there are constants $b, c > 0$ satisfying $|f(x)| \geq c \cdot |x| - b$ for all $x \in X$. Let us characterize $\mathrm{asdim}_L(X)$ in terms of $\tilde{\mathcal{A}}$. Namely, $\mathrm{asdim}_L(X) \leq n$ if and only if $\mathbf{R}^{n+1}$ is an absolute extensor of X in the category $\tilde{\mathcal{A}}$.

Proposition 6.1. *Suppose (X, d) is a pointed proper metric space and $s\colon X \to \mathbf{R}_+$ is a morphism of $\tilde{\mathcal{A}}$. If $f\colon X \to S^n$, then the following conditions are equivalent:*

a. *$f'\colon X \to \mathbf{R}^{n+1}$, $f'(x) = |x| \cdot f(x)$, is asymptotically Lipschitz.*
b. *$F\colon X \to \mathbf{R}^{n+1}$ is asymptotically Lipschitz, where $F(x) = s(x) \cdot f(x)$.*

Proof. Suppose $\lambda \cdot |x| + M \geq s(x) \geq 2c \cdot |x| - b$ for some constants $\lambda, M, c, b > 0$. Also, $|s(x) - s(y)| \leq \lambda \cdot d(x, y) + M$.

(a) $\Longrightarrow$ (b). Assume $|f'(x) - f'(y)| \leq u \cdot d(x, y) + v$ for some $u, v > 0$. If $|y| \leq b/c$, then $|F(x) - F(y)| \leq s(x) + s(y) \leq \lambda \cdot |x| + M + \lambda \cdot b/c + M \leq \lambda \cdot d(x, y) + \lambda \cdot b/c + \lambda \cdot b/c + 2M$, so assume $|x| \geq |y| > b/c$. Notice $f'(x) - f'(y) = |x| \cdot (f(x) - f(y)) + (|x| - |y|) \cdot f(y)$, so $|x| \cdot |f(x) - f(y)| \leq u \cdot d(x, y) + v + d(x, y)$. Similarly, $F(x) - F(y) = s(x) \cdot (f(x) - f(y)) + (s(x) - s(y)) \cdot f(y)$ and $|F(x) - F(y)| \leq \frac{s(x)}{|x|} \cdot (u \cdot d(x, y) + v + d(x, y)) + \lambda \cdot d(x, y) + M \leq (\lambda + Mc/b) \cdot (u \cdot d(x, y) + d(x, y) + v) + \lambda \cdot d(x, y) + M$.

(b) $\Longrightarrow$ (a). By enlarging λ and M, if necessary, assume $|F(x) - F(y)| \leq \lambda d(x, y) + M$ for all $x, y \in X$. If $|y| \leq b/c$, then $|f'(x) - f'(y)| \leq |x| + |y| \leq d(x, y) + 2|y| \leq d(x, y) + 2b/c$, so assume $|x| \geq |y| > b/c$ and observe $s(x) \geq c \cdot |x|$. Notice $F(x) - F(y) = s(x) \cdot (f(x) - f(y)) + (s(x) - s(y)) \cdot f(y)$, so $s(x) \cdot |f(x) - f(y)| \leq 2\lambda \cdot d(x, y) + 2M$. Similarly, $f'(x) - f'(y) = |x| \cdot (f(x) - f(y)) + (|x| - |y|) \cdot f(y)$, so $|f'(x) - f'(y)| \leq \frac{|x|}{s(x)} (2\lambda \cdot d(x, y) + 2M) + d(x, y) \leq (2\lambda \cdot d(x, y) + 2M)/c + d(x, y)$. □

Corollary 6.2. *Suppose (X, d) is a pointed proper metric space. The dimension of the sublinear Higson corona of X is the smallest integer $n \geq 0$ such that $\mathbf{R}^{n+1}$ is an absolute extensor of X in the category $\tilde{\mathcal{A}}$.*

PROOF. Suppose $\dim(\nu_L(X)) \leq n$ and $F\colon A \subset X \to \mathbf{R}^{n+1}$ is a morphism of $\tilde{\mathcal{A}}$, where A is a closed subset of X. Since $\mathbf{R}_+$ is an absolute extensor of X in $\tilde{\mathcal{A}}$ by Theorem 3.1 of [5] (p.1095), there is a morphism $s\colon X \to \mathbf{R}_+$ of $\tilde{\mathcal{A}}$ such that $s(x) = |F(x)|$ for all $x \in A$. Pick $f\colon A \to S^n$ such that $F(x) = s(x) \cdot f(x)$ for all $x \in A$ ($f(x)$ is not uniquely defined if $s(x) = 0$ in which case pick a value for $f(x)$ in an arbitrary fashion). Extend $f'\colon A \to \mathbf{R}^{n+1}$ to a norm-preserving asymptotically Lipschitz $g'\colon X \to \mathbf{R}^{n+1}$ induced by $g\colon X \to S^n$ (use 6.1 and 5.2). Define $G\colon X \to \mathbf{R}^{n+1}$ by $G(x) = s(x) \cdot g(x)$. It is asymptotically Lipschitz by 6.1.

Suppose $\mathbf{R}^{n+1}$ is an absolute extensor of X in $\tilde{\mathcal{A}}$. Pick a discrete 1-net X_1 in X. Use 6.1 to conclude that any norm-preserving Lipschitz $A \subset X_1 \to \mathbf{R}^{n+1}$ extends over X_1 to a norm-preserving Lipschitz function. By 4.8 and 5.2, $\dim(\nu_L(X)) \leq n$. $\square$

Notice in the proof of 6.2 we had to insist that A is a closed subset of X. It is so because that is how absolute extensors of categories $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are defined in [5]. And that makes sense since, for $f\colon A \subset X \to Y$ to be a morphism in any of those categories, A needs to be proper metric, hence closed in X. However, once one leaves the language of categories, the requirement of A to be closed can be dropped as shown in the next proposition.

Proposition 6.3. *Suppose (X, d_X) and (Y, d_Y) are pointed metric spaces and $f\colon A \subset X \to Y$ is an asymptotically Lipschitz function such that $|f(x)| \geq c \cdot |x| - b$ for some constants $c, b > 0$. If $R > 0$, then there is an asymptotically Lipschitz extension $g\colon B(A, R) \to Y$ of f such that $|g(x)| \geq c \cdot |x| - v$ for some constant $v > 0$.*

PROOF. Suppose f is (λ, M)-asymptotically Lipschitz. Pick a retraction $r\colon B(A, R) \to A$ such that $d_X(x, r(x)) < R$ for all $x \in B(A, R)$ and define $g(x) = f(r(x))$ for $x \in B(A, R)$. Using $|r(x)| \geq |x| - R$ one easily gets $|g(x)| = |f(r(x)| \geq c|r(x)| - b \geq c|x| - cR - b$. Also, $d_Y(g(x), g(y)) = d_Y(f(r(x)), f(r(y))) \leq \lambda \cdot d_X(r(x), r(y)) + M \leq \lambda \cdot d_X(x, y) + 2\lambda R + M$. $\square$

Using 6.3 one can extend any asymptotically Lipschitz function from A to its closure since the ball $B(A, R)$ contains that closure.

References

[1] G. Bell and A. N. Dranishnikov, *On asymptotic dimension of groups acting on trees* Geom. Dedicata **103** (2004), 89–101.

[2] N. Brodskiy and J. Dydak, *Coarse dimensions and partitions of unity*, preprint arXiv:math.GT/0506547.

[3] N. Brodskiy, J. Dydak, J. Higes, A. Mitra, *Nagata-Assouad dimension via Lipschitz extensions*, preprint arXiv:math.MG/0601226.

[4] S. Buyalo and V. Schroeder, *Hyperbolic dimension of metric spaces*, arXiv:math.GT/0404525

[5] A. N. Dranishnikov, *Asymptotic topology*, Russian Math.Surveys 55 (2000), no.6, 1085–1129.

[6] A. N. Dranishnikov and J. Smith, *On asymptotic Assouad-Nagata dimension*, preprint ArXiv: math.MG/0607143

[7] J. Dydak and C. S. Hoffland, *An alternative definition of coarse structures*, preprint ArXiv: math.MG/0605562.

[8] M. Gromov, *Asymptotic invariants for infinite groups*, in Geometric Group Theory, vol. 2, 1–295, G. Niblo and M. Roller, eds., Cambridge University Press, 1993.

[9] U. Lang and T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, IMRN International Mathematics Research Notices (2005), no. 58, 3625–3655.

[10] J. Roe, *Lectures on coarse geometry*, University Lecture Series, 31. American Mathematical Society, Providence, RI, 2003.

(M. Cencelj) Pedagoška fakulteta, Univerza v Ljubljani, Kardeljeva pl. 16, SI-1000 Ljubljana, Slovenija

E-mail address: matija.cencelj@guest.arnes.si

(J. Dydak) University of Tennessee, Knoxville, TN 37996, USA

E-mail address: dydak@math.utk.edu

(J. Smrekar) Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska ulica 19, SI-1111 Ljubljana, Slovenija

E-mail address: jaka.smrekar@fmf.uni-lj.si

(A. Vavpetič) Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska ulica 19, SI-1111 Ljubljana, Slovenija

E-mail address: ales.vavpetic@fmf.uni-lj.si