CW towers and mapping spaces

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Abstract

Let $\cdots \to Z_3 \to Z_2 \to Z_1$ be a tower of Hurewicz fibrations where each Z_i is homotopy equivalent to a CW complex. An important question is when also the inverse limit space is homotopy equivalent to a CW complex; we present a sufficient condition that is necessary up to a looping of the tower. When the homotopy groups of the Z_i vanish above a fixed dimension, we obtain simple necessary and sufficient conditions. The main application is $\mathrm{map}(X, Y)$, the space of continuous maps from a countable CW complex X to a CW complex Y. Our results are sharpest when Y is countable with finitely many nontrivial homotopy groups. Then, the path component of $f \in \mathrm{map}(X, Y)$ is homotopy equivalent to a CW complex if and only if all homotopy groups $\pi_k(\mathrm{map}(X, Y), f)$ are countable and f is not a phantom map. Let Y be nilpotent and let $l_*\colon Y \to Y_{(P)}$ be localization at the set of primes P. We show that if $Y_{(P)}$ is a Postnikov section and $\mathrm{map}(X, Y)$ has CW homotopy type, the induced mapping $l_*\colon\ \mathrm{map}(X, Y) \to \mathrm{map}(X, Y_{(P)})$ is localization at P on each path component, thereby extending a classical result from localization theory.

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1. Introduction

Consider a tower (inverse sequence) of fibrations between topological spaces

$$\cdots \to Z_i \xrightarrow{p_i} Z_{i-1} \to \cdots \to Z_1 \tag{1}$$

where each Z_i is homotopy equivalent to a CW complex. We call them *CW towers of fibrations*. They occur naturally in homotopy theory. We investigate conditions under which the inverse limit space also has the homotopy type of a CW complex.

In 1966, Skljarenko [22] proved that if (1) is the Postnikov tower of a CW complex with infinitely many nontrivial homotopy groups, the inverse limit space fails to have CW homotopy type. In relation with representability of singular cohomology in [2], Dydak and Geoghegan made an attempt to find 'algebraic' criteria for CW homotopy type of Z_∞ (see also [3] and [23]). This seems hard to accomplish in general but we do it here in special cases.

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Our motivating example is the following. Let $\operatorname{map}(X, Y)$ denote the space of continuous functions $X \to Y$ with the compact open topology. Assume that X is a countable, and Y an arbitrary CW complex, and let $\{L_i\}$ be an increasing sequence of finite subcomplexes of X with $\cup_i L_i = X$. As subcomplex inclusions are cofibrations, the associated restriction mappings $\operatorname{map}(L_i, Y) \to \operatorname{map}(L_{i-1}, Y)$ are fibrations. By Milnor [19], Theorem 3, the spaces $Z_i = \operatorname{map}(L_i, Y)$ have CW homotopy type. The defining properties of CW topology ensure that $\operatorname{map}(X, Y)$ and the inverse limit space $\lim_i Z_i$ are homeomorphic.

Apparently the first positive result which involved nontrivial infinite domain complexes X came in 1984 from Peter Kahn (see [13]) who showed that $\operatorname{map}(X, Y)$ has CW type if X and Y are CW complexes where X has finite n-skeleton and $\pi_k(Y)$ is trivial for $k > n$. The positive result was subsequently extended in our previous work [23]. In [25], we gave nontrivial positive and negative results depending on the (non)existence of eventual geometric exponents of H-spaces. That shows that the problem of CW homotopy type of mapping spaces contains some deep (and unsolved) problems of algebraic topology.

Here we present a definitive solution to the problem of CW homotopy type of the inverse limit of a CW tower of fibrations (1) in which all spaces Z_i have vanishing homotopy groups above a fixed dimension. In that case, the problem reduces to studying the morphisms $\pi_k(Z_i) \to \pi_k(Z_{i-1})$ induced by the bonding maps p_i on homotopy groups. Necessary and sufficient 'algebraic' conditions for CW type of the inverse limit can be given which can be interpreted very simply in terms of homology groups under additional assumptions. The main application are mapping spaces $\operatorname{map}(X, Y)$ where Y has only finitely many nontrivial homotopy groups. In that case our result can be considered a complete solution of the problem.

For a general CW tower of fibrations we present a sufficient 'geometric' condition on the bonding maps that is necessary up to a single looping. It can be interpreted nicely in terms of the standard model category structure on towers of topological spaces. Our result also settles the problem of the erroneous Theorem B of [2].

We also investigate the effect of localization at a set of primes P on a CW tower of fibrations (1). We show that the localization of the inverse limit space coincides with the inverse limit of localized terms Z_i, *provided* the latter inverse limit space has CW type. The corresponding result for function spaces $\operatorname{map}(X, Y)$ is that the mapping $l_*\colon \operatorname{map}(X, Y) \to \operatorname{map}(X, Y_{(P)})$, induced by localization $l\colon Y \to Y_{(P)}$ at the set of primes P, is itself localization at P if $\operatorname{map}(X, Y)$ has CW type. This seems to be the natural generalization of the classical case (see Theorem B of Hilton, Mislin, Roitberg, Steiner [11]) where X is (homologically) finite. We remark that the dual case of $\operatorname{map}(X_{(P)}, Y) \to \operatorname{map}(X, Y)$ for a P-local space Y was dealt with in Theorem 5.5 of [25].

In [28] we showed that if Y is an absolute neighbourhood retract for metric spaces (shortly ANR) and X is a countable complex, then $\operatorname{map}(X, Y)$ has CW homotopy type if and only if it is an ANR. All results on mapping spaces in this paper can be rephrased to give necessary and/or sufficient conditions for a mapping space of a countable CW complex into an ANR to be an ANR.

Main results. Our main results are Theorem 3.2, Theorem 4.1, Theorem 4.7, Theorem 6.1, and Theorem 6.2. Section 7 contains applications to function spaces of continuous maps.

We proceed to list a handful of important consequences of our main results, mostly as applied to function spaces.

Let C be a path component of Z_∞ and let $\zeta = \{\zeta_i\}$ be a point in C. We assume that ζ_i is nondegenerate in Z_i for each i.

A tower consisting of nullhomotopic fibrations will be called a *contractible tower*, and a tower consisting of identity maps will be called a *constant tower*.

Theorem A. (**a**) *If C has CW type then a subtower of the tower* $\{\Omega(Z_i, \zeta_i) \,|\, i\}$ *homotopy splits into the product of a contractible tower and a constant tower.*

(**b**) *Conversely, if* $\{Z_i \,|\, i\}$ *splits into the product of a contractible tower and a constant tower associated to a space of CW type,* Z_∞ *has CW type.*

A *Postnikov section* is a CW complex whose homotopy groups vanish above some dimension. The following may be viewed as a weak Whitehead theorem for function spaces. It is a sharp generalization of Theorem 1.2 of Kahn [13].

Theorem B. *Let Y be a countable Postnikov section and let* $f\colon \Lambda \to X$ *be a map of countable CW complexes. If the nonempty homotopy fibres of* $f^*\colon \operatorname{map}(X, Y) \to \operatorname{map}(\Lambda, Y)$ *have trivial homotopy groups, they are (genuinely) contractible. Consequently if* $\operatorname{map}(\Lambda, Y)$ *has CW homotopy type,* f^* *is a homotopy equivalence to the union of path components that meet its image. Analogously for spaces of pointed maps.*

Loosely speaking, if the mapping $\operatorname{map}(X, Y) \to \operatorname{map}(\Lambda, Y)$ associated to a map of domains $f\colon \Lambda \to X$ induces isomorphisms on homotopy groups, then it is a homotopy equivalence. The spaces $\operatorname{map}(\Lambda, Y)$ and $\operatorname{map}(X, Y)$ need not have CW homotopy type.

If we let $f\colon \{*\} \to X$ be the base point inclusion, we infer as a special case that (if Y is a Postnikov section) $\operatorname{map}_*(X, Y)$ has trivial homotopy groups if and only if it is contractible.

Let X be a CW complex. Maps $f, g\colon X \to Y$ are said to form a *phantom pair of the second kind* if for each finite subcomplex K of X, the restrictions $f|_K$ and $g|_K$ are homotopic as maps $K \to Y$. The pair is *nontrivial* if f and g are not homotopic. We will call $f\colon X \to Y$ a *phantom map* if f forms a nontrivial phantom pair with some $g\colon X \to Y$. (In the literature, a phantom map is often understood to form a nontrivial phantom pair with a constant map.) Taking skeleta $X^{(n)}$ instead of finite subcomplexes one obtains the notion of phantom pairs *of the first kind*. However, here we only consider phantom maps of the second kind.

Let X be a countable CW complex and let $\operatorname{map}(X, Y)_f$ denote the path component of the map f in $\operatorname{map}(X, Y)$.

Theorem C. (**a**) *If Y is a countable Postnikov section,* $\operatorname{map}(X, Y)_f$ *has CW type if and only if its homotopy groups are countable and f is not phantom.*

(**b**) *If G is a countable abelian group then* $\operatorname{map}(X, K(G, n))$ *has CW type if and only if the groups* $H^k(X; G)$ *are countable for* $k \leqslant n$.

(**c**) *If Y is a simple Postnikov section, then* $\operatorname{map}(X, Y)$ *has CW type if* $\operatorname{map}(X, K(\pi_k(Y), k))$ *has CW type for all* $\pi_k(Y) \neq 0$.

Note that in (**a**) of Theorem C, the statement about f is that it does not form a nontrivial phantom pair with *any* other map $X \to Y$.

Let X be a countable CW complex and let P be a set of primes (possibly empty). Assume that Y is a nilpotent CW complex, i.e. $\pi_1(Y)$ is nilpotent and acts nilpotently on all higher homotopy groups. Let $l\colon Y \to Y_{(P)}$ be localization at P. It can be defined as a map that induces P-localization on all homotopy groups.

Recall that if a space W has CW type then so has $\Omega(W, w)$, for any choice of w.

Theorem D. *If* $\operatorname{map}(X, Y)_f$ *has CW type, then so has* $\Omega(\operatorname{map}(X, Y_{(P)}), lf)$ *and* $\Omega(\operatorname{map}(X, Y), f) \to \Omega(\operatorname{map}(X, Y_{(P)}), lf)$ *is CW localization on path components.*

If, in addition, $Y_{(P)}$ *is a Postnikov section, then also* $\operatorname{map}(X, Y_{(P)})_{lf}$ *has CW type,* $\operatorname{map}(X, Y)_f$ *and* $\operatorname{map}(X, Y_{(P)})_{lf}$ *are nilpotent, and* $l_*\colon \operatorname{map}(X, Y)_f \to \operatorname{map}(X, Y_{(P)})_{lf}$ *is localization at P.*

The analogous results hold for spaces of pointed maps as well. Taking $\mathrm{map}_*(X, Y)$ and $f = *$ we get that if $\mathrm{map}_*(X, Y)$ has CW type, so has $\mathrm{map}_*(X, \Omega Y_{(P)})$ and $\mathrm{map}_*(X, \Omega Y) \to \mathrm{map}_*(X, \Omega Y_{(P)})$ is CW localization on path components.

Other (potential) applications. We mention first that our methods apply also to various function spaces of smooth maps as follows. Let M and N be smooth (C^∞) finite-dimensional manifolds and let $C^k(M, N)$ denote the set of differentiable maps $M \to N$ of class C^k with the (weak) C^k-topology where $1 \leqslant k < \infty$. (See for example Eells [5] or Palais [21] for details.)

Now assume that N is without boundary and M is not compact. The question is what can be said about the homotopy type of $C^k(M, N)$. Let $\rho\colon M \to \mathbb{R}$ be a proper Morse exhaustion function and assume, for simplicity, that the natural numbers in $\mathbb{R}$ are regular values of f. Then the sublevel sets $M_i = \{x \in M \mid f(x) \leqslant i\}$, for $i \in \mathbb{N}$, form an ascending filtration of compact submanifolds with union M. We get an associated tower (1) by setting $Z_i = C^k(M_i, N)$ and letting $Z_i \to Z_{i-1}$ be restrictions. The spaces Z_i are paracompact C^∞-manifolds (see [21], 14.17) and in fact the natural inclusions $Z_i \to \mathrm{map}(M_i, N) = C^0(M_i, N)$ are homotopy equivalences. In particular, Z_i have CW homotopy type. Moreover, restrictions $Z_i \to Z_{i-1}$ are fibrations (see Eells [5], §7C and also Palais [20], §6). Hence we have a CW tower of fibrations with inverse limit $C^k(M, N)$. Actually it follows from Proposition 2.1 below that the the natural inclusion $C^k(M, N) \to \mathrm{map}(M, N)$ is a homotopy equivalence.

Our previous work [28] was applied by Lárusson [14] to spaces of holomorphic maps $\mathcal{H}(M, N)$ from a Stein manifold M with finite topology to an Oka manifold N. The methods of this article do not apply directly to spaces of holomorphic maps. However, it is our hope to find a suitable CW tower of fibrations with inverse limit $\mathcal{H}(M, N)$ in order to investigate the homotopy type of $\mathcal{H}(M, N)$ when M has infinite topology.

To the other extreme, in future work we intend to apply our methods to spaces of continuous functions from domains more general than CW complexes, as for example in [26].

Conventions. A *fibration* is a continuous map that has the homotopy lifting property with respect to all topological spaces. Dually, a *cofibration* is a continuous map whose image is closed, and has the homotopy extension property for all spaces. Spaces X and Y are *homotopy equivalent* if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that the composites gf and fg are homotopic to identity maps. These correspond to fibrations, cofibrations and 'weak equivalences' of a closed model category structure on the category of all topological spaces and continuous maps (see Strøm [31]). We use $\mathcal{T}op$ to refer to this closed model category. We do not pursue abstract model category theory here but we do resort to model theoretic techniques on two occasions to simplify our proofs.

A space is contractible if it is homotopy equivalent to a point. A space is weakly contractible if it is path connected and has trivial homotopy groups.

The point z_0 in Z is *nondegenerate* if $\{z_0\} \hookrightarrow Z$ is a closed cofibration.

For a countable complex X, the compact open topology and its compactly generated refinement are homotopy equivalent in $\mathcal{T}op$. Thus although we nominally use $\mathcal{T}op$, we note that for the purpose of this paper, the category of compactly generated Hausdorff spaces can equally be used. (See also [23], Section 2.3.)

2. Towers, limits, and homotopy groups

By *tower* we generally mean an inverse sequence of objects in a category. The principal category of interest here is $\mathcal{T}op$ but we also use the category of groups.

If (1) is a tower of topological spaces, then (a representative of) its inverse limit is given by the limit space $Z_\infty = \{\{\zeta_i \in Z_i\} \mid p_i(\zeta_i) = \zeta_{i-1}\}$ and *canonical projections* $P^i\colon Z_\infty \to Z_i$. The latter are restrictions of the Cartesian projections and Z_∞ is topologized as subspace of the Cartesian product $\prod_i Z_i$. Explicitly, a basis is given by all possible preimages $(P^i)^{-1}(U_i)$ where U_i are open in Z_i. If the *bonding maps* p_i are fibrations, an easy argument shows that the P^i are also fibrations.

We will find it useful to consider a model category structure on towers in $\mathcal{T}op$. Let $\mathbf{tow}(\mathcal{T}op)$ denote the category whose objects are towers in $\mathcal{T}op$ and morphisms are level-wise continuous functions. That is, a morphism $\{W_i\} \to \{Z_i\}$ in $\mathbf{tow}(\mathcal{T}op)$ is represented with a commutative ladder of the form

$$\begin{array}{cccccc} \cdots \longrightarrow & W_3 & \xrightarrow{p_3} & W_2 & \xrightarrow{p_2} & W_1 \\ & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ \cdots \longrightarrow & Z_3 & \xrightarrow{r_3} & Z_2 & \xrightarrow{r_2} & Z_1 \end{array} \tag{2}$$

We equip $\mathbf{tow}(\mathcal{T}op)$ with the following closed model category structure. Weak equivalences (respectively, cofibrations) of $\mathbf{tow}(\mathcal{T}op)$ are level-wise homotopy equivalences (respectively, cofibrations). Fibrations are morphisms that have the right lifting property with respect to trivial (also called acyclic) cofibrations. Because all objects of $\mathcal{T}op$ are both cofibrant and fibrant, it turns out that the fibrant objects in $\mathbf{tow}(\mathcal{T}op)$ are precisely towers of fibrations while every tower is cofibrant. We refer the reader to Edwards and Hastings [4], and Goerss and Jardine [8] for details concerning the closed model category structure of towers.

The following result (which is the founding stone of the definition of *homotopy inverse limit*), is a basic ingredient of all our 'positive' results, i.e. whenever we prove that an inverse limit in $\mathcal{T}op$ has the homotopy type of a CW complex.

Proposition 2.1. *Assume given a level morphism of towers of fibrations (2) (in* $\mathcal{T}op$*) where the* f_i *are homotopy equivalences. Then the induced map* $f_\infty\colon W_\infty \to Z_\infty$ *of inverse limits is also a homotopy equivalence.* □

To the best of our knowledge, the result is due to Edwards and Hastings, and is inherently model-theoretic; see [4], Theorem 3.2.4, as well as Goerss and Jardine [8], the paragraph following Definition 1.7. (Geoghegan [7] subsequently gave an elegant proof without explicit use of model categories.) Model category machinery considerably simplifies our proofs of Theorem 3.2 and Proposition 5.3.

Corollary 2.2. *If in the tower of fibrations* $\{(Z_i, p_i)\}$ *in* $\mathcal{T}op$ *all bonds* p_i *are also homotopy equivalences, then so are all canonical projections* $Z_\infty \to Z_i$. □

2.1. Homotopy groups of inverse limits in $\mathcal{T}op$

Let $\{(Z_i, p_i)\}$ be a tower of fibrations in $\mathcal{T}op$ with limit space Z_∞. Pick a point $\zeta = \{\zeta_i\} \in Z_\infty$. The following basic result gives a description of the homotopy groups $\pi_k(Z_\infty, \zeta)$. For a proof, see for example Mardešić and Segal [16], Theorem 1 on page 178.

Proposition. *For each number* $k \geqslant 0$ *there exists a natural exact sequence*

$$* \to \lim_i{}^1 \pi_{k+1}(Z_i, \zeta_i) \xrightarrow{\phi} \pi_k(Z_\infty, \zeta) \xrightarrow{\lim \pi_k(P^i)} \lim \pi_k(Z_i, \zeta_i) \to * \tag{3}$$

of pointed sets for $k = 0$ *and of groups for* $k \geqslant 1$*;* ϕ *is an injection even for* $k = 0$.

Here, $\lim^1$ denotes the first right derived functor of the inverse limit functor as defined by Bousfield and Kan [1]. It generally takes values in the category of sets.

We put a name to path components of Z_∞ that get identified by the natural map $\pi_0(Z_\infty) \to \lim_i \pi_0(Z_i)$.

Definition. Path components C, D of Z_∞ form *a phantom pair* (with respect to the tower) if the images of C and D under $P^i \colon Z_\infty \to Z_i$ coincide for each i. This makes sense because fibrations map path components onto path components.

Clearly the relation 'phantom pair' is an equivalence relation. The equivalence class of C will be called the *phantom class* of C and denoted by $\mathrm{Ph}(C)$. We call C *a phantom component* if its phantom class is nontrivial (that is, if it has more than one element). If $\{\zeta_i\} \in C$ then by (3), we may identify $\mathrm{Ph}(C) = \lim^1 \pi_1(Z_i, \zeta_i)$.

2.2. The Mittag-Leffler property

In light of the above, criteria for the vanishing of $\lim^1$ are of particular importance. A property sufficient (and sometimes also necessary) for the vanishing of $\lim^1$ is the Mittag-Leffler condition defined below.

Definition. Let $\{(G_i, p_i \colon G_i \to G_{i-1})\}$ be a tower of groups with inverse limit G_∞ and canonical projections $P^i \colon G_\infty \to G_i$. For $i < j$ let p_{ij} denote the composite $p_{i+1} \dots p_{j-1} p_j$, and let $S_i = \bigcap_{j>i} \mathrm{im}(p_{ij})$. The tower $\{G_i\}$ satisfies *the Mittag-Leffler condition* if for each i there exists $J > i$ such that $S_i = \mathrm{im}(p_{iJ})$. This is to say that the descending chain $\mathrm{im}\, p_{i,i+1} \supset \mathrm{im}\, p_{i,i+2} \supset \dots$ stabilizes at $\mathrm{im}\, p_{iJ}$.

We say that $\{G_i\}$ is *injectively Mittag-Leffler* if it satisfies the Mittag-Leffler condition and the canonical projections P^i are injective for all big enough i. Injective Mittag-Leffler towers are called *stable* in shape theory; see [4] and [16].

Note that in a Mittag-Leffler tower the restriction of p_i to S_i is an epimorphism $S_i \to S_{i-1}$ for each i, and therefore $P^i \colon G_\infty \to S_i$ is always onto.

Lemma 2.3. *Let $\{G_i\}$ be a tower of groups.*

(**i**) *If $\{G_i\}$ satisfies the Mittag-Leffler condition,* $\lim^1 G_i$ *is trivial.*

(**ii**) *If the G_i are countable, then* $\lim^1 G_i$ *is trivial if and only if $\{G_i\}$ is Mittag-Leffler. If* $\lim^1 G_i$ *is nontrivial, then it is uncountable.*

Proof. For (**i**), see [1], Corollary 3.5, as well as [16], Theorems 10 and 11 on pages 173 and 174. For (**ii**), see [18], Theorem 2. □

Lemma 2.4. *Let $\{G_i\}$ be a Mittag-Leffler tower of countable groups. If $G_\infty = \lim_i G_i$ is also countable then $\{G_i\}$ is injectively Mittag-Leffler.*

Proof. Note first that $\{G_i\}$ is (injectively) Mittag-Leffler if and only if some, and hence every, subtower is injectively Mittag-Leffler.

Let S_i denote the stable image $\bigcap_{j>i} p_{ij}(G_j) \leqslant G_i$. By replacing $\{G_i\}$ with a subtower if necessary, we may assume that S_i is the image of G_{i+1} for each i; this is possible by virtue of the Mittag-Leffler property. Let K_i denote the kernel of $G_i \to S_{i-1}$. Clearly, the induced morphism $K_i \to K_{i-1}$ is trivial. In particular, $\{K_i\}$ is Mittag-Leffler (hence $\lim^1_i K_i$ is trivial) and $\lim_i K_i \cong \{1\}$. By [16], Theorem 8 on page 168, the short exact sequence of towers $\{1\} \to K_* \to G_* \to S_{*-1} \to \{1\}$ yields an isomorphism $G_\infty \to \lim S_{i-1} =: S_\infty$. If infinitely many of the epimorphisms $S_i \to S_{i-1}$ were noninjective, S_∞ would clearly be uncountable. Hence, all but finitely many of the morphisms $S_i \to S_{i-1}$ are bijective. Consequently all but finitely many of the projections $S_\infty \to S_i$ are bijective. This proves our claim. □

3. Geometric characterization

Let $\{Z_i\}$ be a tower of fibrations in $\mathcal{T}op$ with inverse limit space Z_∞. Let C be a path component of Z_∞ and let $\zeta = \{\zeta_i\}$ be a point in C. We assume that $\zeta_i \in Z_i$ is nondegenerate for each i. Let C_i denote the image of C under canonical projection $P^i \colon Z_\infty \to Z_i$. Note that C_i is a path component and that for each i, the restricted map $p_i|_{C_i} \colon C_i \to C_{i-1}$ is a fibration. Let F_i denote the fibre of $p_i|_{C_i}$ over ζ_{i-1}.

Let $p \colon E \to B$ be a fibration and e belong to $F = p^{-1}(b)$. It is well known that if the inclusion $F \hookrightarrow E$ is nullhomotopic, then there exists a homotopy equivalence $F \times \Omega(E, e) \xrightarrow{\mu} \Omega(B, b)$. The following result discusses naturality of μ.

Lemma 3.1. *Suppose given a map of fibre sequences as follows:*

$$\begin{array}{ccccc} F' = p'^{-1}(b') & \longrightarrow & E' & \xrightarrow{\quad p' \quad} & B' \\ {\scriptstyle \phi|_{F'}}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \varphi} \\ F = p^{-1}(b) & \longrightarrow & E & \xrightarrow{\quad p \quad} & B \end{array} \tag{4}$$

Assume that basepoints $e' \in E$, $e \in E$, $b' \in B'$ and $b \in B$ have been chosen coherently. Suppose that $k \colon F \times [0,1] \to E$ is a homotopy between the inclusion and the constant map to e, and analogously for $k' \colon F' \times [0,1] \to E'$. If $\phi \circ k'$ and $k' \circ (\phi|_{F'}) \times \mathrm{id}$ equal pointwise, then there exists a strictly commutative diagram

$$\begin{array}{ccc} F' \times \Omega(E', e') & \xrightarrow{\mu'} & \Omega(B', e') \\ {\scriptstyle \phi|_{F'} \times \Omega\phi}\downarrow & & \downarrow{\scriptstyle \Omega\varphi} \\ F \times \Omega(E, e) & \xrightarrow{\mu} & \Omega(B, b) \end{array}$$

in which the horizontal arrows μ' and μ are homotopy equivalences.

This result is plausible but the author hasn't found it in the literature. It is given in Appendix A. Precisely, Lemma 3.1 is a consequence of Proposition A.3.

The following result implies Theorem A. It shows just how restrictive it is for the inverse limit of a tower of fibrations to have CW homotopy type; up to a single looping it can only be a product tower of two 'trivial' towers.

Theorem 3.2. **(i)** *Let all the Z_i have CW type and assume that C has CW type as well. Then there exist integers $i_0 < i_1 < i_2 < \ldots$, a tower of nullhomotopic fibrations $\cdots \to \Phi_2 \xrightarrow{q_2} \Phi_1 \xrightarrow{q_1} \Phi_0$, and a level-morphism*

$$\begin{array}{ccccc} \cdots \to \Phi_2 \times \Omega(Z_\infty, \zeta) & \xrightarrow{q_2 \times \mathrm{id}} & \Phi_1 \times \Omega(Z_\infty, \zeta) & \xrightarrow{q_1 \times \mathrm{id}} & \Phi_0 \times \Omega(Z_\infty, \zeta) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \cdots \longrightarrow \Omega(Z_{i_2}, \zeta_{i_2}) & \xrightarrow{\Omega p_{i_1 i_2}} & \Omega(Z_{i_1}, \zeta_{i_1}) & \xrightarrow{\Omega p_{i_0 i_1}} & \Omega(Z_{i_0}, \zeta_{i_0}) \end{array} \tag{5}$$

where the vertical arrows are homotopy equivalences. Thus $\{\Omega(Z_{i_k}, \zeta_{i_k}) \,|\, k\}$ splits up to homotopy into the product of a contractible and a constant tower.

(ii) *Conversely, assume that in the tower* $\{(Z_i, p_i)\}$ *(with no assumptions regarding homotopy type), the fibrations* p_i *split up to homotopy as* $Z_i \xrightarrow{d_i} Y_{i-1} \xrightarrow{u_{i-1}} Z_{i-1}$ *where for each i, the composite* $d_i u_i \colon Y_i \to Y_{i-1}$ *is a homotopy equivalence. The situation is best described with a diagram:*

$$\begin{array}{ccccccccc} \cdots \longrightarrow Z_4 & & \xrightarrow{\quad p_4 \quad} & & Z_3 & \xrightarrow{\quad p_3 \quad} & Z_2 & \xrightarrow{\quad p_2 \quad} & Z_1 \\ & \searrow^{d_4} & & \nearrow^{u_3} \; \searrow^{d_3} & & \nearrow^{u_2} \; \searrow^{d_2} & & \nearrow^{u_1} & \\ & \cdots \longrightarrow & Y_3 & \xrightarrow{\quad \simeq \quad} & Y_2 & \xrightarrow{\quad \simeq \quad} & Y_1 & & \end{array} \tag{6}$$

Then Z_∞ *has the homotopy type of* Y_1*. (Precisely, if* $P^2 \colon Z_\infty \to Z_2$ *is the canonical projection, then the composite* $d_2 P^2 \colon Z_\infty \to Y_1$ *is a homotopy equivalence.) In particular, if* Y_1 *has CW homotopy type, then so has* Z_∞*.*

The assumptions of (**ii**) are an analogue of the injective Mittag-Leffler property for towers of spaces: a tower of groups $\{G_j\}$ is injectively Mittag-Leffler if and only if the associated tower of Eilenberg-MacLane spaces $\{K(G_j, n)\}$ (where $n = 1$ if the groups are not abelian) admits a subtower $Z_i = K(G_{j_i}, n)$ satisfying (**ii**) of 3.2.

Clearly, those assumptions are satisfied if all maps $p_i \colon Z_i \to Z_{i-1}$ are equivalent to product maps $A_i \times B_i \xrightarrow{a_i \times b_i} A_{i-1} \times B_{i-1}$ where the maps a_i are homotopy equivalences and the b_i are nullhomotopic. This would be an analogue of the conclusion of (**i**) (without the looping). However, the assumption of (**ii**) above is more general. This is exhibited by Example 2 on page 9 as well as Proposition 5.1 (and consequently (**ii**) of Theorem 4.1) which apply (**ii**) of 3.2 in full strength.

In model category language, the conclusion of (**i**) can be reworded to say that a subtower of the looped tower $\{\Omega(Z_i, \zeta_i)\}$ is weakly equivalent in $\mathbf{tow}(\mathcal{T}op)$ to the product of a contractible tower and the constant tower $\mathbf{const}(\Omega(Z_\infty, \zeta))$.

It follows from (**ii**) that the inverse limit of a contractible tower is a contractible space. This phenomenon is probably the core of the flaw in Lemma 7 of [2].

Proof. We prove (**i**) first. An inductive application of [25], Theorem 3.3 yields a sequence of nullhomotopic inclusions $\cdots \to F_{i_2} \hookrightarrow F_{i_1} \hookrightarrow F_{i_0} \hookrightarrow C$. Fixing a contraction $F_{i_0} \times [0,1] \to C$ yields compatible contractions $F_{i_j} \times [0,1] \to C$ by restricting. By applying Lemma 3.1 to the maps between fibre sequences $F_{i_j} \hookrightarrow C \to C_{i_j}$ we get the following strictly commutative ladder:

$$\begin{array}{ccccccc} \cdots \longrightarrow & F_{i_2} \times \Omega(Z_\infty, \zeta) & \longrightarrow & F_{i_1} \times \Omega(Z_\infty, \zeta) & \longrightarrow & F_{i_0} \times \Omega(Z_\infty, \zeta) \\ & \big\downarrow \mu_2 & & \big\downarrow \mu_1 & & \big\downarrow \mu_0 \\ \cdots \longrightarrow & \Omega(Z_{i_2}, \zeta_{i_2}) & \xrightarrow{\Omega p_{i_1 i_2}} & \Omega(Z_{i_1}, \zeta_{i_1}) & \xrightarrow{\Omega p_{i_0 i_1}} & \Omega(Z_{i_0}, \zeta_{i_0}) \end{array}$$

The top horizontal arrows are inclusions, i.e., Cartesian products of (nullhomotopic) inclusions $F_{i_k} \to F_{i_{k-1}}$ with the identity map on $\Omega(Z_\infty, \zeta)$. Taking a fibrant replacement for $\{F_{i_k} \mid k\}$ in $\mathbf{tow}(\mathcal{T}op)$, i.e. replacing inclusions $F_{i_k} \to F_{i_{k-1}}$ inductively with fibrations, we obtain a contractible tower $\{\Phi_k\}$. Up to homotopy, $\{\Phi_k\}$ maps to $\{F_{i_k}\}$, and since $\Omega p_{i_{k-1} i_k}$ are fibrations, we can change the composite homotopy equivalences $\Phi_k \times \Omega(Z_\infty, \zeta) \to F_{i_k} \times \Omega(Z, \zeta) \xrightarrow{\mu_k} \Omega(Z_{i_k}, \zeta_{i_k})$ for homotopic maps inductively to obtain diagram (5).

We turn to (**ii**). Replacing the Y_i with homotopy equivalent spaces, we may assume that the maps $u_i \colon Y_i \to Z_i$ are fibrations. Using the homotopy lifting property on $d_i \colon Z_i \to Y_{i-1}$, we may assume that $u_{i-1}d_i = p_i$ pointwise for all i. Thus we may understand towers $\{Z_i\}$ and $\{(Y_i, d_i u_i) \mid i\}$ as subtowers of a bigger tower $\{W_i\}$ with $W_{2i-1} = Y_i$, $W_{2i} = Z_i$ and the obvious bonding maps. Let $\{W'_i\}$ be a fibrant replacement of $\{W_i\}$ in the model category $\mathbf{tow}(\mathcal{T}op)$, i.e. a morphism of towers consisting of homotopy equivalences $W_i \to W'_i$ so that $\{W'_i\}$ is a tower of fibrations. By restricting to $\{Z_i\}$ and $\{Y_i\}$, we get compatible morphisms of towers $\{Z_i\} \to \{Z'_i\}$ and $\{Y_i\} \to \{Y'_i\}$. Write $Z_\infty = \lim_i Z_i$, $Z'_\infty = \lim_i Z'_i$, and $Y'_\infty = \lim_i Y'_i$. By Proposition 2.1, the induced map $Z_\infty \to Z'_\infty$ is a homotopy equivalence. As $\{Z'_i\}$ and $\{Y'_i\}$ are subtowers of a bigger tower, Z'_∞ and Y'_∞ are homeomorphic. By Corollary 2.2, the canonical projection $Y'_\infty \to Y'_1$ is also a homotopy equivalence. Note that $Y_1 \to Y'_1$ may be taken to be the identity on Y_1. By commutativity, the composite $Z_\infty \to Z'_\infty \to Y'_\infty \to Y'_1 = Y_1$ equals precisely $d_2 P^2$ with $P^2 \colon Z_\infty \to Z_2$ the canonical projection. □

Remark 3.3. In the above proof the limit map $Z_\infty \to Y'_\infty$ is a homotopy equivalence. In particular, it is injective on path components and as $\{Y'_i\}$ is a tower of homotopy equivalences, commutativity implies that the canonical projections $P^i \colon Z_\infty \to Z_i$ are also injective on path components. Hence the next example shows that (**ii**) of Theorem 3.2 does not exhaust all possible towers with limits of CW type.

Example 1. Let $Z_n = \mathbb{N}$ for all n. For $n \geqslant 2$, let $p_n \colon \mathbb{N} \to \mathbb{N}$ be the identity on $\mathbb{N} \setminus \{2n-2\}$, and let $p_n(2n-2) = 2n-3$. Obviously, Z_∞ can be identified with $\mathbb{N}$ which is a CW complex. The canonical projections $P^n \colon Z_\infty \to \mathbb{N}$ satisfy $P^n(2n-1) = P^n(2n)$. Hence none are injective on the set of path components. □

Example 2. Let $f \colon Z \to Z$ be a self-map on a space Z. The *mapping microscope*[2] of f, denoted $\mathrm{Mic}(f)$, is the homotopy inverse limit of the tower

$$\cdots \xrightarrow{f} Z \xrightarrow{f} Z \xrightarrow{f} Z, \tag{7}$$

that is, the inverse limit space of a fibrant replacement for the tower (7) in $\mathbf{tow}(\mathcal{T}op)$. (This has a well-defined homotopy type.) We note that a particularly pleasant fibrant replacement exists by [27]. Namely, $f \colon Z \to Z$ is homotopy equivalent to a self-fibration $g \colon W \to W$ via a homotopy equivalence $Z \to W$ which is also a cofibration under a mild additional (separation) requirement on Z. To ease the notation, we assume at the onset that $f \colon Z \to Z$ is a self-fibration in which case we are talking about the inverse limit space of (7).

We say that the self-map f *factors* through Y if there are maps $d \colon Z \to Y$ and $u \colon Y \to Z$ such that the composite ud is homotopic to f. If, in addition, the composite du is homotopic to the identity on Y, we say that f *splits* through Y. In that case Y is a homotopy retract of Z.

If f splits through the space Y of CW type, then $\mathrm{Mic}(f)$ has CW type by (**ii**) of Theorem 3.2. There are many such examples where Y is not a factor in a product decomposition. In such a case, (7) is not equivalent to the product of a constant and a contractible tower.

For the converse, (**i**) of Theorem 3.2 implies that if $\mathrm{Mic}(f)$ has CW type, then for some $p > 0$, the iterate $\Omega f^p \colon \Omega(Z, z_0) \to \Omega(Z, z_0)$ is equivalent to a product map $\iota \times \mathrm{id} \colon F' \times \Omega\,\mathrm{Mic}(f) \to F \times \Omega\,\mathrm{Mic}(f)$ where ι is nullhomotopic. In particular, Ωf^p factors through $\Omega\,\mathrm{Mic}(f)$. A careful

[2]In May and Ponto [17], the term mapping microscope is attributed to stem from a joke name of Peter May's. In [17], the authors use it for the homotopy limit of a general tower. We use it here only for a tower of self-maps.

application of Proposition A.3 actually implies that the factorization is splitting. However, to obtain information about f itself, we need additional assumptions as we explain.

We assume for convenience that $f(z_0) = z_0$ for a distinguished nondegenerate base point z_0 (if f has had to be changed to a self-fibration this is feasible by Addendum 4 of [27]).

Let p be a natural number. We call $f\colon Z \to Z$ a *p-idempotent* if the iterate f^{p+1} is homotopic to f. If $\alpha \in \pi_1(Z, z_0)$ is the element represented by the loop traced out by a homotopy $f \simeq f^{p+1}$, then the induced morphisms on the fundamental group are conjugate: $f_\#^{p+1} = \alpha^{-1} \cdot f_\# \cdot \alpha$. As usual, we employ the notation $\alpha^{-1} \cdot f_\# \cdot \alpha =: f_\#^\alpha$.

Suppose that $\mathrm{Mic}(f)$ has CW type where f is a p-idempotent. We show that f splits. By hypothesis, $f_\#^{p+1} = f_\#^\alpha = \alpha^{-1} \cdot f_\# \cdot \alpha$ for some $\alpha \in \pi_1(Z, z_0)$. It follows from Corollary 3.4 of [25] (restated as (**i**) of Theorem 4.1 below) that for some $r \geqslant 0$ and $q \geqslant 1$, the images of $f_\#^r$ and $f_\#^{r+q}$ coincide. Consequently, $\mathrm{im}(f_\#^{kp+1}) = \mathrm{im}(f_\#^{kp+1+q})$ for a suitable k. Since $f_\#^{kp+1} = f_\#^{\alpha^k}$ and $f_\#^{kp+1+q} = (f_\#^{q+1})^{\alpha^k}$, it follows that the images of $f_\#$ and $f_\#^{q+1}$ coincide, and consequently coincide with that of $f_\#^2$. Hence f splits by Theorems 2.1 and 2.3 of [24].

More generally, suppose that $f\colon Z \to Z$ is an eventually periodic homotopy idempotent, that is, the iterates f^r and f^{r+p} are homotopic for some $r \geqslant 0$ and $p > 1$. Note that $\mathrm{Mic}(f)$ is homotopy equivalent with $\mathrm{Mic}(f^r)$ and that f^r is homotopic to $(f^r)^{p+1}$, that is, f^r is a p-idempotent. Assume that Z is (homotopy equivalent to) a finite-dimensional CW complex. By Corollary 2.4 of [24] f^r splits, and hence $\mathrm{Mic}(f^r)$ has CW type. □

4. Algebraic characterization

Consider a CW tower of fibrations (1) with inverse limit space Z_∞. The following theorem gives a necessary condition for CW homotopy type of Z_∞ which is also sufficient if the homotopy groups of all the Z_i vanish above a fixed dimension.

Note that a space has CW type if and only if its path components are open and have CW type.

Theorem 4.1. *Let C be a path component of the limit space Z_∞ and let $\zeta = \{\zeta_i\}$ be a point in C. Assume that ζ_i is a nondegenerate base point in Z_i for each i and let C_i denote the image of C under $Z_\infty \to Z_i$.*

(**i**) *If C has CW homotopy type then the towers $\{\pi_k(Z_i, \zeta_i) \,|\, i\}$ are injectively Mittag-Leffler for all $k \geqslant 1$. Consequently, the morphisms $\pi_k(Z_\infty, \zeta) \to \lim_j \pi_k(Z_j, \zeta_j)$ are bijections for $k \geqslant 1$, and C is isomorphic with the inverse limit of $\left\{C_j\right\}$. If, in addition, C is open, then the preimage of C_i under $Z_\infty \to Z_i$ equals C for all big enough i.*

(**ii**) *If Z_∞ itself has CW type then also $\pi_0(Z_\infty) \to \lim_j \pi_0(Z_j)$ is bijective.*

(**iii**) *For the converse, assume that there exists a number N so that $\pi_k(Z_i, \zeta_i) = 0$ for $k > N$ and all i. If for each $k \geqslant 1$ the tower $\{\pi_k(Z_i, \zeta_i) \,|\, i\}$ is injectively Mittag-Leffler, then C has CW homotopy type. If, in addition, the preimage of C_i under $Z_\infty \to Z_i$ equals C then C is open in Z_∞.*

Proof. Corollary 3.4 of [25] states (**i**) and (**ii**), while (**iii**) will be proved in Section 5. □

As Example 1 shows, the fibrations $Z_\infty \to Z_i$ need not be injective on path components even if Z_∞ has CW homotopy type.

Remark 4.2. In (**i**) of 4.1, the injective Mittag-Leffler property is *uniform* in k: for each i there exists a J such that *for all k* the morphisms $\pi_k(Z_\infty, \zeta) \to \pi_k(Z_J, \zeta_J)$ are injective and the image of $\pi_k(Z_J, \zeta_J) \to \pi_k(Z_i, \zeta_i)$ is stable in $\pi_k(Z_i, \zeta_i)$ (and equals the image of $\pi_k(Z_\infty, \zeta) \to \pi_k(Z_i, \zeta_i)$). This is stronger than dimensionwise injective Mittag-leffler property and was exploited profitably in [25]. If all the Z_i are Postnikov pieces as in (**iii**) of 4.1 then there is no difference.

Example 3. Consider the tower $\cdots \to S^1 \xrightarrow{p_2} S^1 \xrightarrow{p_1} S^1$ where S^1 is the unit circle in the complex plane and p_i denotes the covering map $\zeta \mapsto \zeta^{n_i}$ for a nonzero integer n_i. The induced tower on π_1 has the Mittag-Leffler property if and only if $n_i = \pm 1$ for all but finitely many i in which case the limit is S^1. In particular, consider the p-adic solenoid T_p for a prime p, obtained by setting $n_i = p$ for all i. Then by (**i**) of Theorem 4.1, no path component of T_p has CW type. It follows from (3) that the group of path components is isomorphic to the uncountable group $\hat{\mathbb{Z}}_p/\mathbb{Z}$ (p-adic integers modulo the integers) which forms a single phantom class, and that each path component is weakly contractible but not contractible. □

Topology on the set of path components

It is difficult to get a hold on the set of path components of Z_∞ if no group structure is present. The characterization of openness of a path component from (**i**) of 4.1 and a counting argument produce (**i**) of 4.4 below. Additional insight, however, can be gained by topologizing.

For a topological space W, let $\pi_0(W)$ be the quotient space of W obtained by identifying path components with points. If the path components of W are open, then $\pi_0(W)$ is discrete. Let $\bar{p}_i\colon \pi_0(Z_i) \to \pi_0(Z_{i-1})$ denote the function induced by p_i. Since $\pi_0(Z_i)$ are discrete, there exist obvious sections of the quotient maps $Z_i \to \pi_0(Z_i)$. Using homotopy lifting property, sections $s_i\colon \pi_0(Z_i) \to Z_i$ can be constructed inductively so that $p_i s_i = s_{i-1}\bar{p}_i$ for all i. These define a function $s\colon \lim_i \pi_0(Z_i) \to \lim_i Z_i = Z_\infty$. Composing s with the quotient map $Z_\infty \to \pi_0(Z_\infty)$ yields a section for the natural map $l\colon \pi_0(Z_\infty) \to \lim_i \pi_0(Z_i)$.

A space is *completely metrizable* if its topology is induced by a complete metric.

Lemma 4.3. *The space* $\lim_i \pi_0(Z_i)$ *is a completely metrizable space and the natural surjection* $l\colon \pi_0(Z_\infty) \to \lim_i \pi_0(Z_i)$ *admits a section* $s\colon \lim_i \pi_0(Z_i) \to \pi_0(Z_\infty)$*. In particular, if l is bijective, it is a homeomorphism.*

Proof. Since the spaces $\pi_0(Z_i)$ are discrete, they are completely metrizable, and therefore so is the countable product $\prod_{i\in\mathbb{N}} \pi_0(Z_i)$. Since $\lim_i \pi_0(Z_i)$ is a closed subspace, it is itself completely metrizable. □

Addendum 4.4 (to Theorem 4.1)**.** *Let β be an infinite cardinal.*

(**i**) *If Z_∞ has CW type and for each i,* $\operatorname{card}\pi_0(Z_i) \leqslant \beta$*, also* $\operatorname{card}\pi_0(Z_\infty) \leqslant \beta$.

(**ii**) *If Z_∞ has CW type and for each i, $\pi_0(Z_i)$ is finite, then so is $\pi_0(Z_\infty)$.*

Proof. Assume that Z_∞ has CW type. Then (**i**) of Theorem 4.1 implies that for each $C \in \pi_0(Z_\infty)$, there exists an i for which $C = (P^i)^{-1}(P_i C)$. Taking the minimal such well-defines a function $\Phi\colon \pi_0(Z_\infty) \to \mathbb{N}$. Obviously, the canonical function $\pi_0(Z_\infty) \to \pi_0(Z_i)$ is injective on $\Phi^{-1}(i)$. Hence $\Phi^{-1}(i)$ has cardinality smaller than $\pi_0(Z_i)$. Since $\sqcup_i \Phi^{-1}(i) = \pi_0(Z_\infty)$, (**i**) follows.

For (**ii**), note that by (**i**) of Theorem 4.1 and Lemma 4.3, $\pi_0(Z_\infty)$ is an inverse limit of finite spaces. Thus it is a discrete compactum and hence must be finite. □

The most aesthetically pleasing results are obtained when the spaces Z_i have countable CW type (in particular, the sets $\pi_0(Z_i)$ are countable).

Definition. If W is a space, we say that $\pi_*(W)$ is countable if the groups $\pi_k(W, w)$ are countable for all base points $w \in W$ and all $k \geqslant 1$, and, in addition, the set $\pi_0(W)$ is countable. Equivalently, a countable CW approximation exists for W.

Proposition 4.5. *Let $\{(Z_i, p_i)\}$ be a tower of fibrations between spaces of countable CW type, and let C be the path component of $\zeta = \{\zeta_i\} \in Z_\infty$. Then the towers $\{\pi_k(Z_i, \zeta_i) \mid i\}$ are injectively Mittag-Leffler for all $k \geqslant 1$ if and only if $\{\pi_1(Z_i, \zeta_i) \mid i\}$ is Mittag-Leffler and $\pi_*(C)$ is countable.*

Proof. Clearly, if $\{\pi_k(Z_i, \zeta_i) \mid i\}$ is injectively Mittag-Leffler, the natural morphisms $\pi_k(Z_\infty, \zeta) \to \pi_k(Z_i, \zeta_i)$ are injective, which proves the necessity part.

For sufficiency, let $\{\pi_k(Z_i, \zeta_i) \mid i\}$ be Mittag-Leffler for some $k \geqslant 1$. As $\pi_k(Z_\infty, \zeta)$ is countable, exact sequence (3) and Lemma 2.3 render $\{\pi_{k+1}(Z_i, \zeta_i) \mid i\}$ a Mittag-Leffler tower. In particular, $\pi_k(Z_\infty, \zeta)$ is isomorphic with $\lim_i \pi_k(Z_i, \zeta_i)$. Hence by Lemma 2.4, $\{\pi_k(Z_i, \zeta_i) \mid i\}$ is injectively Mittag-Leffler. Proceed by induction. □

Proposition 4.6. *Assume that the spaces Z_i have countable CW type. If the limit space Z_∞ has countably many path components, then:*

(i) *For each $\zeta = \{\zeta_i\} \in Z_\infty$, the tower $\{\pi_1(Z_i, \zeta_i) \mid i\}$ is Mittag-Leffler.*

(ii) *The set of isolated points of $\pi_0(Z_\infty)$ is dense in $\pi_0(Z_\infty)$.*

(iii) *If $\pi_0(Z_\infty)$ is homogeneous, then it is discrete.*

Proof. As the groups $\pi_1(Z_i, \zeta_i)$ are countable, Lemma 2.3 and exact sequence (3) imply that $\{\pi_1(Z_i, \zeta_i) \mid i\}$ is a Mittag-Leffler tower, and the natural morphism $l\colon \pi_0(Z_\infty) \to \lim_i \pi_0(Z_i)$ is bijective. By Lemma 4.3, $\pi_0(Z_\infty)$ is completely metrizable. The set of isolated points D in $\pi_0(Z_\infty)$ can be expressed as $\cap_{x \notin D} \pi_0(Z_\infty) \setminus \{x\}$. By the Baire category theorem, D is dense. This proves (**ii**) and hence (**iii**). □

Theorem 4.7. *Assume that Z_i have countable CW type and that for some N, $\pi_k(Z_i) = 0$ for $k > N$, all i, and all base points.*

(i) *Let C be the path component of $\{\zeta_i\} \in Z_\infty$. Then C has CW type if and only if it has countable CW type which is if and only if $\pi_*(C)$ is countable and C is not a phantom component.*

(ii) *If Z_∞ has CW type then $\pi_*(Z_\infty)$ is countable.*

(iii) *If $\pi_*(Z_\infty)$ is countable, then all path components of Z_∞ have countable CW type, and densely many are open. If, in addition, $\pi_0 Z_\infty$ is homogeneous (for example, if it is finite or if Z_∞ is an H-group), then Z_∞ has countable CW homotopy type.*

Proof. To prove (**i**), note first that if C has CW type then by (**i**) of Theorem 4.1, C has countable CW type and it is not a phantom component. Next, if $\pi_*(C)$ is countable then C is not a phantom component if and only if $\{\pi_1(Z_i, \zeta_i) \mid i\}$ is Mittag-Leffler. (See Lemma 2.3 and the definition on page 6.) An application of Proposition 4.5 and (**iii**) of Theorem 4.1 completes the proof of (**i**).

Addendum 4.4 and (**i**) of this theorem imply (**ii**).

If $\pi_*(Z_\infty)$ is countable, then (**i**) of Proposition 4.6 and (**i**) of this theorem show that all path components of Z_∞ have countable CW type. The rest of (**iii**) follows immediately from (**ii**) and (**iii**) of Proposition 4.6. □

Example 4. Let $Z_0 = \{0\}$ and let $Z_n = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}\right\}$ for $n \geqslant 1$. Define maps $p_n\colon Z_n \to Z_{n-1}$ by letting $p_n(\frac{1}{n}) = 0$ and $p_n(z) = z$ for $z \neq \frac{1}{n}$. Clearly, the maps p_n are fibrations and $Z_\infty = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ with the usual topology. Hence, $\pi_*(Z_\infty)$ is countable, but Z_∞ does not have CW homotopy type. There can be distinct path components $C_0, C_1, \ldots$ such that for each n, $P^n(C_0) = P^n(C_n)$, where $P^n\colon Z_\infty \to Z_n$ are canonical projections. Hence the additional assumptions of (**iii**) of Theorem 4.7 cannot be omitted. □

Corollary 4.8. *If the Z_i have countable CW homotopy type and $\pi_k(Z_i, \zeta_i) = 0$ for $k > N$ and all i, then Z_∞ is contractible if and only if it is weakly contractible.*

Proof. Weak contractibility and the sequence (3) render the $\lim^1$ terms trivial. Lemma 2.3 and (**iii**) of Theorem 4.7 imply contractibility. □

Remark. The simplicity of the conclusion given the assumptions in Corollary 4.8 is in sharp contrast with the general case. If the restriction on the homotopy groups of the spaces Z_i is dropped, then Z_∞ can be weakly contractible but neither Z_∞ nor any of its loop spaces have CW homotopy type. (See Example 1 in [25].)

5. Proof of (iii) of Theorem 4.1

Let $k \geqslant 1$ and let $\cdots \to K_i \xrightarrow{p_i} K_{i-1} \to \cdots \to K_1$ be a tower of fibrations where the K_i are Eilenberg-MacLane spaces of type $K(G_i, k)$. Let $P^i\colon K_\infty \to K_i$ be the projections from the inverse limit K_∞. Pick a base point $\zeta = \{\zeta_i\} \in K_\infty$ and let $G_i = \pi_k(K_i, \zeta_i)$. Denote the induced morphism $G_i \to G_{i-1}$ simply by p_i and let G_∞ denote the inverse limit of $\{G_i, p_i\}$.

Proposition 5.1. *Assume that the tower $\{G_i\}$ is injectively Mittag-Leffler, and, in case $k = 1$, that the point ζ_i is nondegenerate in K_i for each i.*

Then the limit space K_∞ has the homotopy type of $K(G_\infty, k)$. In particular, it has the homotopy type of a CW complex.

Proof. As $\pi_{k+1}(K_i, \zeta_i) = 0$ for all i, it follows from (3) in conjunction with Lemma 2.3 that the single nonvanishing homotopy group of K_∞ is $\pi_k(K_\infty, \zeta) \cong G_\infty$.

More precisely, since $\{G_i, p_i\}$ has the Mittag-Leffler property, the images of $G_j \to G_i$ stabilize for large enough j. By replacing the tower $\{G_i\}$ with an appropriate subtower we may assume that for each i the image S_{i-1} of $G_i \to G_{i-1}$ equals that of $G_j \to G_{i-1}$ for all $j \geqslant i$ (including $j = \infty$). In addition, we may assume that for each i, the morphism $G_\infty \to G_i$ is injective. This implies that the morphisms $G_\infty \to S_i$ are bijective and consequently so are $S_i \to S_{i-1}$, for all i.

Each map $K_i \to K_{i-1}$ splits as $K_i = K(G_i, k) \xrightarrow{d_i} K(S_{i-1}, k) \xrightarrow{u_i} K(G_{i-1}, k) = K_{i-1}$ with d_i induced by the epimorphism $G_i \to S_{i-1}$ and u_i by the monomorphism $S_{i-1} \to G_{i-1}$. Setting $Y_i = K(S_{i-1}, k)$ we see that the conditions of (**ii**) of Theorem 3.2 are satisfied and we conclude that K_∞ has CW homotopy type. □

We recall an important result of Stasheff expressing the relation between CW homotopy type of the base space, total space, and fibres of a fibration.

Proposition 5.2 (Stasheff [30], Propositions (0) and (12))**.** *Let $p\colon E \to B$ be a fibration where the base space B has the homotopy type of a CW complex. Then E has the homotopy type of a CW complex if and only if all fibres of p have.* □

Proposition 5.3. *Let $k \geqslant 1$ and let $\cdots \to Z_i \xrightarrow{p_i} Z_{i-1} \to \cdots \to Z_1$ be a tower of fibrations with a coherent set of base points $\{\zeta_i\}$, assumed to be nondegenerate if $k = 1$. Assume that the Z_i are $(k-1)$-connected spaces of CW type and let $G_i = \pi_k(Z_i, \zeta_i)$. Further let G_∞ denote the limit group of the induced tower $\{G_i\}$.*

There exists a CW tower of fibrations $\left\{\bar{Z}_i\right\}$ and a morphism of towers $\left\{\bar{Z}_i\right\} \to \{Z_i\}$ which is a fibration in **tow**$(\mathcal{T}op)$ *so that each map $\bar{Z}_i \to Z_i$ is a k-connected cover. The limit map $\bar{Z}_\infty \to Z_\infty$ is a principal fibration obtained as the homotopy fibre of a map $Z_\infty \to K_\infty$. If $\{G_i\}$ is injectively Mittag-Leffler, K_∞ has CW homotopy type. Hence Z_∞ has CW homotopy type if and only if $\bar{Z}_\infty$ has.*

Proof. First inductively construct a tower of fibrations

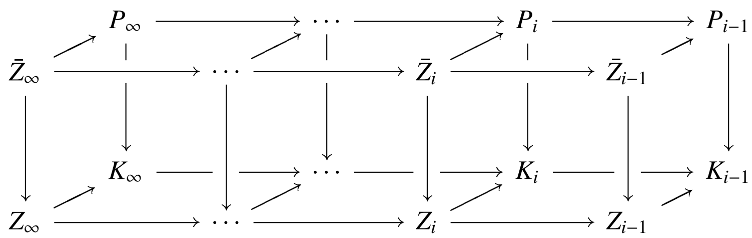
$$\cdots \to (K_i, \kappa_i) \xrightarrow{\alpha_i} (K_{i-1}, \kappa_{i-1}) \to \cdots \to (K_1, \kappa_1)$$

where κ_i is nondegenerate in K_i, the pair (K_i, κ_i) has the type of a $(K(G_i, k), *)$, and the morphism $\alpha_{i\#}\colon \pi_k(K_i, \kappa_i) \to \pi_k(K_{i-1}, \kappa_{i-1})$ realizes $G_i \to G_{i-1}$.

Let const$\{*\}$ be a tower of points and let const$\{*\} \to \{P_i\} \to \{K_i\}$ be a factorization of the obvious morphism const$\{*\} \to \{K_i\}$ into the composite of a trivial cofibration followed by a fibration in **tow**$(\mathcal{T}op)$.

Pick maps $Z_i \to K(G_i, k)$ inducing isomorphisms on the homotopy group π_k. Using homotopy lifting, inductively replace them with homotopic maps to obtain a level-morphism of towers $\{Z_i\} \to \{K_i\}$. Let $\left\{\bar{Z}_i\right\}$ denote the pullback tower of $\{Z_i\} \to \{K_i\} \leftarrow \{P_i\}$.

As limits of general diagrams are obtained by taking 'pointwise' limits (see [15], page 112), each $\bar{Z}_i$ is the pullback of $Z_i \to K_i \leftarrow P_i$ in $\mathcal{T}op$, and consequently $\bar{Z}_i \to Z_i$ is a k-connected cover. Adding the limit square, we obtain the following tower of pullback squares.



Being a right adjoint to the constant tower functor, the inverse limit functor **tow**$(\mathcal{T}op) \to \mathcal{T}op$ preserves fibrations (see [4], Theorem 3.2.4). Thus as $\{P_i\} \to \{K_i\}$ and $\left\{\bar{Z}_i\right\} \to \{Z_i\}$ are fibrations of fibrant objects in **tow**$(\mathcal{T}op)$, also the limit maps $P_\infty \to K_\infty$ and $\bar{Z}_\infty \to Z_\infty$ are fibrations. Moreover, since right adjoints always preserve limits of general diagrams (see [15], pages 114-115), the limit square is a pullback square. Finally, by Corollary 2.2, $P_\infty = \lim_i P_i$ is contractible.

If the tower $\{G_i\}$ is injectively Mittag-Leffler, K_∞ has the homotopy type of a $K(G_\infty, k)$ by Proposition 5.1. In particular, it has CW homotopy type. Hence Z_∞ has CW homotopy type if and only if $\bar{Z}_\infty$ has, by Proposition 5.2. □

Proof of (**iii**) *of Theorem 4.1.* Since $\{\pi_1(C_i, \zeta_i)\}$ is Mittag-Leffler by assumption, the limit space C_∞ of $\{C_i\}$ is path connected by (3). Clearly $C_\infty = C$ since C is a path component with $C \subset C_\infty$. Inductive application of Proposition 5.3 shows that C has CW type if and only if Z has where Z is the limit space of N-connected covers of C_i. As Z is contractible by Corollary 2.2, it has CW type. □

6. Nilpotency and localization

Definition. Let Z be a topological space. Then Z is **nilpotent** if for every base point $z \in Z$, $\pi_1(Z,z)$ is nilpotent and $\pi_k(Z,z)$ is a nilpotent $\pi_1(Z,z)$-module for all $k \geqslant 2$. It is sufficient to consider one base point for each path component of Z.

Let P be a set of primes (possibly vacuous), and let P' denote the complement of P. A group G is said to be P-local if the morphism $g \mapsto g^p$ is bijective for each $p \in P'$. If G is abelian, this means that G admits a $\mathbb{Z}_{(P)}$-module structure where $\mathbb{Z}_{(P)}$ is the subring of the rationals consisting of fractions $\frac{a}{b}$ where the prime divisors of b belong to P'.

A nilpotent space Z of CW homotopy type is called ***P*-local** if its homotopy groups are P-local (at all possible base points if Z is not connected). It follows from the theory of Serre classes that Z is P-local if and only if its homology groups are P-local. (See [10], page 72.)

Finally, a map $l\colon Z \to Z'$ of nilpotent spaces of CW type is said to *P-localize* if Z' is P-local and for every P-local space W every map $Z \to W$ factors, uniquely up to homotopy, through $l\colon Z \to Z'$. In this case, either the map l or just the space Z' is called a ***P*-localization** of Z. It turns out that $l\colon Z \to Z'$ is a P-localization if and only if it P-localizes homology groups.

Every nilpotent space Z of CW type admits a P-localization which is unique up to homotopy type and is denoted by $Z_{(P)}$. By abuse of notation, $Z_{(P)}$ also stands for any specific space ('model') within the homotopy type. See [10] for details.

Let $\{Z_i\}$ be a CW tower of fibrations with limit space Z_∞. Let C be a path component of Z_∞ and let $\zeta = \{\zeta_i\}$ be a point in C. As usual, we assume that ζ_i is nondegenerate in Z_i for each i. Let C_i denote the path component of ζ_i in Z_i, i.e. C_i is the image of C under $Z_\infty \to Z_i$. Let P be a set of primes and let

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_{3(P)} & \longrightarrow & C_{2(P)} & \longrightarrow & C_{1(P)} \end{array} \tag{8}$$

be the map of towers which amounts to (a model for) localization $C_i \to C_{i(P)}$ at each level. Here, the arrows $C_{i+1(P)} \to C_{i(P)}$ have been turned inductively into fibrations which means that (8) is assumed to be strictly commutative, i.e. a morphism of fibrant objects in $\mathbf{tow}(\mathcal{T}op)$. Let ζ_i' be the image of ζ_i under $C_i \to C_{i(P)}$. We may assume that ζ_i' is nondegenerate in $C_{i(P)}$. Taking limits in (8) we obtain a map

$$L\colon C \subset \lim_i C_i \to C' := \lim_i C_{i(P)}. \tag{9}$$

Theorem 6.1. *Assume that the Z_i are nilpotent. If C has CW type, then C and C' are also nilpotent. The towers $\left\{\pi_k(C_{i(P)}, \zeta_i') \,|\, i\right\}$ are injectively Mittag-Leffler, C' is path connected, and* (9) *localizes homotopy groups.*

Proof. Assume that C has CW type. By (**i**) of Theorem 4.1, the towers $\{\pi_k(C_i,\zeta_i)\,|\,i\}$ are injectively Mittag-Leffler. In particular, the morphisms

$$\pi_k(C,\zeta) \rightarrowtail \pi_k(C_i,\zeta_i). \tag{10}$$

are injective for all big enough i. For $k = 1$, monomorphism (10) implies that $\pi_1(C,\zeta)$ is nilpotent since it is a subgroup of a nilpotent group. For $k \geqslant 2$, (10) is a monomorphism of $\pi_1(C,\zeta)$-modules. As the $\pi_1(C,\zeta)$-structure on $\pi_k(C_i,\zeta_i)$ is obtained by restriction of the nilpotent $\pi_1(C_i,\zeta_i)$-structure, it is nilpotent. Hence $\pi_k(C,\zeta)$ is a nilpotent $\pi_1(C,\zeta)$-module as well.

For the sake of brevity we omit base points when referring to homotopy groups.

As every morphism of nilpotent groups $A \to B$, where B is P-local, factors uniquely through localization $A \to A_{(P)}$, diagram (8) induces morphisms of towers

$$\{\pi_k(C_i) \,|\, i\} \to \{\pi_k(C_i)_{(P)} \,|\, i\} \xrightarrow{\cong} \{\pi_k(C_{i(P)}) \,|\, i\} \tag{11}$$

where the last two towers are isomorphic since the maps $C_i \to C_{i(P)}$ localize homotopy groups. Since localization at P on the category of nilpotent groups is exact (and preserves subgroups), it preserves the injective Mittag-Leffler property. Therefore, the towers $\{\pi_k(C_i)_{(P)} \,|\, i\} \cong \{\pi_k(C_{i(P)}) \,|\, i\}$ are injectively Mittag-Leffler. On the one hand, (3) implies that C' is path connected. On the other hand, for $k \geqslant 1$, we obtain commutative diagrams as follows.

$$\begin{array}{ccc} \pi_k(C) & \xrightarrow{\quad\cong\quad} & \lim\limits_i \pi_k(C_i) \\ \Big\downarrow{\scriptstyle \text{localization}} & & \Big\downarrow \\ \pi_k(C)_{(P)} & \xrightarrow{\quad\cong\quad} & \lim\limits_i \pi_k(C_i)_{(P)} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\ \pi_k(C') & \xrightarrow{\quad\cong\quad} & \lim\limits_i \pi_k(C_{i(P)}) \end{array} \tag{12}$$

This diagram is induced by morphisms (11). The outer square is induced by (8), and the top and bottom horizontal arrows are isomorphisms by virtue of (3). The middle horizontal arrow is an isomorphism because of the injective Mittag-Leffler property; $\pi_k(C)_{(P)}$ maps isomorphically onto the stable image in $\pi_k(C_i)_{(P)}$ for all large enough i. By commutativity, the map $\pi_k(C)_{(P)} \to \pi_k(C')$ is an isomorphism. Since the composite $\pi_k(C) \to \pi_k(C)_{(P)} \to \pi_k(C')$ is induced by (9), it follows that (9) localizes homotopy groups.

Finally, canonical projections $\pi_k(C') \to \pi_k(C_{i(P)})$ are injective for all large enough i (being isomorphic with $\pi_k(C)_{(P)} \to \pi_k(C_i)_{(P)}$). As above for C, we infer that C' is nilpotent. □

Theorem 6.2. *If C has CW type, then so has the loop space $\Omega C'$. Therefore, $\Omega C \to \Omega C'$ is CW localization on each path component. If there exists some N such that $\pi_k(C_{i(P)}) = 0$ for all $k > N$ and all i, then in fact C' has CW type and $C \to C'$ is CW localization.*

Proof. By Theorem 6.1, we know that (9) localizes. Hence the second statement follows immediately from (**iii**) of Theorem 4.1.

Thus we have to show that in general, the loop space $\Omega C'$ has CW type. To this end, note that $\Omega C'$ is naturally homeomorphic with the inverse limit space of

$$\cdots \to \Omega C_{3(P)} \to \Omega C_{2(P)} \to \Omega C_{1(P)}. \tag{13}$$

As $\left\{\pi_1(C_{i(P)}, \zeta_i') \,|\, i\right\}$ is injectively Mittag-Leffler, $\Omega C' \to \Omega C_{i(P)}$ is injective on path components for all large enough i. In particular, the path components of $\Omega C'$ are open. Since all are homotopy equivalent, it suffices to show that $\Omega_0 C'$, the path component of the constant loop, has CW homotopy type.

The Mittag-Leffler property and (3) ensure that $\Omega_0 C'$ is homeomorphic with the inverse limit space of the tower

$$\cdots \to \Omega_0 C_{3(P)} \to \Omega_0 C_{2(P)} \to \Omega_0 C_{1(P)}. \tag{14}$$

We exploit the fact that for any path connected nilpotent space of CW type Z, the natural map $\Omega_0 Z \to \Omega_0[Z_{(P)}]$ induces a homotopy equivalence $[\Omega_0 Z]_{(P)} \to \Omega_0[Z_{(P)}]$ which is natural up to homotopy.

Let F_i denote the fibre of $C \to C_i$ over ζ_i. By (**i**) of Theorem 3.2, there is a morphism from the product tower $\{F_{i_k} \times \Omega C \,|\, k\}$ (with bonding maps of the form inclusion×identity) to $\{\Omega C_{i_k} \,|\, k\}$ that is a level-wise homotopy equivalence. Here, $\{i_k \,|\, k\}$ is a strictly increasing sequence.

It follows that for a path component D' of ΩC and coherent path components F'_{i_k} of F_{i_k}, the tower $\{\Omega_0 C_{i_k} \,|\, k\}$ is level-wise homotopy equivalent to $\{F'_{i_k} \times D' \,|\, k\}$. Localizing, we see that the tower $\{\Omega_0[C_{i_k(P)}] \,|\, k\}$ is equivalent to the product tower $\{F'_{i_k(P)} \times D'_{(P)} \,|\, k\}$. Hence by (**ii**) of Theorem 3.2, $\Omega_0 C'$ has CW type. □

7. Applications to function spaces

The space $\mathrm{map}\,((X,A),(Y,B))$ of maps $f\colon X \to Y$ such that $f(A) \subset B$ is topologized as subspace of $\mathrm{map}(X,Y)$. Taking the respective base points $A = *$, $B = *$ we obtain the space of base point preserving maps $\mathrm{map}_*(X,Y)$.

Let $\{L_i\}$ be an increasing sequence of subcomplexes of the countable CW complex X with $\cup_i L_i = X$. Set $Z_i = \mathrm{map}(L_i, Y)$ and let the bonding maps $Z_i \to Z_{i-1}$ be the restriction fibrations. As stated in the introduction, the mapping space $\mathrm{map}(X,Y)$, together with restriction maps $\mathrm{map}(X,Y) \to \mathrm{map}(L_i,Y)$ as canonical projections, is the inverse limit. Similarly, if A is a subcomplex of X, and $B \subset Y$, then $\mathrm{map}\,((X,A),(Y,B))$ can be identified with the limit of $\{\mathrm{map}\,((L_i, A \cap L_i),(Y,B)) \,|\, i\}$.

If the L_i are *finite*, then $\{\mathrm{map}(L_i,Y)\}$ is a CW tower of fibrations by Theorem 3 of Milnor [19], and $f, g\colon X \to Y$ form a phantom pair (of the second kind) if and only if the respective path components form a phantom pair with respect to the tower $\{\mathrm{map}(L_i,Y)\}$. Also, $f\colon X \to Y$ is a phantom map if and and only if its path component in $\mathrm{map}(X,Y)$ is a phantom component with respect to $\{\mathrm{map}(L_i,Y)\}$.

Let x_0 be a base point in X. As x_0 is nondegenerate in X, evaluation $Y^X \to Y$ at x_0 is a fibration. Proposition 5.2 immediately implies the following

Proposition 7.1. *The space* $\mathrm{map}(X,Y)$ *has CW type if and only if for one point y in each path component of Y, the space* $\mathrm{map}\,((X,x_0),(Y,y))$ *has CW type.* □

Our requirements of $Z_i = \mathrm{map}(L_i,Y)$ will be provided by the following.

Lemma 7.2. *Let L be a finite complex. Then* $\mathrm{map}(L,Y)$ *has the homotopy type of a CW complex W and every point of* $\mathrm{map}(L,Y)$ *is nondegenerate. Additionally:*

- (**i**) *If Y is countable, then W may be taken to be countable as well. In particular, the set of path components* $\pi_0(\mathrm{map}(K,Y))$ *is countable.*
- (**ii**) *If the homotopy groups of Y vanish above dimension N for any choice of base point, then the same holds for* $\mathrm{map}(L,Y)$.
- (**iii**) *If the homotopy groups of Y are finitely generated, so are those of* $\mathrm{map}(L,Y)$.
- (**iv**) *If Y is nilpotent and $l\colon Y \to Y_{(P)}$ is localization at the set of primes P, then also* $\mathrm{map}(L,Y)$ *and* $\mathrm{map}(L,Y_{(P)})$ *are nilpotent, and* $l_*\colon \mathrm{map}(L,Y) \to \mathrm{map}(L,Y_{(P)})$ *is localization at P on each path component.*

The statements apply to $\mathrm{map}_*(L,Y)$ *as well.*

Proof. For (countable) CW type, one can use Corollary 2 and Theorem 3 of Milnor [19]. For nondegenerate base points, see [23], Lemmas 3.2 to 3.4. Statements (**ii**) and (**iii**) are readily proved by considering cell adjunction cofibrations $S^{d-1} \xrightarrow{\text{gluing map}} K \hookrightarrow K \cup e^d$ and the homotopy exact sequences of the associated fibrations $\mathrm{map}(K \cup e^d, Y) \to \mathrm{map}(K, Y)$; the details will be left to the reader.

For (**iv**), see Hilton, Mislin, Roitberg, Steiner [11], Theorems A and B, for spaces of free maps, and Hilton, Mislin, Roitberg [10], Theorem 3.11, for pointed maps. □

Proof of Theorem D. Let $f_i = f|_{L_i}$, $C = \mathrm{map}(X, Y)_f$, and $C_i = \mathrm{map}(L_i, Y)_{f_i}$. By Lemma 7.2, $l_*\colon \mathrm{map}(L_i, Y)_{f_i} \to \mathrm{map}(L_i, Y_{(P)})_{lf_i}$ is CW localization at the set of primes P. As restrictions $\mathrm{map}(L_i, Y_{(P)})_{lf_i} \to \mathrm{map}(L_{i-1}, Y_{(P)})_{lf_{i-1}}$ are fibrations, we may take $C_{i(P)} = \mathrm{map}(L_i, Y_{(P)})_{lf_i}$ in (8). Clearly, then, C' of Section 6 may be identified with $\mathrm{map}(X, Y_{(P)})_{lf}$, and the map $C \to C'$ with the map $l_*\colon \mathrm{map}(X, Y)_f \to \mathrm{map}(X, Y_{(P)})_{lf}$ induced by l. By Theorem 6.2, therefore, if $\mathrm{map}(X, Y)_f$ has CW type, so has $\Omega(\mathrm{map}(X, Y_{(P)}), lf)$, and the map

$$\Omega(\mathrm{map}(X, Y), f) \to \Omega(\mathrm{map}(X, Y_{(P)}), lf)$$

is localization at P. If $Y_{(P)}$ is a Postnikov section, so is $\mathrm{map}(X, Y_{(P)})_{lf}$ by Lemma 7.2, hence the second statement of Theorem 6.2 finishes the proof of Theorem D. □

Proof of Theorem B. With no loss of generality we may assume that Λ is a subcomplex of X. Let $\{K_i \,|\, i\}$ be an increasing filtration of finite subcomplexes of X with $\Lambda \cup (\cup_i K_i) = X$. Write $\Lambda_i = \Lambda \cup K_i$. Fixing $g\colon X \to Y$ let F_i denote the fibre of $\mathrm{map}(\Lambda \cup K_i, Y) \to \mathrm{map}(\Lambda, Y)$ over $g|_\Lambda$. Note that F_i is homeomorphic with the fibre Φ_i of $\mathrm{map}(K_i, Y) \to \mathrm{map}(\Lambda \cap K_i, Y)$ over $g|_{\Lambda \cap K_i}$ and that we can apply Lemma 7.2 to the spaces $\mathrm{map}(K_i, Y)$ and $\mathrm{map}(\Lambda \cap K_i, Y)$. Being the preimage of a nondegenerate point under a fibration, Φ_i is cofibered in $\mathrm{map}(K_i, Y)$. Since $g|_{K_i}$ is nondegenerate as a point in $\mathrm{map}(K_i, Y)$, it is nondegenerate as a point of Φ_i as well. This is to say that $g|_{\Lambda_i}$ is a nondegenerate point in F_i. As the preimage of F_{i-1} under $\mathrm{map}(\Lambda_i, Y) \to \mathrm{map}(\Lambda_{i-1}, Y)$ equals F_i, the map $F_i \to F_{i-1}$ is a fibration. By Theorem 3 of Milnor [19] and Proposition 5.2, the Φ_i, and hence F_i, have CW homotopy type. The homotopy exact sequence of a fibration implies that the homotopy groups of F_i are countable and vanish uniformly above dimension N. It is trivial to check that the limit space of $\{F_i\}$ is the fibre F_g of $R_\Lambda\colon \mathrm{map}(X, Y) \to \mathrm{map}(\Lambda, Y)$ over $g|_\Lambda$. As R_Λ induces isomorphisms on all homotopy groups (and is injective on π_0), the limit F_g is weakly contractible. By Corollary 4.8, F_g is contractible. This holds for any $g\colon X \to Y$.

If $\mathrm{map}(\Lambda, Y)$ has CW type, then R_Λ is a homotopy equivalence onto image by virtue of Proposition 5.2. The proof in the case of base point-preserving maps is identical. □

Addendum 7.3 (to Theorem B)**.** *The induced map* $\Omega R_\Lambda\colon \Omega(\mathrm{map}(X, Y), g) \to \Omega(\mathrm{map}(\Lambda, Y), g|_\Lambda)$ *is a homotopy equivalence for each* $g \in \mathrm{map}(X, Y)$.

Proof. Given a fibration $E \to B$, if the fibre F over b_0 is contractible and $e_0 \in F$, the induced map $\Omega(E, e_0) \to \Omega(B, b_0)$ is a homotopy equivalence. □

Proposition 7.4. *Let X be a countable CW complex and let $Y = K(G, n)$ where G is a countable (respectively finitely generated) abelian group.*

(**i**) *Path components of* $\mathrm{map}(X, Y)$ *have CW type if and only if for $k \leqslant n-1$, the cohomology groups $H^k(X; G)$ are countable (respectively finitely generated) and for any (and hence every) ascending filtration $\{L_i\}$ of finite subcomplexes for X, the sequence $\{H^{n-1}(L_i; G) \,\big|\, i\}$ is Mittag-Leffler.*

(**ii**) $\operatorname{map}(X,Y)$ *has CW homotopy type if and only if for* $k \leqslant n$*, the cohomology groups* $H^k(X;G)$ *are countable (respectively finitely generated).*

Proof. Note that as $\operatorname{map}(X,Y)$ is an H-group, all path components are homotopy equivalent. In particular, for any $f\colon X \to Y$ and any $k \geqslant 0$, the group $\pi_k(\operatorname{map}(X,Y), f)$ is isomorphic with $\pi_k(\operatorname{map}(X,Y),\mathrm{const}) \cong H^{n-k}(X;G)$. Theorem 4.7 implies both (**i**) and (**ii**) for the case of countable G, as well as sufficiency for the case of finitely generated G. Necessity for the latter follows from the fact that restrictions $\operatorname{map}(X,Y) \to \operatorname{map}(L_i,Y)$ (that are H-maps) induce morphisms $H^{n-k}(X;G) \to H^{n-k}(L_i;G)$ on π_k which are injective for all large enough i and all $k \geqslant 1$ for (**i**), respectively $k \geqslant 0$ for (**ii**). □

Example 5. Let X be a countable CW complex. Applying Proposition 7.4 to $Y = K(\mathbb{Z}, n)$ we obtain that $\operatorname{map}(X,Y)$ has CW type if and only if the cohomology groups $H^k(X;\mathbb{Z})$, $k \leqslant n$, are countable, which is if and only if the cohomology groups $H^k(X;\mathbb{Z})$, $k \leqslant n$, are finitely generated. It follows that if, for some n, the groups $H^k(X;\mathbb{Z})$ are finitely generated for $k \leqslant n$, and $H^{n+1}(X;\mathbb{Z})$ is not finitely generated, $H^{n+1}(X;\mathbb{Z})$ is uncountable. Note that the same statements apply with any finitely generated group of coefficients G in place of $\mathbb{Z}$.

Actually, we can deduce more. Recall that a Moore space of type $M(A,m)$, where A is an abelian group, is any CW complex (or a space of CW type) whose single nonvanishing reduced homology group is A in dimension m.

Assume that for some n, the group $H^n(X;\mathbb{Z})$ is not finitely generated. Applying Proposition 7.4 with $Y = K(\mathbb{Z}, n)$ and the wedge $X' = M(H_nX, n) \vee M(H_{n-1}X, n-1)$ in place of X we infer that at least one of $H^n(X;\mathbb{Z})$ and $\operatorname{Hom}(H_{n-1}X,\mathbb{Z})$ and hence at least one of $H^n(X;\mathbb{Z})$ and $H^{n-1}(X;\mathbb{Z})$ is uncountable.[3] □

We can extract the following pretty characterization for the case of finitely generated G.

Proposition 7.5. *Suppose that in 7.4,* $G \cong \mathbb{Z}^r \oplus F$ *where* F *is finite and* $r > 0$*.*

(**i**) *Path components of* $\operatorname{map}(X, K(\mathbb{Z}^r \oplus F, n))$ *have CW type if and only if homology groups* $H_k(X)$ *are finitely generated for* $k \leqslant n-2$*, and* $H_{n-1}(X)$ *is isomorphic with* $\mathbb{Z}^p \oplus T$ *where* T *is a torsion group with* $\operatorname{Hom}(T,F)$ *finite.*

(**ii**) $\operatorname{map}(X, K(\mathbb{Z}^r \oplus F, n))$ *has CW type if and only if homology groups* $H_k(X)$ *are finitely generated for* $k \leqslant n-1$ *and there exists a decomposition* $H_n(X) \cong \mathbb{Z}^d \oplus H$ *with* $\operatorname{Hom}(H,\mathbb{Z})$ *trivial and* $\operatorname{Hom}(H,F)$ *finite.*

Proof. To prove (**ii**) we recall that an abelian group A is finitely generated if and only if both $\operatorname{Hom}(A,\mathbb{Z})$ and $\operatorname{Ext}(A,\mathbb{Z})$ are. Also, if $\operatorname{Hom}(A,\mathbb{Z})$ is finitely generated, there exists a decompositon $A \cong \mathbb{Z}^d \oplus H$ with $\operatorname{Hom}(H,\mathbb{Z}) = 0$.

To prove (**i**), fix an ascending filtration $\{L_i\}$ of finite subcomplexes of X. As $H^{n-1}(L_i;\mathbb{Z}^r \oplus F)$ splits naturally as $H^{n-1}(L_i;\mathbb{Z})^r \oplus H^{n-1}(L_i;F)$ and $\{H^{n-1}(L_i;F)\}$ is a tower of finite groups, $\{H^{n-1}(L_i;\mathbb{Z}^r \oplus F)\}$ is Mittag-Leffler if and only if $\{H^{n-1}(L_i;\mathbb{Z})\}$ is. Universal coefficients yield a short exact sequence of towers

$$0 \to \{\operatorname{Ext}(H_{n-2}L_i,\mathbb{Z}) \,|\, i\} \to \left\{H^{n-1}(L_i;\mathbb{Z}) \,|\, i\right\} \to \{\operatorname{Hom}(H_{n-1}L_i,\mathbb{Z}) \,|\, i\} \to 0.$$

As $\{\operatorname{Ext}(H_{n-2}L_i,\mathbb{Z})\}$ is a tower of finite groups, the well-known six-term derived exact sequence of the inverse limit functior yields an isomorphism $\lim^1_i H^{n-1}(L_i;\mathbb{Z}) \cong \lim^1_i \operatorname{Hom}(H_{n-1}L_i,\mathbb{Z})$.

[3] Compare with the remark in Hatcher [9], the paragraph following the proof of 3F.12.

If $T(A)$ denotes the torsion subgroup of the abelian group A, we have a natural isomorphism $\mathrm{Hom}(A/T(A), \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Z})$. Set $T_i = T(H_{n-1}L_i)$. As $H_{n-1}L_i/T_i$ is free, $\lim^1_i \mathrm{Hom}(H_{n-1}L_i, \mathbb{Z})$ is isomorphic with $\mathrm{Ext}(\mathrm{colim}_i(H_{n-1}L_i/T_i), \mathbb{Z})$. (See Jensen [12], page 16.) In turn, $\mathrm{colim}_i(H_{n-1}L_i/T_i)$ is isomorphic with the quotient $H_{n-1}X/T$ where $T = T(H_{n-1}X)$. We have shown that $\{H^{n-1}(L_i; G)\}$ is Mittag-Leffler if and only if $\mathrm{Ext}(H_{n-1}X/T, \mathbb{Z})$ is trivial. Sufficiency now follows from Proposition 7.4.

For necessity, assume that path components of $\mathrm{map}(X, K(\mathbb{Z}^r \oplus F, n))$ have CW type. It follows from 7.4 that cohomology groups $H^k(X; \mathbb{Z})$ are finitely generated for $1 \leqslant k \leqslant n-1$ and $\mathrm{Hom}(H_{n-1}X, F)$ is finite. In addition, $\mathrm{Ext}(H_{n-1}X/T, \mathbb{Z})$ is trivial as shown above. Consequently, homology groups $H_k(X)$ are finitely generated for $k \leqslant n-2$, and so also is $\mathrm{Hom}(H_{n-1}X, \mathbb{Z}) \cong \mathrm{Hom}(H_{n-1}X/T, \mathbb{Z})$. Therefore, $H_{n-1}X/T$ must be finitely generated and since it is torsionfree, it is free, say $H_{n-1}X/T \cong \mathbb{Z}^\rho$. In particular, the short exact sequence $0 \to T \to H_{n-1}X \to H_{n-1}X/T \to 0$ splits. Finally, $\mathrm{Hom}(H_{n-1}X, F)$ is finite if and only if $\mathrm{Hom}(T, F)$ is. This concludes the proof. ☐

Lemma 7.6. *Let $p\colon E \to B$ be the principal fibration obtained by pulling back the path fibration $PY \to Y$ (where Y has a base point) along $k\colon B \to Y$, and let X be a countable complex. Then $p_*\colon \mathrm{map}(X, E) \to \mathrm{map}(X, B)$ is a principal fibration whose nonempty fibres are homotopy equivalent to $\mathrm{map}(X, \Omega Y)$. If E, B, and Y have coherent base points, the analogous result holds for spaces of pointed maps.*

Proof. The functor $\mathrm{map}(X, \_)$ respects pullbacks and fibrations. (See for instance A.3 and A.4 of [25].) Hence p_* is a principal fibration and the nonempty fibres are exactly those of $\mathrm{map}(X, PY) \to \mathrm{map}(X, Y)$. As $\mathrm{map}(X, PY)$ is contractible, its image in $\mathrm{map}(X, Y)$ is precisely the path component of the constant map. The statement of the lemma follows. Analogously for spaces of pointed maps. ☐

Proposition 7.7. *Let Y be a connected simple CW complex with $\pi_k(Y) = 0$ for $k \geqslant n+1$ and let X be a countable CW complex. Then if $\mathrm{map}(X, K(\pi_k Y, k))$ has CW type for all $k \leqslant n$, also $\mathrm{map}(X, Y)$ has CW type.*

Proof. We may assume that $Y = Y_n$ where $Y_n \to Y_{n-1} \to \cdots \to Y_1$ is the Postnikov tower of Y. By hypothesis, $\mathrm{map}(X, Y_1)$ has CW type. For a proof by induction, assume that $\mathrm{map}(X, Y_{i-1})$ has CW type. The map $Y_i \to Y_{i-1}$ is obtained by pulling back the path fibration over $K(\pi_i Y, i+1)$ along the k-invariant $k_{i+1}\colon Y_{i-1} \to K(\pi_i Y, i+1)$. By Lemma 7.6, $\mathrm{map}(X, Y_i) \to \mathrm{map}(X, Y_{i-1})$ is a principal fibration whose nonempty fibres are homotopy equivalent to $\mathrm{map}(X, \Omega K(\pi_i Y, i+1)) \simeq \mathrm{map}(X, K(\pi_i Y, i))$. By assumption, the latter has homotopy CW type. Hence so has $\mathrm{map}(X, Y_i)$, by inductive hypothesis and Proposition 5.2. ☐

Remark. Clearly, Proposition 7.7 may be suitably extended to nilpotent Y as well.

Proof of Theorem C. (**a**) follows from (**i**) of Theorem 4.7 by virtue of Lemma 7.2, (**b**) is the same as (**ii**) of Proposition 7.4, while (**c**) follows from Proposition 7.7. ☐

We conclude the section with some additional examples illustrating our results and methods.

Example 6. We consider mapping spaces from certain Moore spaces to certain Eilenberg-MacLane spaces to illustrate applications of Theorem C.

(**i**) We use $\mathbb{Z}_r$ for the group of integers mod r. Let p be a prime. The quasicyclic p-group $\mathbb{Z}_{p^\infty}$ can be viewed as the colimit of the sequence of finite cyclic groups $\mathbb{Z}_p \leqslant \mathbb{Z}_{p^2} \leqslant \mathbb{Z}_{p^3} \leqslant \dots$.

A cohomology computation and Theorem C render $\mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m), K(\mathbb{Z}_p, n))$ a space of CW type (contractible for $n < m+1$, homotopy discrete for $n = m+1$, a $K(\mathbb{Z}_p, n-m-1)$ otherwise).

Note that for any CW complex representative of $M(\mathbb{Z}_{p^\infty}, m)$ and an arbitrary finite subcomplex L, the restriction fibration $R_L\colon\ \mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m), K(\mathbb{Z}_p, n)) \to \mathrm{map}_*(L, K(\mathbb{Z}_p, n))$ fails to be a homotopy equivalence if $n > m+1$. In this sense, mapping spaces of CW type from infinite domain complexes are not 'covered' by mapping spaces with finite domain complexes. Thus the application range of Theorem C is really much more general than that of [13] or [23].

(**ii**) Let $\mathbb{Z}[p^{-1}]$ be the set of those rational numbers whose denominators are powers of p. This is just localization of the integers *away from* the prime p. For any m, n, the homotopy groups of $\mathrm{map}_*(M(\mathbb{Z}[p^{-1}], m), K(\mathbb{Z}_p, n))$ are trivial, hence by Theorem C, the space is contractible.

(**iii**) If $n \geqslant m+1$, $\mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m), K(\mathbb{Z}, n))$ has uncountable π_{n-m-1} which is isomorphic to $\hat{\mathbb{Z}}_p$, the p-adic integers. Thus by Theorem C, the space does not have CW homotopy type.

(**iv**) Similarly as in (**iii**), if $n \geqslant m+1$, the space $Z = \mathrm{map}_*(M(\mathbb{Z}[p^{-1}], m), K(\mathbb{Z}, n))$ has uncountable $\pi_{n-m-1} \cong \hat{\mathbb{Z}}_p/\mathbb{Z}$ and hence does not have CW type. If $m = n-1$, then Z is homotopy equivalent to the p-adic solenoid of Example 3. If $m > n-1$, then Z is contractible. □

Example 7. We now illustrate the scope and limitations of Theorem B and the usefulness of Theorem 4.7 as well as Theorem 7.7.

Suppose that Y is the total space of a fibration $K(\mathbb{Z}_p, n) \to Y \xrightarrow{q} K(\mathbb{Z}, r)$ where p is a prime and $n > r \geqslant 2$. That is, $q\colon Y \to K(\mathbb{Z}, r)$ is obtained by pulling back the path fibration over $K(\mathbb{Z}_p, n+1)$ along the sole k-invariant $k\colon K(\mathbb{Z}, r) \to K(\mathbb{Z}_p, n+1)$.

Let $m \geqslant 2$. To the short exact sequence $\mathbb{Z} \to \mathbb{Z}[p^{-1}] \to \mathbb{Z}_{p^\infty}$ is associated a Puppe sequence

$$S^{m-1} \to M(\mathbb{Z}[p^{-1}], m-1) \to M(\mathbb{Z}_{p^\infty}, m-1) \to \Sigma S^{m-1} \to \Sigma M(\mathbb{Z}[p^{-1}], m-1) \to \dots$$

where every map is thought of, up to homotopy, as inclusion into the reduced homotopy cofibre (mapping cone) of the previous one. We consider the map on mapping spaces induced by $f\colon M(\mathbb{Z}_{p^\infty}, m-1) \to \Sigma S^{m-1} = S^m$ in the sense of Theorem B.[4] As f is a principal cofibration,

$$f^*\colon\ \mathrm{map}_*(S^m, Y) \to \mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m-1), Y) \tag{15}$$

is a principal fibration whose nonempty fibres are all equivalent to $\mathrm{map}_*(\Sigma M(\mathbb{Z}[p^{-1}], m-1), Y) = \mathrm{map}_*(M(\mathbb{Z}[p^{-1}], m), Y)$. We deliberately forget the fact that f^* is a principal fibration and just focus on what one can infer from Theorem B by knowing what its fibres are.

Let X be a CW complex. The induced fibration $q_\#\colon\ \mathrm{map}_*(X, Y) \to \mathrm{map}_*(X, K(\mathbb{Z}, r))$ is obtained as the homotopy fibre of $k_\#\colon\ \mathrm{map}_*(X, K(\mathbb{Z}, r)) \to \mathrm{map}_*(X, K(\mathbb{Z}_p, n+1))$ (see Lemma 7.6).

If $X = M(\mathbb{Z}[p^{-1}], m)$, then $\mathrm{map}_*(X, K(\mathbb{Z}_p, n+1))$ is contractible (see (**ii**) of Example 6), and hence $\mathrm{map}_*(M(\mathbb{Z}[p^{-1}], m), Y)$ is homotopy equivalent to $\mathrm{map}_*(M(\mathbb{Z}[p^{-1}], m), K(\mathbb{Z}, r))$ which, in turn, is (weakly) contractible if $m \geqslant r$.

Thus for $m \geqslant r$, the typical nonempty homotopy fibre of (15) is weakly contractible and Theorem B (see Addendum 7.3) implies that (15) induces homotopy equivalences on the loop-spaces. However, whether or not f^* itself is a homotopy equivalence to the union of path components that meet its image, depends on the homotopy type of $\mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m-1), Y)$.

To clarify the case at hand, we need to investigate $\mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m-1), Y)$ a bit further. If $m > r$, it follows immediately from Theorem 7.7 and Example 6 that $\mathrm{map}_*(M(\mathbb{Z}_{p^\infty}, m-1), Y)$ has CW type and hence f^* is a homotopy equivalence.

[4] If $m < n$ then Theorem 1.2 of [13] can't apply here as f is then not n-connected.

The case $m = r$ is more delicate. Using the long homotopy exact sequence of $q_\#$ for $X = M(\mathbb{Z}_{p^\infty}, m-1)$ one obtains easily that $\pi_0(\operatorname{map}_*(M(\mathbb{Z}_{p^\infty}, r-1), Y)) \cong \hat{\mathbb{Z}}_p$. Hence by Theorem 4.7, the entire space $\operatorname{map}_*(M(\mathbb{Z}_{p^\infty}, r-1), Y)$ certainly does not have CW type.

However, note that $M(\mathbb{Z}_{p^\infty}, r-1)$ can be viewed as the colimit of an ascending filtration of finite subcomplexes $L_1 \leqslant L_2 \leqslant \dots$ where each L_i is of type $M(\mathbb{Z}_{p^i}, r-1)$. As the spaces $Z_i = \operatorname{map}_*(L_i, Y)$ have trivial π_1 (at all base points), the inverse limit $Z_\infty = \operatorname{map}_*(M(\mathbb{Z}_{p^\infty}, r-1), Y)$ has no phantom components. Since its only other nontrivial homotopy group is $\pi_{n-r} \cong \mathbb{Z}_p$, it follows from Theorem 4.7 that each path-component of Z_∞ has CW homotopy type (and is equivalent to a $K(\mathbb{Z}_p, n-r)$). Finally, f^* corresponds to the standard inclusion $\mathbb{Z} \to \hat{\mathbb{Z}}_p$ on π_0 (we note that the spaces involved are H-spaces since the domains are suspensions). By Lemma 4.3, the topology on $\pi_0(Z_\infty)$ coincides with the profinite topology of the p-adic integers. As the subgroup of integers $\mathbb{Z}$ is not a discrete subspace, the set of path components in Z_∞ in question does not have CW type and hence f^* is not a homotopy equivalence to the set of path components that meet its image. In this sense the conclusion of Theorem B cannot be improved. ☐

I do not know if the converse of Proposition 7.7 holds generally even for a simply connected target space Y with only two nontrivial homotopy groups. The problem derives from the fact that when Y is not an H-space, it is hard to get a grip on the set $[X, Y]$ of homotopy classes of maps $X \to Y$. The following is an example where the converse does hold.

Example 8. Let Y be as in Example 7 and let X be a connected countable CW complex. We claim that if $\operatorname{map}(X, Y)$ has CW type, then so have $\operatorname{map}(X, K(\mathbb{Z}, r))$ and $\operatorname{map}(X, K(\mathbb{Z}_p, n))$.

Consider the induced fibration $q_\# \colon \operatorname{map}(X, Y) \to \operatorname{map}(X, K(\mathbb{Z}, r))$ which, by Lemma 7.6, is principal with nonempty fibres homotopy equivalent to $\operatorname{map}(X, K(\mathbb{Z}_p, n))$. Pick $f \in \operatorname{map}(X, Y)$ and set $\pi_j = \pi_j(\operatorname{map}(X, Y), f)$. Identifying $\pi_j(\operatorname{map}(X, K(G, m)), \text{any map}) \cong H^{m-j}(X; G)$ as in the proof of Proposition 7.4, we transcribe the homotopy long exact sequence of $q_\#$ as follows.

$$\cdots \to H^{r-j-1}(X;\mathbb{Z}) \xrightarrow{\delta_{j+1}} H^{n-j}(X;\mathbb{Z}_p) \to \pi_j \to H^{r-j}(X;\mathbb{Z}) \xrightarrow{\delta_j} H^{n+1-j}(X;\mathbb{Z}_p) \to \dots \tag{16}$$

The map δ_j is the cohomology operation induced by $\Omega^j(k) \colon \Omega^j K(\mathbb{Z}, r) \to \Omega^j K(\mathbb{Z}_p, n+1)$, composed by a change-of-basepoint isomorphism if $q \circ f$ is not nullhomotopic. For this reason, δ_j is a morphism of abelian groups for $j \geqslant 1$ while δ_0 need not be.

For $j \geqslant 1$, therefore, the homotopy groups π_j sit in short exact sequences (written multiplicatively on account of π_1) $1 \to \operatorname{coker} \delta_{j+1} \to \pi_j \to \ker \delta_j \to 1$. Moreover, $\operatorname{coker} \delta_1$ acts effectively on the set $[X, Y] = \pi_0(\operatorname{map}(X, Y))$. Theorem 4.7 implies that for all $j \geqslant 1$, both $\ker \delta_j$ and $\operatorname{coker} \delta_j$ are countable.

For $j \geqslant r-1$, the group $H^{r-j-1}(X;\mathbb{Z})$ is trivially countable and hence $\operatorname{coker} \delta_{j+1}$ is countable if and only if $H^{n-j}(X;\mathbb{Z}_p)$ is. Thus, $H^i(X;\mathbb{Z}_p)$ is countable for $0 \leqslant i \leqslant \min\{n-r+1, n\}$ (in fact finite but we will not need that), and the cohomology exact sequence associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ implies that $p \cdot H^i(X;\mathbb{Z})$ is of countable index in $H^i(X;\mathbb{Z})$ for $0 \leqslant i \leqslant \min\{n, n-r+1\}$.

Since $H^*(X;\mathbb{Z}_p)$ is bounded by p, $\ker \delta_j$ contains $p \cdot H^{r-j}(X;\mathbb{Z})$, and therefore $\ker \delta_j$ contains a subgroup of countable index in $H^{r-j}(X;\mathbb{Z})$ whenever $0 \leqslant r-j \leqslant n-r+1$ and $j \geqslant 1$. We conclude that $H^i(X;\mathbb{Z})$ is countable for $0 \leqslant i \leqslant \min\{n-r+1, r-1\}$.

If $n-r+1 < r-1$ we can repeat the above process with the newly obtained information, concluding first from countability of the cokernels that the groups $H^i(X;\mathbb{Z}_p)$ are countable for $i \leqslant \min\{2(n-r+1), n\}$ and next from that of the kernels that $H^i(X;\mathbb{Z})$ are countable for

$i \leqslant \min\{2(n-r+1), r-1\}$ and so on. It follows by induction that the groups $H^i(X;\mathbb{Z}_p)$ are countable for $i \leqslant n$ and $H^i(X;\mathbb{Z})$ are countable for $i \leqslant r-1$.

Now we investigate the tail of (16), which for $f = \text{const}$ reads as follows.

$$1 \to \operatorname{coker}\delta_1 \implies [X,Y] \xrightarrow{q_\#} [X, K(\mathbb{Z}, r)] \xrightarrow{k_\#} [X, K(\mathbb{Z}_p, n+1)].$$

As noted above, $\operatorname{coker}\delta_1$ acts effectively on $[X,Y]$ and $q_\#\colon [X,Y] \to [X, K(\mathbb{Z}, r)]$ collapses the orbits of the action. The image of $q_\#$ equals the preimage under $k_\#$ of the homotopy class of a constant map, i.e. of $0 \in H^{n+1}(X;\mathbb{Z}_p)$. Since $[X,Y]$ is countable, so also is the preimage $k_\#^{-1}(0)$.

Let $\hat{k}$ of $H^{n+1}(K(\mathbb{Z}, r);\mathbb{Z}_p)$ represent the homotopy class of k. By results of Cartan and Borel, the mod-p cohomology algebra $H^*(K(\mathbb{Z}, r);\mathbb{Z}_p)$ has a p-simple system of generators which are transgressive with respect to the Serre cohomology spectral sequence of the fibration

$$K(\mathbb{Z}, r) \to PK(\mathbb{Z}, r+1) \to K(\mathbb{Z}, r+1). \tag{17}$$

Using the associated vector-space basis for $H^*(K(\mathbb{Z}, r);\mathbb{Z}_p)$ we expand $\hat{k} = \sum_i \lambda_i \cdot \alpha_i \cup \beta_i$ where $\lambda_i \in \mathbb{Z}_p$, and $\alpha_i \in H^{a_i}(K(\mathbb{Z}, r);\mathbb{Z}_p)$ of degree $a_i \leqslant n+1$ is transgressive.

An application of $H^*(\_;\mathbb{Z}_p)$ to some $f\colon X \to K(\mathbb{Z}, r)$ yields the induced homomorphism $f^*\colon H^*(K(\mathbb{Z}, r);\mathbb{Z}_p) \to H^*(X;\mathbb{Z}_p)$; we represent $k \circ f\colon X \to K(\mathbb{Z}, r) \to K(\mathbb{Z}_p, n+1)$ as

$$f^*(\hat{k}) = \sum_i \lambda_i f^*(\alpha_i) \cup f^*(\beta_i) \in H^{n+1}(K(\mathbb{Z}, r);\mathbb{Z}_p). \tag{18}$$

Transgressive elements are in the image of the cohomology suspension with respect to (17), hence for each i there exists an element $\eta_i \in H^{a_i+1}(K(\mathbb{Z}, r+1);\mathbb{Z}_p)$ such that $\sigma^*(\eta_i) = \alpha_i$ where

$$\sigma^*\colon [K(\mathbb{Z}, r+1), K(\mathbb{Z}_p, a_i+1)] \to [\Omega K(\mathbb{Z}, r+1), \Omega K(\mathbb{Z}_p, a_i+1)]$$

is induced by the adjoint of the natural map $l\colon S\Omega K(\mathbb{Z}, r+1) \to K(\mathbb{Z}, r+1)$. This implies that $f^*(\alpha_i)$ may be represented in $[X, K(\mathbb{Z}_p, a_i)]$ as the composite

$$X \xrightarrow{f} K(\mathbb{Z}, r) \xrightarrow{\simeq} \Omega K(\mathbb{Z}, r+1) \xrightarrow{\Omega\psi_i} \Omega K(\mathbb{Z}_p, a_i+1) \xrightarrow{\simeq} K(\mathbb{Z}_p, a_i)$$

for some map $\psi_i\colon K(\mathbb{Z}, r+1) \to K(\mathbb{Z}_p, a_i+1)$. Now we view $f^*(\alpha_i)$ rather as $\Omega\psi_{i\#}([f])$. Since $\Omega\psi_i$ is an H-group morphism, the induced function $\Omega\psi_{i\#}$ is a group homomorphism. So we consider $\Omega\psi_{i\#}\colon [X, K(\mathbb{Z}, r)] \to [X, K(\mathbb{Z}_p, a_i)]$.

We reason as above that since $H^r(X;\mathbb{Z}_p)$ is countable and $[X, K(\mathbb{Z}_p, a_i)] \cong H^{a_i}(X;\mathbb{Z}_p)$ is p-bounded, the kernel A_i of $\Omega\psi_{i\#}$ is a subgroup of (at most) countable index in $H^r(X;\mathbb{Z})$. The finite intersection $A = \cap_i A_i$ is also a subgroup of countable index in $H^r(X;\mathbb{Z})$. By (18), A is contained in $k_\#^{-1}(0)$. Hence A is countable and therefore so is $H^r(X;\mathbb{Z})$.

By Proposition 7.4, the established properties of $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z})$ imply that both $\operatorname{map}(X, K(\mathbb{Z}, r))$ and $\operatorname{map}(X, K(\mathbb{Z}_p, n))$ have CW type. □

Example 9. We show how to apply Theorem D to study CW type 'one prime at a time'.

Let p and q be distinct primes and let Y have three nontrivial homotopy groups: $\pi_r(Y) \cong \mathbb{Z}$, $\pi_m(Y) \cong \mathbb{Z}_p$, and $\pi_n(Y) \cong \mathbb{Z}_q$ where $n > m > r$. Assume that $\operatorname{map}(X, Y)$ has CW type for some X.

We localize Y at p to obtain a fibration $K(\mathbb{Z}_p, m) \to Y_{(p)} \to K(\mathbb{Z}_{(p)}, r)$. By Theorem D, also $\operatorname{map}(X, \Omega Y_{(p)})$ has CW type. As the algebra $H^*(K(\mathbb{Z}_{(p)}, r);\mathbb{Z}_p)$ is isomorphic with $H^*(K(\mathbb{Z}, r);\mathbb{Z}_p)$

(this is because the cyclic group $\mathbb{Z}_p$ itself is p-local), Example 8 may be reread almost verbatim to yield the groups $H^i(X;\mathbb{Z}_p)$ and $H^j(X;\mathbb{Z}_{(p)})$ countable for $i \leqslant m-1$ and $j \leqslant r-1$, respectively.

Next we localize Y away from p to obtain a fibration $K(\mathbb{Z}_q, n) \to Y[p^{-1}] \to K(\mathbb{Z}[p^{-1}], r)$. By Theorem D, also $\mathrm{map}(X, \Omega Y[p^{-1}])$ has CW type and as $H^*(K(\mathbb{Z}[p^{-1}], r);\mathbb{Z}_q) \cong H^*(K(\mathbb{Z}, r);\mathbb{Z}_q)$, Example 8 yields the groups $H^i(X;\mathbb{Z}_q)$ and $H^j(X;\mathbb{Z}[p^{-1}])$ countable for $i \leqslant n-1$ and $j \leqslant r-1$.

Finally, we rationalize: $Y_{(0)} \simeq K(\mathbb{Q}, r)$. By Theorem D, $\mathrm{map}(X, \Omega K(\mathbb{Q}, r))$ has CW type and hence by Theorem C, also the groups $H^j(X;\mathbb{Q})$ are countable for $j \leqslant r-1$.

From the cohomology long exact sequence associated to the short exact sequence of coefficients $0 \to \mathbb{Z}[p^{-1}] \to \mathbb{Q} \to P \to 0$ (where P is the direct sum of quasicyclic groups at all primes different from p) we infer immediately that $H^j(X;P)$ are countable for $j \leqslant r-2$. Thereby we extract from the cohomology long exact sequence associated to $0 \to \mathbb{Z} \to \mathbb{Z}_{(p)} \to P \to 0$ that $H^j(X;\mathbb{Z})$ are countable for $j \leqslant r-1$.

By Theorem C, $\mathrm{map}(X, K(\mathbb{Z}, r-1))$, $\mathrm{map}(X, K(\mathbb{Z}_p, m-1))$, and $\mathrm{map}(X, K(\mathbb{Z}_q, n-1))$ have CW type.

We note that an enhancement of the argument in Example 8, which we omit, actually shows that $\mathrm{map}(X, K(\mathbb{Z}, r))$, $\mathrm{map}(X, K(\mathbb{Z}_p, m))$, and $\mathrm{map}(X, K(\mathbb{Z}_q, n))$ all have CW type. □

A. Pullbacks and lifting functions

Let $p\colon E \to B$ be a fibration with $p(e) = b$ where $F = p^{-1}(b)$ contracts in E. The homotopy splitting $\Omega(B, b) \simeq F \times \Omega(E, e)$ leans on the following pullback diagram:

$$\begin{array}{ccc} \Lambda_f & \longrightarrow & \mathrm{P}E \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_1} \\ F & \xrightarrow{f} & E \end{array} \tag{A.1}$$

Here $\mathrm{P}E$ is the contractible path space $\mathrm{P}E = \mathrm{map}\big((I,0),(E,e)\big) = \mathrm{map}_*(I,E)$ and ε_1 denotes evaluation at 1. When f is the inclusion, the pullback fibration $\Lambda_{\mathrm{incl}} \to F$ is the homotopy fibre of p. Now on the one hand, Λ_{incl} is functorially equivalent to the loop space $\Omega(B, b)$. On the other hand, the fibre homotopy type of $\Lambda_f \to F$ depends only on the homotopy class of f. Hence if the inclusion $F \hookrightarrow E$ is nullhomotopic, Λ_{incl} is fibre homotopy equivalent to $F \times \Omega(E, e)$. The essence of Proposition 3.1 is the fact that the path-loop fibration construction $\mathrm{P}E \to E$ admits *canonical homotopy liftings* which are natural with respect to maps $E' \to E$. To make this precise, we need to employ *lifting functions*.

In contrast to $\mathrm{P}E$, let $\mathrm{FP}\,E = \mathrm{map}(I, E)$ denote the space of free paths into E.

Definition. Let $p\colon E \to B$ be a map and let $\varepsilon_0\colon \mathrm{FP}\,B \to B$ denote evaluation at 0. Let $\bar{E} = E\,{}_p\sqcap_{\varepsilon_0} \mathrm{FP}\,B = E \sqcap \mathrm{FP}\,B$ be the pullback of $E \xrightarrow{p} B \xleftarrow{\varepsilon_0} \mathrm{FP}\,B$. A lifting function for p is a map $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ that makes the following diagram commute.

$$\begin{array}{ccccc} \bar{E} & & \longrightarrow & & \mathrm{FP}\,B \\ \big\downarrow & \searrow^{\lambda} & & \nearrow_{p_*} & \big\downarrow{\scriptstyle \varepsilon_0} \\ & & \mathrm{FP}\,E & & \\ & \swarrow^{\varepsilon_0} & & & \\ E & & \xrightarrow{p} & & B \end{array} \tag{A.2}$$

Note that since the square is a pullback, there is a natural map $\nu\colon \mathrm{FP}\,E \to \bar{E}$, and the universal property forces the pointwise identity $\nu\lambda = \mathrm{id}$. Recall that p has a lifting function if and only if it is a fibration. (See Fadell [6].)

Following the proof of Theorem 2.8.14 of Spanier [29] we extract

Lemma A.1. *Let $p\colon E \to B$ be a fibration and let $f_0, f_1\colon X \to B$ be homotopic maps. Set $E_0 = f_0^*E$, and $E_1 = f_1^*E$. The fibrations $E_0 \to X$ and $E_1 \to X$ are fibre homotopy equivalent.*

As above let $\bar{E} = E \sqcap_B \mathrm{FP}\,B$ and let $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ be a lifting function for p. Let $\tilde{h}\colon X \to \mathrm{FP}\,B$ denote the adjoint of a homotopy between f_0 and f_1. The map

$$E_0 \to E_1, \quad (x, e) \mapsto (x, [\varepsilon_1 \circ \lambda](e, \tilde{h}(x))) \tag{A.3}$$

is a fibre homotopy equivalence. The inverse is obtained by symmetry. □

Lemma A.2. *In the situation described in the beginning of this appendix, we may view $\Lambda_{\mathrm{incl}} = \mathrm{map}\,((I, 0, 1), (E, e, F)) \subset \mathrm{FP}\,E$ and $\Omega(B, b) \equiv \{e\} \times \Omega(B, b) \subset \bar{E} = E \sqcap \mathrm{FP}\,B$. In this way composition with p yields a map $\Lambda \to \Omega(B, b)$, while the lifting function $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ maps $\Omega(B, b)$ to Λ.*

The maps $\Lambda \to \Omega(B, b)$ and $\Omega(B, b) \to \Lambda$ are mutual homotopy inverses.

Proof. This is well known. It is contained implicitly for example in Fadell [6]. □

Proposition A.3. *Let $p\colon E \to B$ be a fibration and let $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ be a lifting function for p as in diagram* (A.2). *Assume that $F = p^{-1}(b)$ contracts in E to e and let $k\colon F \to \mathrm{FP}\,E$ denote the adjoint of a contracting homotopy. Further let $\varepsilon_1\colon \mathrm{FP}\,E \to E$ denote evaluation at* 1. *Then the map*

$$\Omega(B, b) \to F \times \Omega(E, e), \quad \gamma \mapsto (\varepsilon_1[\lambda(e, \gamma)], \lambda(e, \gamma) \star k(\varepsilon_1[\lambda(e, \gamma)]))$$

is a homotopy equivalence with inverse

$$F \times \Omega(E, e) \to \Omega(B, b), \quad (x, \omega) \mapsto p \circ [\omega \star k^{-1}(x)] = (p \circ \omega) \star (p \circ k^{-1}(x)).$$

Here $\star$ denotes concatenation of paths.

Proof. Let $\rho\colon I \times I \to I \times \{0\} \cup \{1\} \times I$ be a retraction satisfying $\rho(s, 1) = (2s, 0)$ for $s \leqslant \frac{1}{2}$ and $\rho(s, 1) = (1, 2s - 1)$ for $s \geqslant \frac{1}{2}$. Note that evaluation $\varepsilon_1\colon \mathrm{FP}\,E \to E$ can be indentified with the restriction $\mathrm{map}((I, 0), (E, e)) \to \mathrm{map}((1, \emptyset), (E, e))$ and by [25], Lemma A.2, the mapping $\mu\colon \mathrm{map}(I \times 0 \cup 1 \times I, (0, 0)), (E, e)) \to \mathrm{map}((I \times I, 0 \times I), (E, e))$, defined by $\mu(f) = f \circ \rho$, can be viewed as a lifting function for ε_1.

We are going to apply Lemma A.1 to the situation depicted in diagram (A.1) to find an explicit map $\Lambda_{\mathrm{incl}} \to \Lambda_{\mathrm{const}} = F \times \Omega(E, e)$. Display (A.3) transcribes into

$$\Lambda_{\mathrm{incl}} \to \Lambda_{\mathrm{const}}, \quad (x, \eta) \mapsto (x, \varepsilon_1[\mu(\eta, k(x))]).$$

Here $x \in F$ and $\eta \in \mathrm{P}E$ is a path based at e with $\eta(1) = x$. Also, $k(x)$ is a path in E beginning in x and ending in e. Then $\varepsilon_1[\mu(\eta, k(x))]$ has the following interpretation. Concatenate η and $k(x)$ to yield a map $I \times \{0\} \cup \{1\} \times I \to E$, precompose with ρ, restrict to $I \times \{1\}$, and view the result again as a path in E. The special requirement on ρ makes the result exactly the concatenation $\eta \star k(x) = \eta \star k(\eta(1))$.

By Lemma A.2, the map $\Omega(B, b) \to \Lambda_{\mathrm{incl}}$ is given by $\gamma \mapsto (x, \eta)$ with $\eta = \lambda(e, \gamma)$ and $x = \varepsilon_1(\eta) = \eta(1)$. This accounts for the map $\Omega(B, b) \to F \times \Omega(E, e)$. The inverse is obtained similarly. □

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