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PERIODIC HOMOTOPY AND CONJUGACY IDEMPOTENTS

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ABSTRACT. A self-map f on the CW complex Z is a *periodic homotopy idempotent* if for some $r \geqslant 0$ and $p > 0$ the iterates f^r and f^{r+p} are homotopic. Geoghegan and Nicas defined the *rotation index* $RI(f)$ of such a map. They proved that for $r = p = 1$, the homotopy idempotent f splits if and only if $RI(f) = 1$, while for $r = 0$, the index $RI(f)$ divides p^2. We extend this to arbitrary p and r, and generalize various results related to the splitting of homotopy idempotents on CW complexes and conjugacy idempotents on groups.

1. INTRODUCTION. PERIODIC HOMOTOPY IDEMPOTENTS

Let p be a natural number. In any category, it makes sense to call the morphism $f\colon Z \to Z$ a *p-idempotent* if the iterate f^{p+1} is equal to f. There is a distinguished class of p-idempotents $Z \to Z$, namely those that factor as $f = u \circ d$ for some morphisms $d\colon Z \to Y$ and $u\colon Y \to Z$ satisfying $(du)^p = \mathrm{id}_Y$. These are called *split* p-idempotents, and any factorization $f = ud$ with this property is called *splitting*. Note that the isomorphism class of Y is unique in this case. When emphasis is required, we call 1-idempotents *ordinary idempotents*. Note that if f is a p-idempotent, then f splits if and only if $f = ud$ with du an isomorphism.

The categories we are interested in here are those of groups (with morphisms conjugacy classes of group homomorphisms) and of CW complexes (with morphisms free homotopy classes of continuous maps).

If $f\colon Z \to Z$ is an ordinary homotopy idempotent on the connected CW complex Z (this is to say f^2 is freely homotopic to f), then the induced morphism on the fundamental group $\phi = f_{\#}\colon \pi_1(Z) \to \pi_1(Z)$ is an ordinary conjugacy idempotent (i.e. ϕ^2 is conjugate to ϕ). Freyd and Heller ([4], Main Lemma) have shown that f splits up to homotopy if and only if ϕ splits up to conjugacy. Moreover, this happens if and only if the kernel of a certain universal morphism $\kappa_\phi\colon F \to \pi_1(Z)$ from the Thompson group F has nontrivial kernel (see Brown and Geoghegan [3]).

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In [6] Geoghegan and Nicas have given another necessary and sufficient condition for an ordinary homotopy idempotent to split, in relation with their one-parameter fixed point theory on mapping tori (see [5], §6). We need a definition.

Let $I = [0,1]$. For a self-map f on the topological space Z, its *mapping torus* T_f is the quotient space of $Z \times I$ modulo the relations $(z,1) \sim (f(z),0)$. It admits a projection $p_f \colon T_f \to S^1$ defined by $[z,t] \mapsto e^{2\pi i t}$. Let $T_f^{T_f}$ denote the space of self-maps $T_f \to T_f$ with the compact-open topology. Pick a base-point z_0 in Z and set $\zeta_0 = [z_0, 0]$. Let $\mathrm{eval} \colon (T_f)^{T_f} \to T_f$ denote evaluation at ζ_0. Denote the composite $P_f = p_f\,\mathrm{eval} \colon (T_f)^{T_f} \to S^1$. Set $\Gamma_f = \pi_1\big((T_f)^{T_f}, \mathrm{id}\big)$. Identifying $\pi_1(S^1,1)$ with the integers $\mathbb{Z}$ via degree, P_f induces a morphism $P_{f\#} \colon \Gamma_f \to \mathbb{Z}$. Each element of Γ may be represented (when the topology on Z is reasonable) with a *cyclic* homotopy $h \colon T_f \times I \to T_f$, i.e., one beginning and ending in the identity. Evaluation then corresponds to taking the *trace* of the homotopy, i.e., the path $t \mapsto h(\zeta_0, t)$. The *rotation index* (Geoghegan and Nicas, [5]) $RI(f)$ of the self-map f is the least positive integer in the image of $P_{f\#}$. If $P_{f\#}$ is trivial, we set $RI(f) = \infty$.

Geoghegan and Nicas have shown that an ordinary homotopy idempotent $f \colon Z \to Z$ on the connected CW complex Z splits if and only if $RI(f) = 1$. An obvious problem is to find a characterization of $RI(f) = q$ for any q. If for some $r \geqslant 0$ and $p > 0$ the iterates f^r and f^{r+p} are homotopic, the self-map f is called a *periodic homotopy idempotent*. On nice spaces (e.g., compacta), $RI(f) = q$ implies $f^i \simeq f^{i+q}$ for some i. ($\simeq$ denotes the 'free' homotopy relation.) Thus the interesting problems are the meaning of $RI(f) = q$ for $f^r \simeq f^{r+p}$, and also the relation between $RI(f)$ and $RI(f^r)$ in this case. Our main results are the following solutions:

Theorem 1.1. *Let Z be a connected space of CW homotopy type and let f be a self-map on Z with f^r freely homotopic to f^{r+p}. Then $RI(f) < \infty$ if and only if f^r splits as a homotopy p-idempotent. In that case $RI(f)$ divides p^2.*

Corollary 1.2. *Let $f \colon Z \to Z$ be a periodic homotopy idempotent, $f^r \simeq f^{r+p}$. If for some $j \geqslant 1$ we have $RI(f^j) < \infty$ it follows that $RI(f^i) < \infty$, $\forall i \geqslant 1$. Moreover, this is true if and only if on the fundamental group $\mathrm{im} f^r_\# = \mathrm{im} f^{r+1}_\# = \cdots$.*

Theorem 1.1 generalizes Theorems 2.4 and 3.4 of Geoghegan and Nicas [6] and in conjunction with Corollary 3.3 answers all questions of [6], Introduction. (See also [5], §6, page 35.) Moreover, in Theorem 4.3 below we characterize a finite rotation index by means of a universal morphism from a generalized Thompson group.

Theorem 1.1 depends on the investigation of split p-idempotents in section 2. In section 3 we list the basic properties of the rotation index, particulary in relation to *eventual coherence* where we point out an error in [5]. In section 4 we focus on periodic conjugacy idempotents. Section 5 contains the lengthier proofs.

2. Split homotopy and conjugacy p-idempotents

As for $p = 1$ (Freyd and Heller [4], Main Lemma), the question of splitting homotopy p-idempotents on CW complexes can be reduced to groups.

Theorem 2.1. *Suppose f is a base-point preserving self-map on the CW complex Z, and that f^{p+1} is freely homotopic to f. Then f splits up to homotopy if and only if $f_\# \colon \pi_1(Z) \to \pi_1(Z)$ splits up to conjugacy.*

Thus we turn to groups and consider an endomorphism $f \colon G \to G$ on the group G. For α in G we let f^α denote the conjugate of f by α. Also, for elements ξ and η

of G we denote $\xi^\eta = \eta^{-1}\xi\eta$, so that $f^\alpha(\xi) = (f(\xi))^\alpha$. We denote the commutator $[\xi, \eta] = \xi\eta\xi^{-1}\eta^{-1}$. Recall the generalized *Thompson group* F_p defined as

$$F_p = \left\langle x_0,\, x_1,\, x_2, \dots \,\middle|\, x_i^{-1} x_j x_i = x_{j+p},\ i < j \right\rangle .$$

It admits a shift $\phi\colon F_p \to F_p$, defined by $\phi(x_i) = x_{i+1}$. The defining relations imply $\phi^{p+1} = \phi^{x_0}$ which is to say that ϕ is a conjugacy p-idempotent.

Lemma 2.2. *Suppose $f\colon G \to G$ has $f^{p+1} = f^\alpha$. Then there exists a unique homomorphism $\kappa = \kappa_f\colon F_p \to G$ such that $\kappa(x_0) = \alpha$ and $\kappa\phi = f\kappa$.* □

This means that as for $p = 1$, the p-idempotent ϕ is universal (cf. Brown and Geoghegan [3], Proposition 1.1). Moreover, as for $p = 1$ (cf. Freyd and Heller [4], Main Theorem), ϕ is the universal nonsplitting p-idempotent. To wit, we prove

Theorem 2.3. *Let $f\colon G \to G$ satisfy $f^{p+1} = f^\alpha$. The following are equivalent:*

(**i**) *f splits.*
(**ii**) *The images of f and f^2 coincide.*
(**iii**) $[\alpha, f(\alpha)] = 1$.
(**iv**) *The kernel of the universal morphism $\kappa_f\colon F_p \to G$ is nontrivial.*
(**v**) *The kernel of $\kappa_f\colon F_p \to G$ contains the commutator subgroup.*

The case $p = 1$ is somewhat special in that there are critical points in the slick arguments of Freyd and Heller [4] that do not carry over to the case $p > 1$. So Theorems 2.1 and 2.3 are proven here using a different, geometric, approach.

Corollary 2.4. (**i**) *Let f be either a homotopy or a conjugacy p-idempotent. Then f splits if and only if f^p splits as an idempotent.*
(**ii**) *Every homotopy p-idempotent on a finite-dimensional CW complex splits.*

Proof. For (**i**) it suffices to consider groups, by virtue of Theorem 2.1. Let $f\colon G \to G$ satisfy $f^{p+1} = f^\alpha$. We note $(f^p)^2 = f^{p+1}f^{p-1} = f^\alpha f^{p-1} = (f^p)^\alpha$. If f^p splits as an idempotent, then $[\alpha, f^p(\alpha)] = 1$. Hence $\kappa_f\colon F_p \to G$ has nontrivial kernel (because $x_0 x_p x_{2p}^{-1} x_0^{-1}$ is the normal form for $[x_0, x_p]$; see Lemma 5.5 below), and f splits.

As every ordinary homotopy idempotent on a finite dimensional CW complex splits (cf. Hastings and Heller [7]), (**ii**) follows from (**i**). □

3. Properties of the rotation index. Eventual coherence

The rotation index does not depend on the choice of base-point z_0 in Z; see [6], Proposition 1.2. We recall the basic result on homotopy invariance.

Proposition 3.1 (Geoghegan and Nicas [6], Proposition 1.6)**.** *Let $h\colon Z \to Z'$ be a homotopy equivalence, and let $f\colon Z \to Z$ and $f'\colon Z' \to Z'$ be self-maps for which the composites $h \circ f$ and $f' \circ h$ are homotopic. Then $RI(f) = RI(f')$.* □

Let W be a path-connected space that is homotopy equivalent to the CW complex Z, and let g be a self-map on W. By Proposition 3.1, $RI(g) = RI(g')$ for an obvious self-map g' on Z. If $z_0 \in Z$ is any point, then g' is homotopic to some f with $f(z_0) = z_0$. Again, $RI(g') = RI(f)$. Thus for our purposes it suffices to focus on base-point preserving self-maps on CW complexes.

The notion of eventual coherence of periodic homotopy idempotents has been introduced by Geoghegan and Nicas (see [5], Section 6, as well as [6]).

Definition. A self-map f on the path-connected space Z is an *eventually coherent periodic homotopy idempotent* if, for some $i \geqslant 0$ and $q > 0$, there is a homotopy $N\colon Z \times [0,1] \to Z$ between f^i and f^{i+q} such that fN is homotopic to $N(f(\_),\_)$ via a homotopy $Z \times [0,1] \times [0,1] \to Z$ that is constant on $Z \times \{0,1\}$.

The *period* of f is the least q for which there exists an i with the above property. If such an i does not exist, we let the period be infinite in accordance with $\inf \emptyset = \infty$.

The following proposition is the content of [5], Theorem 6.3.

Proposition 3.2 (Geoghegan and Nicas [6])**.** *Let f be a self-map on the connected space Z.*

(**i**) *If Z is compact, then the period of f equals its rotation index $RI(f)$.*
(**ii**) *In general, if f has period $q < \infty$, then $RI(f)$ is finite and divides q.* □

Corollary 3.3. *If Z is a compact space of CW type, then every periodic homotopy idempotent on Z is eventually coherent.*

Proof. If $f^r \simeq f^{r+p}$, then f^r splits by (**ii**) of Corollary 2.4. By Theorem 1.1, $RI(f)$ divides p^2. Proposition 3.2 renders f eventually coherent of period $RI(f)$. □

Example. In [5], Theorem 6.3 (and [6], Proposition 1.5) the authors claim that for any space Z and any self-map f the period and rotation index coincide, although at some point their proof requires compactness. The following is a counterexample. The idea is that in general $RI(f) < \infty$ does not imply that the period is finite.

Let $n \geqslant 2$ and construct the Eilenberg-MacLane complex $K(\mathbb{Z}, n)$ with base-point ξ. For $i \geqslant 1$ set $Z_i = K(\mathbb{Z}, n)^{p+ip}$ and define the self-map f_i on Z_i by

$$f_i(x_1, \dots, x_{p-1}, x_p; y_1, \dots, y_{ip-1}, y_{ip}) = (x_2, \dots, x_p, x_1; y_2, \dots, y_{ip}, \xi).$$

Evidently $f_i^{ip+p} = f_i^{ip}$. On $\pi_n \cong \mathbb{Z}^{p+ip}$, the induced morphism f_{i*} permutes the 'first' p generators, and acts nilpotently with order ip on the 'remaining' ip generators. Therefore clearly $f_i^{j_1} \simeq f_i^{j_2}$ for $j_1 < j_2$ implies $j_1 \geqslant ip$ and $j_2 - j_1 = kp$ for some positive k. In particular, the period of f_i is p.

Since f_i fixes the base-point $(\xi, \dots, \xi) = \{\xi\}^{p+ip}$, the f_i induce an obvious self-map f on $Z = \bigvee_{i=1}^{\infty} Z_i$. Note that f cannot be a periodic homotopy idempotent.

Let $\hat{f}_i$ denote the induced map on the mapping torus T_{f_i}. Evidently $\hat{f}_i^{p+ip} = \hat{f}_i^{ip}$. The concatenation $R_{p+ip} * R_{ip}^{-1}$ of the corresponding 'tumbles' (see the notes before Lemma 5.6) is a cyclic homotopy $\bar{h}_i\colon T_{f_i} \times I \to T_{f_i}$ of rotation degree p (see also the discussion in [5] preceding Theorem 6.3). We want to glue these to get a cyclic homotopy $h\colon T_f \times I \to T_f$ of degree p. The torus T_f is the 'union' (=colimit) of the tori T_{f_i} meeting in the common circle $S^1 = \{[*, t] \,|\, t \in [0,1]\}$ where $*$ is the generic base-point. The $\bar{h}_i$ do not agree along S^1. However, for each i the restriction $\bar{h}_i|_{S^1 \times I}$ is a map $S^1 \times I \to S^1$ representing an element in $\pi_1((S^1)^{S^1}, \mathrm{id})$ of degree p. Fix a representative for that map. Each $\bar{h}_i$ is homotopic as a cyclic homotopy to some h_i extending the fixed map $S^1 \times I \to S^1$. The h_i together define a cyclic homotopy $T_f \times I \to T_f$ of degree p. The self-map f is not a periodic homotopy idempotent but has $RI(f) = p$, contradicting Theorem 6.3 of [5]. □

It seems hard to get a grip on the period for noncompact spaces. The rotation index is easier to deal with and adequate for comparison with splitting phenomena.

4. Periodic conjugacy idempotents

We consider the following generalization of the Thompson group F_p:

$$F_{p,r} = \left\langle x_0,\, x_1,\, x_2, \dots \,\middle|\, x_i^{-1} x_j x_i = x_{j+p},\ \ i + r \leqslant j \right\rangle .$$

Also $F_{p,r}$ admits the shift $\phi(x_i) = x_{i+1}$, and evidently $\phi^{r+p} = (\phi^r)^{x_0}$, i.e., ϕ is a periodic conjugacy p-idempotent. Note that $F_p = F_{p,1}$ is a quotient of $F_{p,r}$. As for p-idempotents, $\phi\colon F_{p,r} \to F_{p,r}$ is universal in the following sense.

Lemma 4.1. *Suppose $f\colon G \to G$ has $f^{p+r} = (f^r)^\alpha$. Then there exists a unique homomorphism $\kappa = \kappa_f\colon F_{p,r} \to G$ such that $\kappa(x_0) = \alpha$ and $\kappa\phi = f\kappa$.* □

As $\phi^{r+p} = (\phi^r)^{x_0}$, it follows that $(\phi^r)^{1+p} = \phi^{r+rp} = (\phi^r)^{x_0^r}$, that is, ϕ^r is a p-idempotent. Denote the shift $\psi\colon F_p \to F_p$. The morphism $\kappa_{p,r}\colon F_p \to F_{p,r}$ with $\kappa_{p,r}(x_0) = x_0^r$ and $\kappa_{p,r}\psi = \phi^r \kappa_{p,r}$ of Lemma 2.2 is given by $\kappa_{p,r}(x_i) = x_{ri}^r$. Denote the quotient morphism $q_{p,r}\colon F_{p,r} \to F_p$.

Proposition 4.2. *The composite $F_p \xrightarrow{\kappa_{p,r}} F_{p,r} \xrightarrow{q_{p,r}} F_p$ is injective. Consequently $\kappa_{p,r}$ is injective, $\phi^r\colon F_{p,r} \to F_{p,r}$ does not split, and $RI(\phi) = \infty$.*

(For a morphism $\vartheta\colon G \to G$ we can define $RI(\vartheta)$ to be $RI(f)$ for any self-map on the Eilenberg-MacLane space $K(G, 1)$ inducing ϑ on π_1.)

We infer the following analogue of equivalence (**i**) $\Longleftrightarrow$ (**iv**) of Theorem 2.3.

Theorem 4.3. *Let G be a group and $f\colon G \to G$ a morphism with f^{r+p} conjugate to f^r. Let $\kappa_f\colon F_{p,r} \to G$ be the associated morphism. Then $RI(f) < \infty$ if and only if the restriction of κ_f to the image of $\kappa_{p,r}$ has nontrivial kernel.*

Proof. Let $\kappa'\colon F_p \to G$ be the morphism associated to the p-idempotent f^r. Then $\kappa' = \kappa_f \circ \kappa_{p,r}$, and the assertion follows from Proposition 4.2 and Theorem 2.3. □

Example. The shift ϕ on $F_{p,r}$ has $\phi^{r+p} = (\phi^r)^{x_0}$; hence $f = \phi^r$ satisfies $f^{r+p} = (f^r)^\alpha$ for $\alpha = x_0^r$. Corollary 1.2 implies that $RI(f) = \infty$. Let $\kappa_f\colon F_{p,r} \to F_{p,r}$ denote the universal morphism. As f is also a p-idempotent, it has a universal morphism $F_p \to F_{p,r}$ in the sense of Lemma 2.2. It is precisely $\kappa_{p,r}$ which is injective. As $\kappa_f = \kappa_{p,r} q_{p,r}$, the kernel of κ_f equals that of $q_{p,r}$, and is nontrivial. □

5. Proofs

Let $f\colon Z \to Z$ be a self-map. The *mapping telescope* Tel_f of f is the quotient space of the disjoint union $\bigsqcup_{n\in\mathbb{Z}} Z \times \{n\} \times [0, 1]$ modulo the relations $(z, n, 1) \sim (f(z), n+1, 0)$. We use $[z, n, t] \in \mathrm{Tel}_f$ for the point represented by (z, n, t). Another frequently used telescope is the subspace Tel_f^+ of Tel_f consisting of all points $[z, n, t]$ with $n \geqslant 0$. Note that Tel_f^+ is a strong deformation retract of Tel_f.

We say that a continuous map $f\colon X \to Y$ factors *strictly* through the topological space W if there exist $g\colon X \to W$ and $h\colon W \to Y$ such that $h \circ g = f$ pointwise.

For $f\colon (X, x_0) \to (Y, y_0)$ and generic $k \geqslant 1$ we denote the induced morphism on homotopy groups as $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_0)$. When the fundamental group is emphasized, we use $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Our first goal is to prove Theorems 2.3 and 2.1. The first two auxiliary results will also be applied later on in the proof of Theorem 1.1.

Lemma 5.1. *Let $f\colon Z \to Z$ be a self-map. If for some $r \geqslant 0$ and $p \geqslant 1$ the iterates f^r and f^{r+p} are homotopic, then f^r factors strictly through Tel_f^+ (and Tel_f).*

Proof. Assume that $N\colon Z\times I\to Z$ is a homotopy between f^r and f^{r+p}. Let $d\colon Z\to \mathrm{Tel}_f^+$ be the inclusion $z\mapsto [z,0,0]$ and define $u\colon \mathrm{Tel}_f^+\to Z$ by setting $u\big([z,n,t]\big)=f^{n(p-1)}(N(z,t))$. Evidently u is well defined and $ud=f^r$.

For Tel_f precompose u with the retraction $\mathrm{Tel}_f\to\mathrm{Tel}_f^+$. □

Proposition 5.2. *Let $f\colon (Z,z_0)\to(Z,z_0)$ be a self-map on the pointed space of CW type (Z,z_0). Suppose that f is homotopic to f^{p+1} via homotopy $N\colon Z\times I\to Z$. Let $\nu\in\pi_1(Z,z_0)$ denote the class of the loop $t\mapsto N(z_0,t)$ traced out under N by the base-point. Assume the factorization of f^r through Tel_f^+ defined above. Let z_0 and $[z_0,0,0]$ be the base-points in Z and Tel_f^+, respectively.*

Assume that for each $k\geqslant 1$ and each $\gamma\in\pi_k(Z,z_0)$ there exist positive numbers r_0 and q as well as an element $\eta\in\pi_1(Z,z_0)$ with the property that

$$(\dagger)\qquad f_*^{r+\lambda q}(\gamma)=\big[f_*^r(\gamma)\big]^{f_\#^{r+\lambda q}(\eta^\lambda)}$$

for all $r\geqslant r_0$ and all $\lambda\geqslant 0$. Then for every $k\geqslant 1$ the restriction

$$(\star)\qquad d_*|_{\mathrm{im}f_*}\colon \mathrm{im}f_*\to\pi_k(\mathrm{Tel}_f^+)$$

is an isomorphism and the factorization of f through its telescope Tel_f^+ is splitting.

Proof. Let $G_k=\pi_k(Z,z_0)$. Then $\pi_k(\mathrm{Tel}_f^+,z_0)$ is isomorphic with the colimit C_k of

$$G_k^0\xrightarrow{f_*}G_k^1\xrightarrow{f_*}G_k^2\to\cdots,$$

where $G_k^i=G_k$ for all i. The morphism $d_*\colon G_k\to C_k$ corresponds to inclusion on the first level, i.e., $G_k\equiv G_k^0\to C_k$. To determine its kernel suppose that $d_*(\gamma)=1$. This is true if and only if $f^\mu(\gamma)=1$ for all large enough μ. Take $\mu=\lambda p+1$. Then

$$f_*^\mu=f_*^{\lambda p+1}=f_*^{\nu^\lambda},$$

and $f_*^\mu(\gamma)=\big(f_*(\gamma)\big)^{\nu^\lambda}$ is trivial if and only if $f_*(\gamma)$ is. Thus $\ker d_*=\ker f_*$. The same argument shows that f_*, and hence d_*, are injective on the image of f_*.

Next we show that d_* is surjective. Pick an element ξ of C_k. It is represented by an element $\gamma\in G_k^{\mu'}$ for some μ'. Let r_0, q, and η be as in the statement of the proposition. There is an $r\geqslant r_0$ such that $\mu=\mu'+r=\lambda q$ for some λ. Then ξ is also represented by $f_*^r(\gamma)\in G_k^\mu$. We set $\Gamma=f_*^{r-1}\big(\gamma^{\eta^{-\lambda}}\big)$ and rewrite (†) as

$$(\ddagger)\qquad f_*^r(\gamma)=(f_*^{r+\lambda q})^{f_\#^{r+\lambda q}(\eta^{-\lambda})}(\gamma)=f_*^{r+\lambda q}\big(\gamma^{\eta^{-\lambda}}\big)=f_*^\mu(f_*(\Gamma)).$$

But $f_*^\mu(f_*(\Gamma))$ in G_k^μ is in the image of $f_*(\Gamma)\in G_0^k$. This shows that the restriction of d_* to the image of f_* is surjective onto C_k. In particular, the morphism $(\star)$ is an isomorphism, as claimed.

The resulting short exact sequence $1\to\ker d_*\to G_k\to C_k\to 1$ with $\ker d_*=\ker f_*$ induces the isomorphism $h\colon \mathrm{im}f_*\to C_k$ defined by $h(f_*(x))=d_*(x)$.

Thus $f_*=u_*d_*=u_*hf_*$ which is to say that u_*h is the identity on $\mathrm{im}f_*$. Then $d_*u_*h\colon \mathrm{im}f_*\to C_k$ is an isomorphism. Since h is an isomorphism, $d_*u_*\colon C_k\to C_k$ is also an isomorphism. By Whitehead's theorem, the composite $du\colon \mathrm{Tel}_f^+\to\mathrm{Tel}_f^+$ is a homotopy equivalence, hence the factorization of f is splitting. □

Addendum. Assume the notation employed in the statement of the proposition. If ν lies in the image of $f_\#$, i.e., $\nu=f_\#(\eta)$, then (†) holds for this η together with $r_0=1$ and $q=p$ for all γ. Indeed, recall first that $f_*^{\lambda p+1}=f_*^{\nu^\lambda}$. Using $\nu=f_\#(\eta)$, we compute $f_{\#}^{p+1}(\eta) = f_{\#}^{\nu}(\eta) = \nu^{-1} f_{\#}(\eta)\nu = f_{\#}(\eta)$. Thus $f_{\#}(\eta^{\lambda}) = f_{\#}^{\lambda p+1}(\eta^{\lambda})$ for all λ. Substituting this into the equation $f_*^{\lambda p+1} = f_*^{f_{\#}(\eta^{\lambda})}$ yields (†). □

Realizing a group morphism as a map between Eilenberg-MacLane spaces we immediately deduce from Proposition 5.2 and the addendum the following.

Corollary 5.3. *Let $g\colon G \to G$ be a group endomorphism which is a p-idempotent up to conjugation, $g^{p+1} = g^{\nu}$. If ν lies in the image of g, then g splits.* □

Proposition 5.4. *Suppose $g\colon G \to G$ is an endomorphism of groups which is a p-idempotent up to conjugation, $g^{p+1} = g^{\beta}$. If $[\beta, g(\beta)] = 1$, then $h = g^{\beta}$ has $h^{p+1} = h^{h(\eta)}$ for some η (and $[\eta, h(\eta)] = 1$).*

Proof. First we compute

$$(*) \qquad h^{p+1} = g^p g^{(p+1)p+1} = g^p g^{\beta g^p(\beta) g^{2p}(\beta)\cdots g^{pp}(\beta)} = (g^{p+1})^{g^p(\beta) g^{2p}(\beta)\cdots g^{(p+1)p}(\beta)}.$$

From $[\beta, g(\beta)] = 1$ it follows that $g^{p+1}(\beta) = g^{\beta}(\beta) = [g(\beta)]^{\beta} = g(\beta)$. Thus $g^{kp+i}(\beta) = g^i(\beta)$ for all $i, k > 0$ (and $h^i(\beta) = g^i(\beta)$). Using this in $(*)$ we express $h^{p+1} = (g^{p+1})^{g^{(p+1)p}(\beta)} = h^{h^p(\beta)}$, which proves the assertion for $\eta = h^{p-1}(\beta)$. □

Lemma 5.5 (Normal forms)**.** *Any $x \in F_p$ can be written in the form*

$$x = x_{i_1} \cdots x_{i_k} x_{j_m}^{-1} \cdots x_{j_1}^{-1} \quad \text{with} \quad i_1 \leqslant \cdots \leqslant i_k,\ j_1 \leqslant \cdots \leqslant j_m,\ k, m \geqslant 0.$$

This expression for x can be chosen so that if x_i and x_i^{-1} both occur for some i, then x_j or x_j^{-1} also occurs for some $j \in i+1, \ldots, i+p$. Otherwise there would be a subproduct of the form $x_i \phi^{i+p+1}(y) x_i^{-1}$ which we can replace with $\phi^{i+1}(y)$. We call such an expression a normal form for x. Every $x \neq 1$ has a unique normal form.

Proof. See Remarks, (2), in Brown [2], page 58, and Brown and Geoghegan [3], 1.3, as well as the paragraph preceding Proposition 2.1.5 of Brin and Guzmán [1]. □

Proof of Proposition 4.2. Take $x \neq 1$ in F_p and let $x = x_{i_1} \cdots x_{i_k} x_{j_m}^{-1} \cdots x_{j_1}^{-1}$ be the normal form for x (see Lemma 5.5). Denote $\mu = q_{p,r} \circ \kappa_{p,r}\colon F_p \to F_p$. Then

$$(\bullet) \qquad \mu(x) = x_{ri_1}^r \cdots x_{ri_k}^r x_{rj_m}^{-r} \cdots x_{rj_1}^{-r}.$$

If $(\bullet)$ is a normal form, then $\mu(x) \neq 1$. Otherwise, there exists an i such that in $(\bullet)$ there occur both x_{ri}^r and x_{ri}^{-r} but neither x_k^r nor x_k^{-r} for $k \in ri+1, \ldots, ri+p$. We may assume i is maximal with this property.

Let j be the smallest number bigger than i such that one of x_j and x_j^{-1} occurs in x. By assumption, $rj - ri > p$. Divide $rj - ri = \lambda p + \delta$ where $\delta \in \{1, \ldots, p\}$. For all $k > ri$ we substitute $x_k^{\pm r}$ in $(\bullet)$ for $x_{ri}^{-\lambda} x_{k-\lambda p}^{\pm r} x_{ri}^{+\lambda}$, rendering the subexpression with indices $l \geqslant ri$ a normal form (after expansion of powers). As $j - i \leqslant p$, we have $rj - ri \leqslant rp$ and thus $\lambda < r$. Thus in the subexpression in question we couldn't possibly eliminate the pair x_{ri} and x_{ri}^{-1}. Moreover, the number of different generators and their inverses in $(\bullet)$ did not change after this transformation. Repeating this we get the normal form for $\mu(x)$ with the same number of different generators and their inverses as in $(\bullet)$ and hence as in the normal form for x. Thus $\mu(x) \neq 1$. □

Proof of Theorem 2.3. Let $g\colon G \to G$ be a conjugacy p-idempotent, $g^{p+1} = g^{\beta}$.

To prove that (**i**) implies (**ii**), assume that g splits. This is to say that g is conjugate to ud, say $g = (ud)^{\alpha}$, with du an isomorphism. Then $g = u^{\alpha} d$ and

$du^\alpha = (du)^{d(\alpha)}$ is an isomorphism. Thus $g^2 = u^\alpha du^\alpha d = u^\alpha(du^\alpha)d$, and since du^α is an isomorphism, the image of g^2 is the same as that of g.

For (**ii**) $\Longrightarrow$ (**iii**), assume $\mathrm{im}g^2 = \mathrm{im}g$. We have $g^{p+2} = gg^{p+1} = gg^\beta = (g^2)^{g(\beta)}$ and on the other hand, $g^{p+2} = g^{p+1}g = g^\beta g = (g^2)^\beta$. Thus the actions of β and $g(\beta)$ by conjugation on $\mathrm{im}g^2 = \mathrm{im}g$ coincide. In particular, β and $g(\beta)$ act in the same way on $g(\beta)$, i.e., they commute.

Evidently (**iii**) implies (**i**) by Proposition 5.4 and Corollary 5.3.

The existence of normal forms by Lemma 5.5 guarantees that (**iii**) implies (**iv**). By Proposition 2.1.5 of Brin and Guzmán [1] (our F_p is $F^*_{p+1,0}$ there), (**iv**) implies (**v**). By definition, (**v**) implies (**iii**). □

Proof of Theorem 2.1. Let $N\colon f \simeq f^{p+1}$ have trace α. In particular $f_\#^{p+1} = f_\#^\alpha$.

Suppose that $f_\#$ splits. Then by implication (**i**) $\Longrightarrow$ (**iii**) of Theorem 2.3 and Proposition 5.4 the map $h = f^{p+1} \simeq f$ admits a homotopy $M\colon h \simeq h^{p+1}$ with trace $h_\#(\eta)$, for some η. (E.g., we may take for M the concatenation of homotopies $f^pN * f^{2p}N * \cdots * f^{(p+1)p}N$.) By Proposition 5.2, h splits. Hence so does f. □

To prove Theorem 1.1 we will need a handful of auxiliary results. First we discuss the basic properties of mapping telescopes and mapping tori.

The group of integers $\mathbb{Z}$ acts on Tel_f by 'integer translations' $\tau_k\colon \mathrm{Tel}_f \to \mathrm{Tel}_f$, defined by $\tau_k([z,n,t]) = [z, n+k, t]$. This is a properly discontinuous action with orbit space T_f, and the associated projection $\rho\colon \mathrm{Tel}_f \to T_f$, $[z,n,t] \mapsto [z,t]$, is a regular covering with group of covering translations exactly $\{\tau_k \mid k \in \mathbb{Z}\}$.

For $k \geqslant 0$, the *k-th tumble* (see Geoghegan and Nicas [5], page 32) is the homotopy $R_k\colon T_f \times I \to T_f$, defined by $R_k([z,u],s) = [f^{\lfloor u+qs \rfloor}(z), \mod (u+qs)]$ where $\lfloor x \rfloor$ denotes the lower integer part of x and $\mod(x) = x - \lfloor x \rfloor$ its modulus. The standard lifting of R_k to the telescope is the homotopy $\bar{R}_k\colon \mathrm{Tel}_f \times I \to \mathrm{Tel}_f$, given by $([z,n,u],s) \mapsto [f^{\lfloor u+qs \rfloor}(z), n + \lfloor u+sq \rfloor, \mod (u+qs)]$. It is an equivariant homotopy between the identity and the composite $\tau_k \circ \bar{f}^k$. In particular, $\bar{f}$ is homotopic to τ_{-1} which is a homeomorphism. Thus $\bar{f}$ is always a homotopy equivalence.

We define an explicit map $\beta\colon \mathrm{Tel}_f \to \mathrm{Tel}_{f^r}$ by setting

$$[z, rm, u] \mapsto [z, m, u] \text{ and } [z, rm+i, u] \mapsto [f^{r-i}(z), m+1, 0],\ \ 1 \leqslant i \leqslant r-1.$$

Note that β is well-defined and continuous and satisfies $\beta \circ \bar{f} = \bar{f} \circ \beta$ where the latter $\bar{f}$ denotes the self-map on Tel_{f^r} induced by f. Evidently $\beta \circ \tau_r = \tau_1 \circ \beta$.

Next we define an explicit map $\alpha\colon \mathrm{Tel}_{f^r} \to \mathrm{Tel}_f$ by setting

$$\alpha[z,n,u] = [f^{\lfloor ru \rfloor}(z), rn + \lfloor ru \rfloor, \quad \mod (ru)].$$

It is evident upon inspection that $\alpha \circ \tau_1 = \tau_r \circ \alpha$.

It is easy to see that α and β are mutually inverse homotopy equivalences. In particular, the composite $\beta\alpha$ is equivariantly homotopic to the identity on Tel_{f^r}.

Lemma 5.6. *Let $f\colon (Z, z_0) \to (Z, z_0)$ be a self-map on the CW complex Z and $\bar{f}\colon \mathrm{Tel}_f \to \mathrm{Tel}_f$ the induced map. Furthermore, let $d\colon Z \to \mathrm{Tel}_f$ denote the standard inclusion. If for each k there exists a subgroup B_k of $\pi_k(Z)$ such that the restriction $d_*|_{B_k}\colon B_k \to \pi_k(\mathrm{Tel}_f)$ is an isomorphism, then $RI(f) = RI(\bar{f})$.*

Proof. First we define a map from the mapping torus of f to that of $\bar{f}$,

$$j\colon T_f \to T_{\bar{f}},\ \ [z,u] \mapsto \big[[z,0,0],u\big].$$

Note that j is well-defined and continuous and that it lifts to $\bar{j}\colon \mathrm{Tel}_f \to \mathrm{Tel}_{\bar{f}}$, $[z,n,u] \mapsto \big[[z,0,0],n,u\big]$. Denote the projection $p_{\bar{f}}\colon T_{\bar{f}} \to S^1$. As $p_{\bar{f}} \circ j = p_f$, the map j is a homotopy equivalence if and only if its lifting $\bar{j}$ is a homotopy equivalence.

Since $\bar{f}\colon \mathrm{Tel}_f \to \mathrm{Tel}_f$ is a homotopy equivalence, so is the standard inclusion $\bar{d}\colon \mathrm{Tel}_f \to \mathrm{Tel}_{\bar{f}}$, $[z,n,u] \mapsto [[z,n,u],0,0]$ of Tel_f into the telescope of $\bar{f}$. Also denoting the standard inclusion $d\colon Z \to \mathrm{Tel}_f$, defined by $z \mapsto [z,0,0]$, we note that $\bar{j} \circ d = \bar{d} \circ d$. We get the induced equality $\bar{j}_* \circ d_* = \bar{d}_* \circ d_*$ on π_k, and consequently

$$\bar{j}_* \circ d'_* = \bar{d}_* \circ d'_*$$

where d'_* denotes the restriction of d_* to B_k. By hypothesis, d'_* is an isomorphism $B_k \to \pi_k(\mathrm{Tel}_f)$. But $\bar{d}_*$ is also an isomorphism, and the above equation implies that so is $\bar{j}_*$. Thus $\bar{j}$ induces isomorphisms on the homotopy groups. By Whitehead's theorem it is a homotopy equivalence. Hence so is j.

As $j\colon (T_f,[z_0,0]) \to (T_{\bar{f}},[[z_0,0,0],0])$ is a homotopy equivalence, it is a pointed homotopy equivalence with a pointed inverse j' and, in particular, a base-point preserving homotopy $h\colon T_{\bar{f}} \times I \to T_{\bar{f}}$ between the identity and jj'. The map

$$K\colon T_f^{T_f} \to T_{\bar{f}}^{T_{\bar{f}}},\ \ \gamma \mapsto j \circ \gamma \circ j',$$

is also a homotopy equivalence and induces an isomorphism $K_\#\colon \pi_1\big(T_f^{T_f},\mathrm{id}\big) \to \pi_1\big(T_{\bar{f}}^{T_{\bar{f}}},jj'\big)$. Viewing h as a path in the function space $T_{\bar{f}}^{T_{\bar{f}}}$, let $\tau_h\colon \pi_1\big(T_{\bar{f}}^{T_{\bar{f}}},jj'\big) \to \pi_1\big(T_{\bar{f}}^{T_{\bar{f}}},\mathrm{id}\big)$ denote the transfer of base-point isomorphism. Letting P_f (respectively $P_{\bar{f}}$) denote the rotation morphism for f (respectively $\bar{f}$), note that $P_{\bar{f}} \circ \tau_h \circ K_\#$ equals P_f, as the trace of h is constant. Hence the images of P_f and $P_{\bar{f}}$ coincide. □

Taking a cyclic homotopy $T_f \times I \to T_f$ that represents an element of $\pi_1((T_f)^{T_f},\mathrm{id})$ and lifting it to a homotopy $\mathrm{Tel}_f \times I \to \mathrm{Tel}_f$ that begins in the identity one gets

Lemma 5.7. *Let f be a self-map on the connected space Z. There exists an element in $\pi_1((T_f)^{T_f},\mathrm{id})$ of rotation degree q if and only if there exists an equivariant homotopy $\mathrm{Tel}_f \times I \to \mathrm{Tel}_f$ between the identity and the covering translation τ_q.* □

Now we make a precise statement about what can be deduced from the proof of Theorem 6.3 of [5] in the noncompact case. We apply it in Corollary 5.9.

Proposition 5.8. *Let f be a self-map on the connected space Z with $RI(f) = q < \infty$. Then for any compact subset L of Z there exist a nonnegative number r, a homotopy $N\colon L \times I \to Z$ between $f^r|_L$ and $f^{r+q}|_L$, and a homotopy $J\colon L \times I \times I \to Z$ relative to $L \times \partial I$ such that $J(z,s,0) = f(N(z,s))$, $J(z,s,1) = N(f(z),s)$ for all $z \in L$. If we assume r minimal with this property, then for $L' \supset L$ together with corresponding r', N' and J' we have $N'|_L = f^{r'-r} \circ N$ and $J'|_L = f^{r'-r} \circ J$.* □

Corollary 5.9. *Let $f\colon (Z,z_0) \to (Z,z_0)$ be a self-map on the connected space Z of CW type. If $RI(f)$ is finite, then f satisfies the property involving equation (†) of Proposition 5.2. Consequently if f is also a homotopy p-idempotent, $f \simeq f^{p+1}$, then its 'canonical' factorization through the telescope Tel_f^+ is splitting, and the restrictions ($\star$) in Proposition 5.2 are isomorphisms.*

Proof. Taking $L = \{z_0\}$ the corresponding homotopy $N = N_0$ is a loop whose class in $\pi_1(Z, z_0)$ we denote by η. The homotopy $J = J_0$ yields $f_\#(\eta) = \eta$. For any L the trace of N_L is a loop of the form $f^l \circ N_0$, and its class in $\pi_1(Z, z_0)$ equals η.

Let $g\colon (S^k, *) \to (Z, z_0)$ represent $\gamma \in \pi_k(Z, z_0)$. Its image $\mathrm{im} g =: L$ is compact, and applying the proposition we get r and the homotopies N and J. Now $N \circ g$ is a homotopy between $f^r \circ g$ and $f^{r+q} \circ g$ with trace $f^l \circ N_0$. Taking classes we get (†) of Proposition 5.2 (note that we may always enlarge r by composing with f).

The remaining assertions are immediate consequences of Proposition 5.2. □

Lemma 5.10. *Let f be a self-map on the connected space Z with $RI(f) = q$. Then on* Tel_f*, the iterate $\bar{f}^q$ is homotopic to the identity.*

Proof. By Lemma 5.7 there exists an equivariant homotopy on Tel_f between id and τ_q. Composing with $\bar{f}^q$ we get a homotopy between $\bar{f}^q$ and $\tau_q \circ \bar{f}^q$ while the latter is homotopic to the identity via the equivariant lifting $\bar{R}_q$ of the q-th tumble. □

Lemma 5.11. *Let f be a self-map on the connected space Z. If $RI(f)$ is finite, then so is $RI(f^r)$, for all r, and $RI(f^r) \leqslant LCM(r, RI(f))/r$.*

Proof. Let $l = LCM(r, q)$ be the least common multiple of r and q and set $q' = l/r$. If $RI(f) = q$, there exist an element of $\pi_1((T_f)^{T_f}, \mathrm{id})$ of rotation degree l and, by Lemma 5.7, an equivariant homotopy $\bar{F}\colon \mathrm{Tel}_f \times I \to \mathrm{Tel}_f$ between the identity and τ_l. The relations $\alpha\tau_1 = \tau_r\alpha$ and $\beta\tau_r = \tau_1\beta$ render $\beta \circ \bar{F}(\alpha(\_), \_)$ an equivariant homotopy between $\beta\alpha$ and $\beta\tau_l\alpha = \tau_{q'}\beta\alpha$. As $\beta\alpha$ is equivariantly homotopic to the identity, this accounts for an equivariant homotopy between the identity and $\tau_{q'}$ on Tel_{f^r}. A reapplication of Lemma 5.7 yields $RI(f^r)$ finite and a divisor of q'. □

Proof of Theorem 1.1 *and Corollary* 1.2. Let $f^r \simeq f^{r+p}$. Then f^r is a homotopy p-idempotent. If f^r splits, then on homotopy groups, $\mathrm{im} f^r_* = \mathrm{im} f^{r+1}_* = \cdots$, by Theorems 2.3 and 2.1. We can use this in place of (†) in the proof of Proposition 5.2 to infer that for $d_r\colon Z \to Tel_{f^r}$ the restrictions $d_{r*}|_{\mathrm{im} f^r_*}\colon \mathrm{im} f^r_* \to \pi_k(\mathrm{Tel}_{f^r})$ are isomorphisms. The composite $Z \xrightarrow{d} \mathrm{Tel}_f \xrightarrow{\beta} \mathrm{Tel}_{f^r}$ equals $Z \xrightarrow{d_r} \mathrm{Tel}_{f^r}$, and as β is a homotopy equivalence, the restrictions $d_*|_{\mathrm{im} f^r_*}\colon \mathrm{im} f^r_* \to \pi_k(\mathrm{Tel}_f)$ are isomorphisms as well. Lemma 5.6 now yields the equality $RI(f) = RI(\bar{f})$, where $\bar{f}\colon \mathrm{Tel}_f \to \mathrm{Tel}_f$ is the induced map on the telescope of f.

Corollary 2.4, (**i**), and Theorem 2.1 imply that f^{rp} is a split idempotent and thus by Proposition 3.1 of Geoghegan and Nicas [6], $RI(f^{rp}) = 1$. By Lemma 5.10, the map $\overline{f^{rp}}\colon \mathrm{Tel}_{f^{rp}} \to \mathrm{Tel}_{f^{rp}}$ is homotopic to the identity. But $\overline{f^{rp}} = \bar{f}^{rp}$, where $\bar{f}$ is the induced self-map on $\mathrm{Tel}_{f^{rp}}$. As $\mathrm{Tel}_f \xrightarrow{\bar{f}} \mathrm{Tel}_f \xrightarrow{\beta} \mathrm{Tel}_{f^{rp}}$ equals $\mathrm{Tel}_f \xrightarrow{\beta} \mathrm{Tel}_{f^{rp}} \xrightarrow{\bar{f}} \mathrm{Tel}_{f^{rp}}$, the iterate $\bar{f}^{rp}$ on Tel_f is homotopic to the identity.

A conjunction of Theorem 2.1 and the equivalence (**i**) $\Longleftrightarrow$ (**ii**) of Theorem 2.3 also renders f^{r+1} a split p-idempotent. Repeating the above argument we infer that on Tel_f, the iterate $\bar{f}^{(r+1)p}$ is homotopic to the identity.

Therefore $\bar{f}^p$ is homotopic to the identity on Tel_f. By Theorem 2.4 of Geoghegan and Nicas [6], $RI(\bar{f})$ is finite and divides p^2. As $RI(f) = RI(\bar{f})$ we are done.

Conversely, assume that $RI(f)$ is finite. Then by Lemma 5.11 $RI(f^r)$ is also finite, rendering f^r a split p-idempotent by Corollary 5.9. By what we have shown above it follows that $RI(f)$ divides p^2.

The theorem, Lemma 5.11, and Theorems 2.3 and 2.1 imply the corollary. □

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