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Homotopy type of mapping spaces and existence of geometric exponents

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Abstract. Let Y be a simply connected finite complex and let p be a prime. Let $S^m[p^{-1}]$ denote the complex obtained from the m-sphere by inverting p. It is shown in this paper that Y has an eventual H-space exponent at p if and only if the space $\mathrm{map}_*(S^m[p^{-1}], Y)$ of pointed maps $S^m[p^{-1}] \to Y$ has the homotopy type of a CW complex for some (and hence all big enough) m. This makes it possible to interpret the question of eventual H-space exponents in terms of phantom phenomena of mapping spaces.

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Let $S^2[p^{-1}]$ denote the CW complex obtained from the 2-sphere S^2 by inverting a single prime p. Let Y be a simply connected finite CW complex. It turns out that the space $\mathrm{map}_*(S^2[p^{-1}], Y)$ of pointed continuous maps $S^2[p^{-1}] \to Y$ 'contains' all information about exponents of Y. Namely, the 'algebra' of that space carries in some sense the information about a possible homotopy exponent for Y at p, i.e. a possible common exponent for the p-torsion in homotopy groups of Y. What is meant by algebra here is the information that can be deduced from, for example, the corresponding simplicial mapping space formed from $S^2[p^{-1}]$ and Y considered as simplicial sets.

What is perhaps surprising, though, is that the 'topology' of $\mathrm{map}_*(S^2[p^{-1}], Y)$ governs the existence of an eventual H-space (also called geometric) exponent at p. Precisely, it is shown in this paper that Y has an eventual geometric exponent at p if and only if the topological space $\mathrm{map}_*(S^m[p^{-1}], Y) = \Omega^{m-2}\,\mathrm{map}_*(S^2[p^{-1}], Y)$ is homotopy equivalent to a CW complex for all big enough m.

As parallel products we exhibit rather non-trivial examples of CW complexes X and Y where X is infinite and $\mathrm{map}(X, Y)$ either has or does not have the homotopy type of a CW complex. By Theorem 3 of Milnor [22] (1959), $\mathrm{map}(K, Y)$ has the homotopy type of a CW complex if K is finite. Very little has been done in this direction since, the only published results being those of Kahn [15] and the author [28]. The examples given here are quite different, and they depend on the existing results on geometric exponents. The problem

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of CW homotopy type of the space of continuous functions between two CW complexes was the author's motivating problem. Investigating it, the problem of geometric exponents occurred naturally.

1 Introduction

A simple space Z is said to have a homotopy exponent at the prime p if there exists k such that p^k annihilates the p-primary component of $\pi_l(Z)$ for all l.

If Z is a homotopy associative H-space then Z is said to have an H-space (or *geometric*) exponent if there exists a number b such that the map $b\colon Z \to Z$, sending z to z^b, is nullhomotopic. Note that if Z has H-space exponent b, then also ΩZ has H-space exponent b. The space Z is said to have an *eventual H-space exponent* if the iterated loop spaces $\Omega^m Z$ have H-space exponents for all big enough m.

We refer the reader to the survey articles of Neisendorfer and Selick [24], and Cohen [5] for an introduction to (geometric) exponents and closely related problems in homotopy theory.

A simply connected finite complex Y is *elliptic* if $\pi_k(Y)$ is torsion for all but finitely many k. Otherwise it is *hyperbolic*. See Félix, Halperin, Thomas [9] for explanations of this terminology deriving from rational homotopy theory.

A famous conjecture due to John C. Moore asserts a deep relation between homotopy exponents, eventual H-space exponents, and ellipticity. It naturally splits in two separate conjectures (called 'S1' and 'S2' in Selick [26]). The following is one of them. We call it the 'geometric Moore conjecture.'

Geometric Moore conjecture. *Let $Y_{(p)}$ denote the localization of the simply connected finite complex Y at the prime p. The following are equivalent.*

(i) *The complex $Y_{(p)}$ admits an eventual H-space exponent.*
(ii) *The complex Y is elliptic.*

The other conjecture states that if Y is hyperbolic, it does not admit a homotopy exponent at p for any prime p.

The following is our principal result.

Theorem 1.1. *Let Y be a simply connected finite complex and let p be a prime. Let $Y_{(p)}$ denote the localization of Y at p and let $S^m[p^{-1}]$ denote the localization of the m-sphere away from the prime p. Furthermore, let $K(\mathbb{Z}[p^{-1}], m)$ denote the Eilenberg-MacLane space with the single nonvanishing homotopy group in dimension m isomorphic with $\mathbb{Z}[p^{-1}]$. The following are equivalent:*

(i) *The space $\mathrm{map}_*(S^m[p^{-1}], Y)$ has CW homotopy type for all big enough m.*
(ii) *The space $\mathrm{map}_*(S^m[p^{-1}], Y_{(p)})$ is contractible for all big enough m.*
(iii) *The space $\Omega^m Y_{(p)}$ admits an H-space exponent for all big enough m.*
(iv) *The space $\mathrm{map}_*\left(K(\mathbb{Z}[p^{-1}], m), Y_{(p)}\right)$ is contractible for all big enough m.*

Conventions. All topological spaces will be Hausdorff. The terms *map* and *continuous function* will be used synonymously. If X and Y are spaces, $\mathrm{map}(X, Y)$ denotes the space of maps $X \to Y$ equipped with the compact open topology. A fibration (traditionally called *Hurewicz fibration*) is a map with the homotopy lifting property for all spaces. Dually, a cofibration is a map (whose image is closed) with the homotopy extension property for all spaces. A homotopy equivalence is a continuous map which admits a homotopy inverse. Note that this actually defines a closed model category structure on the category of all topological spaces (see Strøm [33] for details). We use also *genuine homotopy equivalence* to stress the difference from *weak homotopy equivalence* which is a map inducing isomorphisms on all homotopy groups. Spaces Z and W are homotopy equivalent if $\mathrm{map}(Z, W)$ contains a homotopy equivalence. We use $\simeq$ for homotopic maps or homotopy equivalent spaces and $\approx$ (or simply $=$) for homeomorphic spaces. The space Z has CW homotopy type if $Z \simeq W$ for some CW complex W. If $Z \simeq \{*\}$ then Z is called contractible.

Let P be a set of primes, and Y a (nilpotent) CW complex. We denote by $Y_{(P)}$ and $Y[P^{-1}]$, respectively, the localization of Y at P and at the complement of P (also called *away* from P). In case of a single prime $P = \{p\}$, we use $Y_{(p)}$ and $Y[p^{-1}]$. Analogously for groups.

For the purpose of this paper, localizations are defined only up to homotopy type. We outline a specific model. In the case of a sphere S^m, we may take for $S^m[P^{-1}]$ the mapping telescope of a sequence of self-maps $S^m \to S^m \to S^m \to \cdots$. The degrees of the maps have to belong to P and each member of P has to occur infinitely many times. If Y is a countable CW complex whose 1-skeleton is a point, then $Y[P^{-1}]$ is constructed by inductively 'localizing' the cells of Y, i.e. by replacing cones of ordinary spheres S^m with cones of localized spheres $S^m[P^{-1}]$. The two constructions will be carried out explicitly at the beginning of §4 and in the proof of Theorem 5.5, respectively. The reader is referred to Sullivan [34] as well as Hilton, Mislin, Roitberg [12] for more details on localization with respect to a set of primes.

We topologize $\mathrm{map}\big((X,A),(Y,M)\big) = \{f \in \mathrm{map}(X,Y) \,|\, f(A) \subset M\}$ as subspace of $\mathrm{map}(X, Y)$. Taking $A = *, M = *$ yields the space of pointed maps denoted by $\mathrm{map}_*(X, Y)$. Suppressing base-points whenever irrelevant, ΣX and ΩY denote, respectively, the reduced suspension of X and the space of pointed loops on Y, i.e. ΣX is the smash product $S^1 \wedge X$ where S^1 is the unit circle in $\mathbb{C}$ and $\Omega Y = \mathrm{map}_*(S^1, Y)$. Note that whenever X is a pointed CW complex, ΣX has a canonical CW structure.

Organization. In Section 2 we present the mapping space with domain a countable CW complex as an inverse limit of a tower of fibrations, and give a brief discussion of invariance of our results for the choice of topology (topological category).

In Section 3, we study the question of CW homotopy type of an abstract inverse limit space. We apply the results to function spaces in Section 4 where we prove the equivalence of (i) – (iii) in Theorem 1.1 as a consequence of Theorem 4.4 (in conjunction with Lemma 4.3).

The equivalence of (ii) and (iv) in Theorem 1.1 is of a different nature, and is contained in Section 5. It relates the problem of eventual geometric exponents to the genuine homotopy type of the spaces $\mathrm{map}_*\left(K(\mathbb{Z}[p^{-1}], m), Y_{(p)}\right)$. Even their *weak* homotopy type was not understood until Zabrodsky's enhancements of Miller's proof of the Sullivan conjecture (see Zabrodsky [35] as well as Roitberg [25]).

In Section 6 we represent our results in light of ‘phantom phenomena’. In Appendix A we prove a handful of technical results concerning the compact open topology.

The theorem of Milnor cited above (Theorem 3 of [22]) will be referred to as *Milnor's theorem*.

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2 The topology and homotopy type of a mapping space

2.1 Inverse limit representation and the loop space of a mapping space

Let X be a countable CW complex and let Y be any space. Let $L_1 \leqslant L_2 \leqslant \cdots$ be an increasing filtration of subcomplexes of X (always understood to satisfy $X = \bigcup_i L_i$). Set $Z_i = \mathrm{map}(L_i, Y)$ and let $p_i\colon Z_i \to Z_{i-1}$ be the restriction. That is, p_i assigns to each map $L_i \to Y$ its restriction to L_{i-1}. The inclusion $L_{i-1} \hookrightarrow L_i$ is a cofibration, hence the restriction p_i is a fibration (this is a byproduct of Lemma A.2 below).

Hence $\{(Z_i, p_i)\}$ is an inverse sequence of fibrations (also called a tower of fibrations). Let Z denote the inverse limit space. This is the set of sequences $\{\zeta_i \in Z_i\}$ such that $p_i(\zeta_i) = \zeta_{i-1}$ for each i, topologized as a subspace of the cartesian product $\prod_{i\in\mathbb{N}} Z_i$. If we denote the canonical projections $P^i\colon Z \to Z_i$ then a basis for the topology of Z is given by all sets of the form $(P^i)^{-1}(U_i)$ where U_i is open in Z_i.

The subcomplexes L_i dominate the compact subsets of the compactly generated space X which easily implies that Z is homeomorphic with $\mathrm{map}(X, Y)$. Similarly, if $x_0 \in L_1$ and we let $Z_i' = \mathrm{map}\big((L_i, x_0), (Y, *)\big) = \mathrm{map}_*(L_i, Y)$, then $\mathrm{map}_*(X, Y)$ is the inverse limit of Z_i'.

If the subcomplexes L_i are *finite* then the spaces Z_i have CW homotopy type by Milnor's theorem. By Corollary 2.2, so have the Z_i'. In this case the mapping spaces under investigation are inverse limits of towers of fibrations between spaces of CW homotopy type.

Let $q\colon E \to X$ be a fibration. Set $E_i := q^{-1}(L_i)$ (the L_i need not be finite) and $W_i = \mathrm{map}(E_i, Y)$. Note that the collection $\{E_i\}$ dominates compact subsets of E. Since L_{i-1} is cofibered in L_i, also E_{i-1} is cofibered in E_i. (See Strøm [32], Theorem 12.) If E is compactly generated then $\mathrm{map}(E, Y)$ is homeomorphic with the inverse limit space of $\{W_i\}$. In fact the induced map $q^*\colon\ \mathrm{map}(X, Y) \to \mathrm{map}(E, Y)$ corresponds to the limit of the maps $q_i^*\colon Z_i \to W_i$.

2.2 A theorem of Stasheff and some basic corollaries

We state a theorem of Stasheff (crucial for the next section), and some corollaries. One is the fact that for a connected target space, the space of free maps has CW type if and only if the space of pointed maps has. Another is that (i) and (ii) of Theorem 1.1 are *eventual notions*; if $\mathrm{map}_*(X, Y)$ has CW type then so have $\mathrm{map}_*(\Sigma^m X, Y)$ for all m.

Theorem 2.1 (Stasheff [30], Propositions (0) and (12)). *Let $p : E \to B$ be a fibration where the base space B has the homotopy type of a CW complex. Then E has the homotopy type of a CW complex if and only if all fibers of p have.* □

Corollary 2.2. *Let X and Y be CW complexes and let x_0 be a base-point in X. Then* $\mathrm{map}(X, Y)$ *has CW homotopy type if and only if* $\mathrm{map}\big((X, x_0), (Y, y)\big)$ *has CW type for one point y in each path component of Y. Consequently if Y is connected this is if and only if the space of pointed maps* $\mathrm{map}_*(X, Y)$ *has CW type.*

Proof. As x_0 is nondegenerate in X, evaluation $Y^X \to Y$ at x_0 is a fibration. □

Lemma 2.3. *Let X, Y be pointed CW complexes, and A a subcomplex of X.*
(i) $\mathrm{map}_*(X/A, Y)$ *and* $\mathrm{map}((X, A), (Y, *))$ *are naturally homeomorphic.*
(ii) $\Omega\big(\mathrm{map}_*(X, Y), *\big)$*,* $\mathrm{map}_*(\Sigma X, Y)$*, and* $\mathrm{map}_*(X, \Omega Y)$ *are homeomorphic.*

Proof. As for (i), the bijection $Q\colon \mathrm{map}_*(X/A, Y) \to \mathrm{map}((X, A), (Y, *))$ is induced by $q\colon X \to X/A$ and as such continuous. Note that for every compactum $L \subset X/A$ there exists a compactum $K \subset X$ with $q(K) = L$. This implies that Q is open.

For (ii), the exponential law (see A.1) implies that both $\Omega\big(\mathrm{map}_*(X, Y), *\big)$ and $\mathrm{map}_*(X, \Omega Y)$ are homeomorphic with $\mathrm{map}((X \times S^1, X \vee S^1), (Y, *))$. The latter is homeomorphic with $\mathrm{map}_*(\Sigma X, Y)$ by (i). □

Corollary 2.4. (i) *For any z_0, the space $\Omega(Z, z_0)$ has CW type if Z has.*
(ii) *In particular, if* $\mathrm{map}_*(X, Y)$ *has CW type then so has* $\mathrm{map}_*(X, \Omega Y)$.

Proof. By Milnor's theorem, the space of free loops $\mathrm{map}(S^1, Z)$ has CW type if Z has. Evaluation $\mathrm{map}(S^1, Z) \to Z$ at $1 \in S^1$ is a fibration, hence its fiber over z_0 has CW type by Theorem 2.1. This proves (i).

Taking $Z = \mathrm{map}_*(X, Y)$ and $z_0 = \mathrm{const}$, (ii) follows from (i) by Lemma 2.3. □

2.3 Homotopy invariance and choice of category

By homotopy invariance we mean the following result. (For a proof see Maunder [16], Theorem 6.2.25.)

Lemma 2.5. *Let $\varphi\colon X' \to X$ and $\psi\colon Y \to Y'$ be homotopy equivalences. The function* $\mathrm{map}(X, Y) \to \mathrm{map}(X', Y')$ *defined by $f \mapsto \psi \circ f \circ \varphi$ is also a homotopy equivalence. Analogously for spaces of pointed maps.* □

There is a 'competing' topology for $\mathrm{map}(X, Y)$, namely $\mathbf{k}(\mathrm{map}(X, Y))$ where $\mathbf{k}$ denotes the compactly generated refinement (see Steenrod [31]). Along with it comes the competing choice of the 'convenient topological category'. (Since our spaces are Hausdorff, compactly generated and weak-Hausdorff $\mathbf{k}$-spaces coincide.)

Proposition 2.6. (i) $\mathbf{k}$ *takes homotopy equivalences to homotopy equivalences.*
(ii) *If a space Z is homotopy equivalent to a compactly generated Hausdorff space, then the natural map $\mathbf{k}(Z) \to Z$ is a homotopy equivalence.*
(iii) *If X has the homotopy type of a countable CW complex and Y of a metrizable space then* $\mathrm{map}(X,Y)$ *and* $\mathbf{k}(\mathrm{map}(X,Y))$ *are homotopy equivalent.*

Proof. (i) is an easy consequence of the fact $\mathbf{k}(Z \times I) = \mathbf{k}(Z) \times I$ since the unit segment I is compact. Then (ii) is a consequence of (i). To prove (iii), use Lemma 2.5 and adapt the proof of (ii) of Lemma 3.4 of [28]. □

As every CW complex has the type of a metric simplicial complex, (iii) of 2.6 shows that our Theorem 1.1 (and other homotopy theoretic results) do not depend on the choice of topological category. The essence lies in the *genuine homotopy type* of the mapping spaces involved. The definition of the convenient topological category (see [31]) makes it clear that for low-level properties, most of the work has to be done in the classical category Top. Here, choosing either of the two would have some merit, but it seems that the classical category is slightly more convenient.

3 Homotopy type of inverse limits

In what follows, we fix an inverse sequence of fibrations

(1) $$\cdots \to Z_3 \xrightarrow{p_3} Z_2 \xrightarrow{p_2} Z_1.$$

Let Z denote the inverse limit space and pick a point $\zeta = \{\zeta_i\} \in Z$. The following basic result gives a description of the homotopy groups $\pi_k(Z,\zeta)$. For a proof, see for example Mardešić and Segal [17], Theorem 1 on page 178.

Proposition 3.1. *For each number $k \geqslant 0$ there exists a natural exact sequence (of pointed sets for $k = 0$ and of groups otherwise)*

(2) $$* \to \lim{}^1 \pi_{k+1}(Z_i,\zeta_i) \xrightarrow{\phi} \pi_k(Z,\zeta) \xrightarrow{\lim \pi_k(P^i)} \lim \pi_k(Z_i,\zeta_i) \to *.$$

In addition, ϕ is an injection even for $k = 0$.

Here $\lim^1$ denotes the first right derived functor of the inverse limit functor. When $k = 0$ in (2) it takes values in the category of sets (it was defined in this generality by Bousfield and Kan [1]). We will be most interested in its vanishing. For this we recall a definition.

Definition. Let $\{(G_i, p_i\colon G_i \to G_{i-1})\}$ be an inverse sequence of groups. For $i < j$ we let p_{ij} denote the composite $p_{i+1}\ldots p_{j-1}p_j$. The sequence $\{G_i\}$ satisfies the *Mittag-Leffler condition* if for each i_0 there is $i \geqslant i_0$ so that for each $j \geqslant i$ the images of $p_{i_0 i}$ and $p_{i_0 j}$ coincide. In this case we call the image of $p_{i_0 i}$ *stable* in G_{i_0}.

Lemma 3.2. *If* $\{G_i\}$ *satisfies the Mittag-Leffler condition,* $\lim^1 G_i$ *is trivial. If the* G_i *are countable, the converse holds as well.*

Proof. See [1], Corollary 3.5, as well as [17], Theorems 10 and 11 on pages 173 and 174. ☐

Let C_i denote the path component of ζ_i in Z_i and let C denote the path component of ζ in the limit space Z. Denoting canonical projections $P^i\colon Z \to Z_i$, note that the restrictions $P^i|_C\colon C \to C_i$ are also fibrations. Let F_i denote the fiber of $C \to C_i$ over $\zeta_i \in C_i$. Explicitly, F_i consists of all the sequences $\eta = \{\eta_j \mid j\}$ for which $\eta \in C$ and $\eta_j = \zeta_j$ whenever $j \leqslant i$. Obviously, therefore, $F_j \subset F_i$ for $j \geqslant i$.

A space X is called *semilocally contractible* if each point of x has a neighborhood U such that the inclusion $U \hookrightarrow X$ is nullhomotopic. Clearly, the endpoint of the homotopy can be assumed to equal x.

Note that semilocal contractibility is a homotopy invariant. In particular, since each CW complex is homotopy equivalent to a metric simplicial complex (see Milnor [22]), each CW complex is semilocally contractible. (In fact CW complexes are locally contractible in a strong sense. See for example [10], Theorem 1.3.2.)

Note also that a space has the homotopy type of a CW complex if and only if its path components are open and each has the homotopy type of a CW complex.

Theorem 3.3. *Assume that all the* Z_i *have CW type. If also* C *has CW type then for each* i_0 *there exists* $i \geqslant i_0$ *such that the inclusions* $F_j \hookrightarrow F_{i_0}$ *(and hence the inclusions* $F_j \hookrightarrow C$*) are nullhomotopic for all* $j \geqslant i$*. If, in addition,* C *is open,* F_j *may be assumed to equal the fiber of* $Z \to Z_j$ *over* ζ_j *for* $j \geqslant i$*.*

Proof. Note first that C_{i_0} necessarily has CW type. Hence by Theorem 2.1, the fiber F_{i_0} has CW homotopy type as well. In particular, it is semilocally contractible. Therefore a basic neighborhood of ζ in F_{i_0} contracts in F_{i_0}. The neighborhood contains one of the form $(P^i)^{-1}(U_i) \cap F_{i_0}$ for some $i \geqslant i_0$. The latter contains all fibers F_j for $j \geqslant i$, which finishes the proof of the first statement.

If C is open, it contains a basic neighborhood of ζ in Z, say $(P^{i_1})^{-1}(V_{i_1})$. The latter contains $(P^{i_1})^{-1}(\zeta_{i_1})$. Now replace i with $\max\{i, i_1\}$. ☐

Corollary 3.4. *Assume the hypotheses of Theorem 3.3. Then*

(i) *There exists* $i \geqslant 1$ *so that the morphisms* $\pi_k(Z,\zeta) \to \pi_k(Z_j,\zeta_j)$ *are injective for* $j \geqslant i$ *and* $k \geqslant 1$*. If* C *is also open, then* $(P^j)^{-1}(C_j) = C$ *for* $j \geqslant i$*.*

(ii) *For each* i_0 *there exists* $i \geqslant i_0$ *such that the images*

$$\pi_k(Z_j,\zeta_j) \to \pi_k(Z_{i_0},\zeta_{i_0})$$

are the same for all $j \geqslant i$ *and all* $k \geqslant 1$*. In particular, the morphisms* $\pi_k(Z,\zeta) \to \lim_j \pi_k(Z_j,\zeta_j)$ *are bijective and* C *is isomorphic with the inverse limit of* $\{C_j\}$*. If, in addition,* Z *has CW type then also the function* $\pi_0(Z) \to \lim_j \pi_0(Z_j)$ *is bijective.*

Proof. (i) follows from the theorem and the long homotopy exact sequence of the fibration $F_j \to C \to C_j$ (respectively $F_j \to Z \to Z_j$ in case C is also open).

(ii) follows from the theorem by using naturality of the long homotopy exact sequence of a fibration on the following morphism of fibrations:

$$\begin{array}{ccccc} F_j & \longrightarrow & C & \longrightarrow & C_j \\ \downarrow & & \| & & \downarrow \\ F_{i_0} & \longrightarrow & C & \longrightarrow & C_{i_0} \end{array} \tag{3}$$

A diagram chase shows that the images $\pi_k(Z_j, \zeta_j) \to \pi_k(Z_{i_0}, \zeta_{i_0})$ coincide for $j \geqslant i$.

This means that the sequences $\{\pi_k(Z_j, \zeta_j) \mid j\}$ satisfy the Mittag-Leffler condition, and consequently the $\lim^1$ term in the short exact sequence of Proposition 3.1 vanishes by Lemma 3.2. Thus the morphisms $\pi_k(Z, \zeta) \to \lim_j \pi_k(Z_j, \zeta_j)$ are bijective for $k \geqslant 1$. The inverse limit of $\{C_i\}$ could possibly contain other path-components of Z. That is not the case, by exactness of (2) for $k = 0$.

If Z has CW homotopy type then all its path-components have CW type. Using exactness of (2) for all possible choices of base-points implies injectivity of $\pi_0(Z) \to \lim_j \pi_0(Z_j)$. This completes the proof. □

Corollary 3.5. *Assume the hypotheses of Theorem 3.3. If $\Omega(Z, \zeta) = \Omega(C, \zeta)$ is contractible then for each i_0 there exists $j \geqslant i_0$ such that the map*

$$\Omega(Z_j, \zeta_j) \to \Omega(Z_{i_0}, \zeta_{i_0})$$

is nullhomotopic.

Proof. As $\Omega(C, \zeta)$ is contractible, C is weakly contractible. Since C has CW type by assumption, it must be contractible. Theorem 3.3 yields $j \geqslant i_0$ such that F_j contracts in F_{i_0}. Diagram (3) depicts a morphism of fibrations which induces a morphism of the associated Puppe sequences. Since C is contractible, the connecting maps $\Omega(C_j, \zeta_j) \to F_j$ and $\Omega(C_{i_0}, \zeta_{i_0}) \to F_{i_0}$ are homotopy equivalences. By naturality of the Puppe sequence, it follows that the restriction fibration $\Omega(C_j, \zeta_j) \to \Omega(C_{i_0}, \zeta_{i_0})$ is homotopy equivalent to the map $F_j \to F_{i_0}$. □

The next result is due to Edwards and Hastings (see Geoghegan [11] for details).

Proposition 3.6. *Assume given a commutative diagram of topological spaces*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & W_3 & \xrightarrow{p_3} & W_2 & \xrightarrow{p_2} & W_1 \\ & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ \cdots & \longrightarrow & Z_3 & \xrightarrow{r_3} & Z_2 & \xrightarrow{r_2} & Z_1 \end{array}$$

where the p_i and r_i are fibrations and the maps f_i are homotopy equivalences. Then the induced inverse limit map $f_\infty\colon W_\infty \to Z_\infty$ is also a homotopy equivalence. □

Corollary 3.7. *Suppose $\cdots \to Z_3 \to Z_2 \to Z_1$ is an inverse sequence. If the maps are both fibrations and homotopy equivalences, so are the projections $Z_\infty \to Z_i$.* □

Corollary 3.8. *In order to conclude that W_∞ and Z_∞ are homotopy equivalent it is enough to assume commutativity only up to homotopy in Proposition 3.6.*

Proof. Construct commutative squares inductively by lifting homotopies. □

The following proposition generalizes Lemma 2.8 of Kahn [15], and is crucial for understanding the problem of CW homotopy type of inverse limits.

Proposition 3.9. *Let $\cdots \to Z_3 \xrightarrow{r_3} Z_2 \xrightarrow{r_2} Z_1$ be an inverse sequence of fibrations and let Z_∞ denote the inverse limit space. If each fibration $r_{i+1}\colon Z_{i+1} \to Z_i$ is homotopic to a constant map, then the limit space Z_∞ is contractible.*

Proof. Let $f_1\colon W_1 = Z_1 \to Z_1$ be the identity, and assume that for some $i \geqslant 1$, a homotopy equivalence $f_i\colon W_i \to Z_i$ has been constructed. Let $r_{i+1}\colon Z_{i+1} \to Z_i$ be homotopic to the constant $\zeta_i \in Z_i$. Let w_i be a point in W_i that is mapped by f_i into the path component of ζ_i. Let PW_i denote the space of paths in W_i that start in w_i and set $W_{i+1} = Z_{i+1} \times PW_i$. Let $p_{i+1}\colon W_{i+1} \to W_i$ be the composite $Z_{i+1} \times PW_i \xrightarrow{\mathrm{pr}} PW_i \xrightarrow{\varepsilon} W_i$ where pr is the obvious projection and ε denotes the evaluation at end point. As a composite of fibrations, p_{i+1} is a fibration. Let $f_{i+1}\colon W_{i+1} \to Z_{i+1}$ be the projection $Z_{i+1} \times PW_i \to Z_{i+1}$. Clearly, f_{i+1} is a homotopy equivalence. The composite $r_{i+1} \circ f_{i+1}$ is homotopic to const_{ζ_i}, and the composite $f_i \circ p_{i+1}$ is homotopic to $\mathrm{const}_{f_i(w_i)}$. The two constant maps are homotopic by choice of w_i. We proceed inductively, and appeal to Corollary 3.8 to conclude that Z_∞ is homotopy equivalent to $W_\infty = \lim W_i$. The latter is homeomorphic to the limit space of the sequence $\cdots \to PW_{i+1} \to PW_i \to \cdots \to PW_1$ where $PW_{i+1} \to PW_i$ are fibrations. By Corollary 3.7, the limit space is contractible. □

4 Spaces of maps out of localized spheres

In this section, let R be a set of primes (possibly empty), and let R' denote the complement of R. A group G is said to be R-local if the morphism $g \mapsto g^p$ is bijective for each $p \in R'$. If G is abelian, this is tantamount to saying that G admits a $\mathbb{Z}_{(R)}$-module structure where $\mathbb{Z}_{(R)}$ is the subring of the rationals consisting of fractions $\frac{a}{b}$ where the prime divisors of b belong to R'.

A nilpotent CW complex T is called R-local if its homotopy groups are R-local (at all possible base-points if T is not connected). The reader uncomfortable with nilpotent spaces may safely assume that T is a simply connected CW complex or an iterated loop space of such a space. A consequence of the theory of Serre classes is the fact that T is R-local if and only if its homology groups are R-local. (See [12], page 72.)

Finally, a map $l\colon T \to T'$ of nilpotent CW complexes is said to *R-localize* if T' is R-local and for every R-local space Y every map $T \to Y$ factors, up to homotopy, uniquely through $l\colon T \to T'$. In this case, either the map l or just the complex T' is called an R-localization of T. It turns out that $l\colon T \to T'$ is an R-localization if and only if it R-localizes homology groups.

Every nilpotent CW complex T admits an R-localization which is unique up to homotopy type and is denoted by $T_{(R)}$. By a slight abuse of notation, $T_{(R)}$ also stands for any specific CW complex within the homotopy type. There are ways of making R-localization functorial but we will not need that.

Let P be a nonempty set of primes and let $S^m[P^{-1}]$ denote the localization of the m-sphere away from the set P.

We construct $S^m[P^{-1}]$ as the reduced infinite mapping telescope of the sequence

$$S^m \xrightarrow{p_1} S^m \xrightarrow{p_2} S^m \xrightarrow{p_3} \cdots$$

where the p_i belong to P, and each prime in P occurs infinitely many times.

More precisely, we assume $S^m[P^{-1}]$ filtered by a based sequence $L_0 \leqslant L_1 \leqslant L_2 \leqslant \cdots$ of finite subcomplexes each of which is homotopy equivalent to S^m, and for all i, the inclusion $L_{i-1} \leqslant L_i$ corresponds to coH-group multiplication by p_i on the sphere S^m. The inclusion $S^m = L_0 \to \operatorname{colim}_i L_i = S^m[P^{-1}]$ is the localization map (this is to say the map that localizes at the complement of P).

If T is any CW complex, then the mapping space $\operatorname{map}_*(S^m[P^{-1}], T)$ can be identified with the inverse limit of the associated inverse sequence of spaces $Z_i = \operatorname{map}_*(L_i, T)$. The bonding fibration $Z_i \to Z_{i-1}$ is the restriction fibration induced by inclusion $L_{i-1} \leqslant L_i$ and is equivalent to H-group multiplication by p_i on $\operatorname{map}_*(S^m, T) = \Omega^m T$.

In particular, the morphisms $\pi_k(Z_i, *) \to \pi_k(Z_{i-1}, *)$ correspond to multiplication by p_i on $\pi_{k+m}(T)$. Here $*$ generically denotes the constant loop.

We call the above inverse limit representation of $\operatorname{map}_*(S^m[P^{-1}], T)$ the *standard representation*. It will be used (for a handful of complexes in place of T) throughout this section.

Lemma 4.1. *Assume that the complex T is R-local where $R \cap P = \emptyset$. The localization induced fibration* $\operatorname{map}_*(S^m[P^{-1}], T) \to \operatorname{map}_*(S^m, T) = \Omega^m T$ *is a homotopy equivalence. In particular,* $\operatorname{map}_*(S^m[P^{-1}], T)$ *has CW homotopy type.*

Proof. As T is R-local, the p-th power map $\Omega^m T \xrightarrow{p} \Omega^m T$ is a homotopy equivalence for any $p \in P$. (This follows from the definition by virtue of Whitehead's theorem.) This renders our standard inverse sequence for $\operatorname{map}_*(S^m[P^{-1}], T)$ one of homotopy equivalences. Now we use Corollary 3.7. $\square$

Another corollary of Stasheff's theorem is the following

Proposition 4.2. *Consider the pullback diagram*

$$\begin{tikzcd} E' \arrow[r, "\bar f"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

where p is a fibration and both B' and B have CW homotopy type.

(i) *If E has CW type then so has E'.*
(ii) *If E' has CW type and $f_*(\pi_0(B')) \supset p_*(\pi_0(E))$ then also E has CW type.*

Proof. (i) is immediate from Theorem 2.1 since for each $b' \in B'$, the fibers $p'^{-1}(b')$ and $p^{-1}(f(b'))$ coincide. The hypotheses of (ii) imply that each nonempty fiber of p is homotopy equivalent to a fiber of p'. Hence (ii) also follows from Theorem 2.1. □

We use Proposition 4.2 to prove the following lemma.

Lemma 4.3. *Let Y be any CW complex. If $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ is path connected, then $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW type if and only if $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ has.*

Proof. The complex Y is homotopy equivalent to the homotopy pullback of the diagram $Y[P^{-1}] \to Y_{(0)} \leftarrow Y_{(P)}$. Not changing the homotopy type of the spaces involved we may assume that the natural maps $Y[P^{-1}] \to Y_{(0)}$ and $Y_{(P)} \to Y_{(0)}$ are fibrations. Since $\mathrm{map}_*(X, \_)$ preserves pull-backs (see Lemma A.3), $\mathrm{map}_*(S^m[P^{-1}], Y)$ is homotopy equivalent to the pullback of

$$\mathrm{map}_*(S^m[P^{-1}], Y[P^{-1}]) \to \mathrm{map}_*(S^m[P^{-1}], Y_{(0)}) \leftarrow \mathrm{map}_*(S^m[P^{-1}], Y_{(P)}).$$

Both maps above are fibrations (see for example Lemma A.4). An application of Lemma 4.1 and Proposition 4.2 finishes the proof. □

Theorem 4.4. *Let P be a nonempty set of primes, and let Y be a simply connected finite complex.*

(i) *If $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW type then Y is elliptic, $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ is contractible, and $\Omega^{m+1} Y_{(P)}$ has an H-space exponent.*
In addition, $\tilde H_*(Y; \mathbb{Z}_p) \neq 0$ for at most finitely many $p \in P$. In particular, if $H_k(Y)$ is infinite for some $k \geqslant 1$, then P is finite.
(ii) *If $\Omega^m Y_{(P)}$ has an H-space exponent, then $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW type.*

Proof. Assume that $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW homotopy type. By (ii) of Corollary 3.4, there exists $i \geqslant 0$ such that the images of $\pi_k(Z_j, *) \to \pi_k(Z_0, *)$ are the same for all $j \geqslant i$ and all $k \geqslant 1$. Set $b = p_i p_{i-1} \dots p_2 p_1$. Then

(4) $$\forall j > i \ \forall k \geqslant 1: \ p_j p_{j-1} \dots p_{i+1}(b \cdot \pi_{k+m} Y) = b \cdot \pi_{k+m} Y.$$

This exhibits the groups $b \cdot \pi_{k+m}(Y)$, for $k \geqslant 1$, as P-divisible. As they are finitely generated, they must be finite groups with trivial P-torsion. This means that for $l \geqslant m+1$, the group $\pi_l(Y)$ is finite and b annihilates its P-primary part. In particular, Y is elliptic.

If $\tilde{H}_*(Y; \mathbb{Z}_p) \neq 0$ then by generalized Serre's theorem (McGibbon and Neisendorfer [19]), infinitely many groups $\pi_l(Y)$ contain a subgroup of order p. There can only be finitely many such p that also belong to P, namely, precisely the prime divisors of b.

We continue the proof of the first statement. By the above, $\pi_{m+k}(Y_{(P)})$ is a finite P-group bounded by b, for every $k \geqslant 1$. Hence Lemma 3.2 and Proposition 3.1 imply that for each $k \geqslant 0$, the group $\pi_k(\mathrm{map}_*(S^m[P^{-1}], Y_{(P)}), *)$ is isomorphic with the inverse limit of

$$\cdots \xrightarrow{p_3} \pi_{m+k} Y_{(P)} \xrightarrow{p_2} \pi_{m+k} Y_{(P)} \xrightarrow{p_1} \pi_{m+k} Y_{(P)}.$$

As no element of $\pi_{m+k} Y_{(P)}$ is infinitely divisible by a prime belonging to P it follows that $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ is weakly contractible.

By Lemma 4.3, the space $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ has CW homotopy type and as such is contractible. By Corollary 3.5, for some j and $\beta = p_j \ldots p_1$, the β-th power map $\Omega^{m+1} Y_{(P)} \xrightarrow{\beta} \Omega^{m+1} Y_{(P)}$ is nullhomotopic. This completes the proof of (i).

For (ii), assume that the power map $\Omega^m Y_{(P)} \xrightarrow{b} \Omega^m Y_{(P)}$ (where all the prime divisors of b belong to P) is nullhomotopic. Then clearly, a subsequence of the standard inverse sequence for $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ is one of nullhomotopic maps. Hence by Proposition 3.9, the space $\mathrm{map}_*(S^m[P^{-1}], Y_{(P)})$ is contractible, and by Lemma 4.3, the space $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW homotopy type. □

We apply Theorem 4.4 to a couple of intriguing examples. The first exploits (i) of the theorem.

Example 1. The mapping space $\mathrm{map}_*(M(\mathbb{Q}, m), S^n)$ does not have CW type for any m. In particular, for $m > n$ the space is weakly contractible but not contractible.

Since $\Omega\big(\mathrm{map}_*(M(\mathbb{Q}, m), S^n), *\big) \approx \mathrm{map}_*(M(\mathbb{Q}, m+1), S^n)$ (see Lemma 2.3), the situation does not 'improve' after looping. □

The next example follows from (i) and (ii) of the theorem by exploiting results from the theory of H-space exponents.

Example 2. For all large enough m, the mapping space $\mathrm{map}_*(S^m[p^{-1}], S^n)$ has the homotopy type of a CW complex (and is homotopy equivalent to $\Omega^m S^n[p^{-1}]$).

This follows from the fact that $\Omega^l S^n_{(p)}$ has an H-space exponent for all large enough l. When n and p are odd, this is due to Cohen, Moore, Neisendorfer (see [3], Corollary 1.6), and the case n odd, $p = 2$ is due to Cohen (see [4], Corollary 5.2). This implies the case of even n: for $p = 2$ using the 2-local EHP sequence of James (see [13] and [14]), and for $p > 2$ using Serre's p-local (homotopy) fibrations $S^{2k-1}_{(p)} \to S^{4k-1}_{(p)} \to S^{2k}_{(p)}$ (see [27]).

Let W denote the localization at p of the n-connected cover of S^n. It was shown by Neisendorfer and Selick [24] that $\Omega^{n-3} W$ does not have an H-space exponent. However, $\Omega^{n-2} W$ does, see Neisendorfer [23].

In particular, the space $\mathrm{map}_*(S^{n-4}[p^{-1}], W)$ does not have the type of a CW complex, and the space $\mathrm{map}_*(S^{n-2}[p^{-1}], W)$ does. □

As a consequence we infer

Proposition 4.5. *Let Y be an elliptic simply connected finite complex. For all large enough m and almost all primes p the mapping space $\mathrm{map}_*(S^m[p^{-1}], Y)$ has the homotopy type of a CW complex.*

Proof. By McGibbon and Wilkerson [20] the loop space ΩY is p-equivalent to a finite product of spheres and loop spaces of spheres, for almost all primes p. Our assertion now follows from Theorem 4.4 and the previous example. □

If P and Q are sets of primes with $Q \subset P$, then a natural question is whether $\mathrm{map}_*(S^m[Q^{-1}], Y)$ has CW homotopy type if $\mathrm{map}_*(S^m[P^{-1}], Y)$ has. The remaining part of this section will be devoted to showing that this is indeed the case. To prove this, we will develop a characterization of CW homotopy type of $\mathrm{map}_*(S^m[P^{-1}], Y)$ for an arbitrary CW complex Y.

Suppose that $m \geqslant 2$. Then the cofibration $S^m \to S^m[P^{-1}]$ is principal, i.e., it can be obtained (up to homotopy) as the mapping cone of a map $\varphi\colon M \to S^m$. We get an induced pullback diagram:

$$\begin{array}{ccc}
\mathrm{map}_*(S^m[P^{-1}], Y) & \longrightarrow & \mathrm{map}_*(CM, Y) \\
\big\downarrow & & \big\downarrow \\
\mathrm{map}_*(S^m, Y) & \xrightarrow{\ \varphi^*\ } & \mathrm{map}_*(M, Y)
\end{array}$$

Here CM denotes the cone of M. By Lemma 2.5, the space $\mathrm{map}_*(CM, Y)$ is contractible. Hence $\mathrm{map}_*(S^m[P^{-1}], Y) \to \mathrm{map}_*(S^m, Y)$ is a principal fibration whose fibers are either empty or homotopy equivalent to the fiber of $\mathrm{map}_*(CM, Y) \to \mathrm{map}_*(M, Y)$ over the constant map. The latter is homeomorphic with $\mathrm{map}_*(\Sigma M, Y)$ by (i) of Lemma 2.3. By Theorem 2.1, therefore, $\mathrm{map}_*(S^m[P^{-1}], Y)$ has CW homotopy type if and only if $\mathrm{map}_*(\Sigma M, Y)$ has.

Let $M(A, m)$ denote the Moore complex with the single nonvanishing homology group A in dimension m. Note that $\Sigma M \simeq S^m[P^{-1}]/S^m$ is a space of type $M(\mathbb{Z}[P^{-1}]/\mathbb{Z}, m)$. Recall that the quotient $\mathbb{Z}[P^{-1}]/\mathbb{Z}$ is isomorphic with the direct sum of quasicyclic groups $\bigoplus_{p\in P} \mathbb{Z}_{p^\infty}$. The composite of homotopy equivalences

$$\Sigma M \to M(\mathbb{Z}[P^{-1}]/\mathbb{Z}, m) \to M(\bigoplus_{p\in P} \mathbb{Z}_{p^\infty}, m) \to \bigvee_{p\in P} M(\mathbb{Z}_{p^\infty}, m)$$

induces a homotopy equivalence $\prod_{p\in P} \mathrm{map}_*\big(M(\mathbb{Z}_{p^\infty}, m), Y\big) \to \mathrm{map}_*(\Sigma M, Y)$. Here it was used that for any family of CW complexes $\{X_\lambda \mid \lambda\}$, the mapping space $\mathrm{map}_*(\bigvee_\lambda X_\lambda, Y)$ is homeomorphic with the cartesian product $\prod_\lambda \mathrm{map}_*(X_\lambda, Y)$. (The evident function $\mathrm{map}_*(\bigvee_\lambda X_\lambda, Y) \to \prod_\lambda \mathrm{map}_*(X_\lambda, Y)$ is bijective by virtue of the weak topology, and open by virtue of closure finiteness of the wedge $\bigvee_\lambda X_\lambda$.)

Proposition 4.6. *Let Y be a CW complex, let P be a set of primes and $m \geqslant 2$. The following are equivalent.*
(i) $\mathrm{map}_*(S^m[P^{-1}], Y)$ *has CW homotopy type.*
(ii) $\prod_{p \in P} \mathrm{map}_*\big(M(\mathbb{Z}_{p^\infty}, m), Y\big)$ *has CW homotopy type.*
(iii) *For each $p \in P$, the space* $\mathrm{map}_*\big(M(\mathbb{Z}_{p^\infty}, m), Y\big)$ *has CW homotopy type, and all but finitely many are contractible.*

Proof. The equivalence of (i) and (ii) has been proved above.

To show that (iii) implies (ii), note first that the product of arbitrarily many contractible spaces is contractible. Now the implication follows from the fact that the product of finitely many spaces of CW homotopy type has itself CW homotopy type (see [22], Proposition 3).

To show that (ii) implies (iii), recall first that a space that is homotopy dominated by a CW complex has itself the homotopy type of a CW complex (see Milnor [22], Theorem 2). Assuming (ii) it follows that $\mathrm{map}_*\big(M(\mathbb{Z}_{p^\infty}, m), Y\big)$ has CW homotopy type for each $p \in P$. Now it is a consequence of Theorem 3.3 that all but finitely many are actually contractible. □

Corollary 4.7. *If P and Q are sets of primes with $Q \subset P$ then* $\mathrm{map}_*(S^m[Q^{-1}], Y)$ *has CW homotopy type if* $\mathrm{map}_*(S^m[P^{-1}], Y)$ *has.* □

Corollary 4.8. *Let Y be a simply connected finite complex, P a set of primes, and $m \geqslant 2$. Then* $\mathrm{map}_*(S^m[P^{-1}], Y)$ *has CW homotopy type if and only if* $\mathrm{map}_*(S^m[p^{-1}], Y)$ *has CW type for each $p \in P$, and $\tilde{H}_*(Y; \mathbb{Z}_p) \neq 0$ for at most finitely many $p \in P$.*

Proof. Use Proposition 4.6 in conjunction with Theorem 4.4. (Observe that if $\tilde{H}_*(Y; \mathbb{Z}_p) = 0$ then the homotopy groups of $\mathrm{map}_*\big(M(\mathbb{Z}_{p^\infty}, m), Y\big)$ vanish.) □

5 Spaces of maps out of Eilenberg-MacLane complexes

Proposition 5.1. *Let $E \to B$ be a fibration where B has the homotopy type of a connected countable CW complex, and E is compactly generated. Let F be a fiber and let X be any space. If $X \to \mathrm{map}(F, X)$ is a homotopy equivalence, then so is $\mathrm{map}(B, X) \to \mathrm{map}(E, X)$.*

The proposition analogous to 5.1 that assumes *weak* contractibility, and infers *weak* homotopy equivalence is known as 'the Zabrodsky lemma' (see Miller [21], §9 or McGibbon [18], Lemma 5.5). Proposition 5.1 assumes genuine contractibility and infers genuine homotopy equivalence and could perhaps be named the 'genuine Zabrodsky lemma.'

Remark 5.2. First note that the hypotheses of Proposition 5.1 imply that F is compactly generated. Let y_0 be a base-point of F, and let $\varepsilon_0\colon \mathrm{map}(F, X) \to X$ denote the evaluation $\varepsilon_0 : f \mapsto f(y_0)$. As ε_0 is a (pointwise) left inverse for the section $X \to \mathrm{map}(F, X)$, one of the two is a homotopy equivalence if and only if the other is. If y_0 is a nondegenerate base-point in F, then ε_0 is a fibration. In this case by Theorem 6.3 of Dold [6] for any connected space X of CW type:

(i) $\mathrm{map}_*(F,X)$ is contractible if and only if ε_0 is a homotopy equivalence.
(ii) $\mathrm{map}(B,X) \to \mathrm{map}(E,X)$ is an equivalence if and only if $\mathrm{map}_*(B,X) \to \mathrm{map}_*(E,X)$ is.

In 'most' cases we may assume that F has a nondegenerate base-point. Notably, if $(Y,y_0) \xrightarrow{g} (B,b_0)$ is a map of well-pointed spaces, then the base-point $(y_0, \mathrm{const}_{b_0})$ is nondegenerate in the homotopy fiber of g (viewed as subspace of $Y \times \mathrm{map}(I,B)$).

Our proof of Proposition 5.1 is essentially that of Lemma 1.5 of Zabrodsky [36] modulo some care about genuine homotopy type.

Proof of Proposition 5.1. Consider first a projection $\mathrm{pr}\colon T \times F \to T$ where T is compact. As F is compactly generated, $\mathrm{map}(T,X) \to \mathrm{map}(T \times F, X)$ is equivalent to $\mathrm{map}(T,X) \to \mathrm{map}\big(T, \mathrm{map}(F,X)\big)$ by Lemma A.1. The latter is a homotopy equivalence by Lemma 2.5. We want to exploit this in conjunction with the fact that every fibration over a contractible base space is trivial, i.e., equivalent to a projection.

To this end, we replace B with a homotopy equivalent locally finite countable simplicial complex B'. (See Fritsch and Piccinini [10], Theorem 5.2.3.) By pulling back over a homotopy equivalence $B' \to B$ and appealing to Corollaries 1.4 and 1.5 of Brown, Heath [2] we get a homotopy equivalent fibration $E' \to B'$. As B' is locally compact and E is compactly generated, the product $B' \times E$ is also compactly generated (see Steenrod [31], 4.3), and hence so is its closed subset E'. By virtue of Lemma 2.5 we may assume that $B' = B$ and $E' = E$.

Pick a filtration $B_0 \leqslant B_1 \leqslant B_2 \leqslant \cdots$ for B where B_0 is a vertex and B_i is obtained from B_{i-1} by adjunction of a simplex. Set $E_i = E|_{B_i}$ where $E|_K$ denotes $p^{-1}(K)$. The map $p^*\colon \mathrm{map}(B,X) \to \mathrm{map}(E,X)$ can be identified with the limit of maps $p_i^*\colon \mathrm{map}(B_i,X) \to \mathrm{map}(E_i,X)$. (Recall the paragraph preceding subsection 2.2 on page 436.) In light of Proposition 3.6 it is therefore enough to prove the following. Suppose L and L' are subcomplexes of B such that $L' = L \cup \sigma$ where σ is a simplex (possibly also a vertex) with $\partial\sigma \subset L$. If $\mathrm{map}(L,X) \to \mathrm{map}(E|_L,X)$ is a homotopy equivalence, then so is $\mathrm{map}(L',X) \to \mathrm{map}(E|_{L'},X)$.

Note $E|_{L'} = E|_L \cup E|_\sigma$. Since σ is contractible, there exists a fiber homotopy equivalence $\sigma \times F \to E|_\sigma$ (over σ). In particular this equivalence restricts to a (fiber) homotopy equivalence $\partial\sigma \times F \to E|_{\partial\sigma}$.

Thus there exists the following commutative diagram.

$$\begin{tikzcd}
& \mathrm{map}(E|_{L'},X) \arrow[rr] \arrow[dd] & & \mathrm{map}(E|_\sigma,X) \arrow[r, "\simeq"] \arrow[dd] & \mathrm{map}(\sigma \times F,X) \arrow[dd] \\
\mathrm{map}(L',X) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{map}(\sigma,X) \arrow[ur] \arrow[dd] & & \\
& \mathrm{map}(E|_L,X) \arrow[rr] & & \mathrm{map}(E|_{\partial\sigma},X) \arrow[r, "\simeq"] & \mathrm{map}(\partial\sigma \times F,X) \\
\mathrm{map}(L,X) \arrow[ur] \arrow[rr] & & \mathrm{map}(\partial\sigma,X) \arrow[ur] & &
\end{tikzcd}$$

Since $\sigma \times F \to E|_\sigma$ is a map over σ, the composite $\operatorname{map}(\sigma,X) \to \operatorname{map}(E|_\sigma,X) \to \operatorname{map}(\sigma \times F,X)$ is induced by the projection $\sigma \times F \to \sigma$. Similarly for the composite $\operatorname{map}(\partial\sigma,X) \to \operatorname{map}(E|_{\partial\sigma},X) \to \operatorname{map}(\partial\sigma \times F,X)$.

Since the projection induced maps $\operatorname{map}(\sigma,X) \to \operatorname{map}(\sigma \times F,X)$, respectively $\operatorname{map}(\partial\sigma,X) \to \operatorname{map}(\partial\sigma \times F,X)$, are homotopy equivalences, so are the maps $\operatorname{map}(\sigma,X) \to \operatorname{map}(E|_\sigma,X)$, respectively $\operatorname{map}(\partial\sigma,X) \to \operatorname{map}(E|_{\partial\sigma},X)$. By assumption, $\operatorname{map}(L,X) \to \operatorname{map}(E|_L,X)$ is a homotopy equivalence. The cube in the diagram is a morphism of pullback diagrams with vertical arrows fibrations, so by coglueing homotopy equivalences [2] also the map (of pullback spaces) $\operatorname{map}(L',X) \to \operatorname{map}(E|_{L'},X)$ is a homotopy equivalence. □

It is not difficult to show that in Proposition 5.3, it suffices for E to be homotopy equivalent to a compactly generated space. However, the targeted result here is Corollary 5.3 (which is a special case). We prove it by a more direct argument.

Corollary 5.3. *Let $f : A \to B$ be a map of pointed countable CW complexes where B is connected. Let F be the homotopy fiber of f with its natural base-point. If X is a pointed space such that* $\operatorname{map}_*(F,X)$ *is contractible, then* $f^* : \operatorname{map}_*(B,X) \to \operatorname{map}_*(A,X)$ *is a homotopy equivalence.*

Proof. The map f is equivalent to a map $f' : A' \to B'$ of locally finite simplicial complexes. (Use [10], Theorem 5.2.3.) The homotopy fibers of f and f' are homotopy equivalent (as pointed spaces). By Lemma 2.5 we may assume $A = A'$ and $B = B'$. Let $A \xrightarrow{g} E \xrightarrow{p} B$ be the canonical factorization of f where g is a homotopy equivalence and p is a fibration. By definition, F is a fiber of p and E is a subspace of $A \times \operatorname{map}(I,B)$. Note that A and B are metrizable. Next, the space of free paths $\operatorname{map}(I,B)$ is metrizable by virtue of the supremum metric. Thus E is metrizable and hence compactly generated. Now apply Proposition 5.1 and Remark 5.2. □

Corollary 5.4. *Let X be a connected countable CW complex and let Y be any CW complex. If* $\operatorname{map}_*(\Omega X,Y)$ *is contractible, so is* $\operatorname{map}_*(X,Y)$. □

Let $X \to X_{(P)}$ be localization at the set of primes P. If Y is a P-local complex, then the natural map $\operatorname{map}(X_{(P)},Y) \to \operatorname{map}(X,Y)$ is a weak homotopy equivalence. This is a basic fact in the theory of homotopy localization. It is not so obvious, however, that the map is actually a genuine homotopy equivalence. This has been proved in Lemma 4.1 for X a sphere which we use to tackle the general case:

Theorem 5.5. *Let P be a set of primes and let X be a simply connected countable CW complex. Let Y be a P-local complex. The mapping* $\operatorname{map}(X_{(P)},Y) \to \operatorname{map}(X,Y)$ *induced by localization $X \to X_{(P)}$ is a genuine homotopy equivalence.*

Remark 5.6. We emphasize here that in Theorem 5.5, the complex X is not assumed to be finite, and that the space $\operatorname{map}(X,Y)$ need not have CW homotopy type.

Proof. Possibly replacing X with a homotopy equivalent CW complex we can assume that X has trivial 1-skeleton and, for simplicity, that all characteristic maps are based. Then

X has a based filtration $L_0 \leqslant L_1 \leqslant L_2 \leqslant \cdots$ where L_0 is a point and $L_{i-1} \to L_i$ is the adjunction of a single cell of some dimension not smaller than 2. Then $X_{(P)}$ may be obtained by successive adjunctions of 'local cells' (see Sullivan [34], proof of Theorem 2.2) which we now describe. Set $L'_0 = L_0$. Assume that we have already constructed a CW complex inclusion $\lambda_{i-1}\colon L_{i-1} \leqslant L'_{i-1}$ which localizes homology and induces a homotopy equivalence $\mathrm{map}(L'_{i-1}, Y) \to \mathrm{map}(L_{i-1}, Y)$. The complex L_i is the mapping cone of the characteristic map $\varphi_i\colon S^m \to L_{i-1}$ of the corresponding cell. As L'_{i-1} is P-local there exists a map (unique up to homotopy) $\varphi'_i\colon S^m_{(P)} \to L'_{i-1}$ making the following square commutative up to homotopy.

$$
\begin{tikzcd}
S^m \arrow[r, "\varphi_i"] \arrow[d] & L_{i-1} \arrow[d, "\lambda_{i-1}"] \\
S^m_{(P)} \arrow[r, "\varphi'_i", "\psi'_i"'] & L'_{i-1}
\end{tikzcd}
\tag{5}
$$

By cellular approximation, φ'_i may be assumed to map into the m-skeleton ${L'_{i-1}}^{(m)}$. Also $\lambda_{i-1} \circ \varphi_i$ maps into ${L'_{i-1}}^{(m)}$, and since $S^m \to S^m_{(P)}$ is a cofibration, the homotopy extension property yields $\psi'_i\colon S^m_{(P)} \to {L'_{i-1}}^{(m)}$ (homotopic to φ'_i) making the diagram strictly commutative. Let L'_i be the mapping cone of $\psi'_i\colon S^m_{(P)} \to L'_{i-1}$ and let $\lambda_i\colon L_i \to L'_i$ denote the induced subcomplex inclusion. A standard application of the five lemma shows that λ_i localizes homology. On mapping spaces, we get an induced map of pullback diagrams (the front and the back square):

$$
\begin{tikzcd}
 & \mathrm{map}(L_i, Y) \arrow[rr] \arrow[dd] & & \mathrm{map}(CS^m, Y) \arrow[dd] \\
\mathrm{map}(L'_i, X) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{map}(CS^m_{(P)}, Y) \arrow[ur] \arrow[dd] & \\
 & \mathrm{map}(L_{i-1}, Y) \arrow[rr] & & \mathrm{map}(S^m, Y) \\
\mathrm{map}(L'_{i-1}, Y) \arrow[ur] \arrow[rr] & & \mathrm{map}(S^m_{(P)}, Y) \arrow[ur] &
\end{tikzcd}
$$

By Lemma 4.1, the mapping $\mathrm{map}(S^m_{(P)}, Y) \to \mathrm{map}(S^m, Y)$ is a homotopy equivalence. The cones $CS^m_{(P)}$ and CS^m are contractible, so by Lemma 2.5, the mapping $\mathrm{map}(CS^m_{(P)}, Y) \to \mathrm{map}(CS^m, Y)$ is equivalent to the identity $Y \to Y$ and as such is a homotopy equivalence. By inductive hypothesis, also $\mathrm{map}(L'_{i-1}, Y) \to \mathrm{map}(L_{i-1}, Y)$ is a homotopy equivalence. All four vertical arrows are fibrations, hence by coglueing homotopy equivalences [2], also $\mathrm{map}(L'_i, Y) \to \mathrm{map}(L_i, Y)$ is a homotopy equivalence. Letting X' denote the union (=colimit) of the complexes L'_i, we see that the colimit map $\lambda\colon X \to X'$ localizes homology which renders X' a valid complex of type $X_{(P)}$. Finally, the mapping $\mathrm{map}(X', Y) \to \mathrm{map}(X, Y)$ is the inverse limit of mappings $\mathrm{map}(L'_i, Y) \to \mathrm{map}(L_i, Y)$. As such, it is a homotopy equivalence by Proposition 3.6. $\square$

Proposition 5.7. *Let p be a prime and let Y be any simply connected CW complex. The following are equivalent.*
(i) *The space* $\mathrm{map}_*(S^m[p^{-1}], Y_{(p)})$ *is contractible for all big enough m.*
(ii) *The space* $\mathrm{map}_*\big(K(\mathbb{Z}[p^{-1}], m), Y_{(p)}\big)$ *is contractible for all big enough m.*

Proof. First note that by Corollary 5.4, also (ii) is an 'eventual property'. That is, if $\mathrm{map}_*\big(K(\mathbb{Z}[p^{-1}], m), Y_{(p)}\big)$ is contractible, so is $\mathrm{map}_*\big(K(\mathbb{Z}[p^{-1}], m+1), Y_{(p)}\big)$. Fixing a large odd integer m we obtain the following sequence of homotopy equivalences.

$$
\begin{aligned}
(6)\qquad \mathrm{map}_*(S^m[p^{-1}], Y_{(p)}) &\simeq \mathrm{map}_*(M(\mathbb{Q}, m), Y_{(p)})\\
&= \mathrm{map}_*(K(\mathbb{Q}, m), Y_{(p)}) \simeq \mathrm{map}_*(K(\mathbb{Z}[p^{-1}], m), Y_{(p)})
\end{aligned}
$$

The first and the last follow from Theorem 5.5 and Remark 5.2. The middle 'equality' holds because m is odd. ☐

Remark. Proposition 5.7 shows that (ii) and (iv) of Theorem 1.1 are equivalent. As the equivalence of (i) – (iii) is contained in Theorem 4.4, this completes the proof of Theorem 1.1.

The short exact sequence $\mathbb{Z} \to \mathbb{Z}[p^{-1}] \to \mathbb{Z}_{p^\infty}$ induces a homotopy fibration

$$
(7)\qquad K(\mathbb{Z}[p^{-1}], m) \to K(\mathbb{Z}_{p^\infty}, m) \xrightarrow{\omega} K(\mathbb{Z}, m+1).
$$

Corollary 5.8. *Let Y be a simply connected finite complex and let p be a prime. If $Y_{(p)}$ has an eventual geometric exponent then the mapping $\omega^*\colon$* $\mathrm{map}_*(K(\mathbb{Z}, m+1), Y_{(p)}) \to \mathrm{map}_*(K(\mathbb{Z}_{p^\infty}, m), Y_{(p)})$ *is a homotopy equivalence for all big enough m.*

Proof. This is an immediate consequence of Theorem 1.1 and Corollary 5.3. ☐

If Y is elliptic, Miller's theorem on maps out of classifying spaces implies that $\mathrm{map}_*(K(\mathbb{Z}_{p^\infty}, m), Y_{(p)})$ is weakly contractible, and, if Y has an eventual geometric exponent at p then by Corollary 5.8, $\mathrm{map}_*(K(\mathbb{Z}_{p^\infty}, m), Y_{(p)})$ is genuinely contractible for all big enough m if and only if $\mathrm{map}_*(K(\mathbb{Z}, m), Y_{(p)})$ is. The author of this paper does not even know if this is true in case Y is a sphere. However, speculation about this is tempting because of the following lemma.

Lemma 5.9. *If for some m, the space* $\mathrm{map}_*(K(\mathbb{Z}, m), Y_{(p)})$ *is contractible, so also is* $\mathrm{map}_*(S^m[p^{-1}], Y_{(p)})$*. Consequently Y has an eventual geometric exponent at p.*

Proof. View $K(\mathbb{Z}[p^{-1}], m)$ as the homotopy colimit of the sequence $K(\mathbb{Z}, m) \xrightarrow{p} K(\mathbb{Z}, m) \xrightarrow{p} K(\mathbb{Z}, m) \xrightarrow{p} \cdots$. Apply $\mathrm{map}(\_, Y_{(p)})$ to the stages of the corresponding telescope and use Corollary 3.5 to infer that $\mathrm{map}_*(K(\mathbb{Z}[p^{-1}], m), Y_{(p)})$ is contractible if $\mathrm{map}_*(K(\mathbb{Z}, m), Y_{(p)})$ is. Proposition 5.7 finishes the proof. ☐

Remark 5.10. If R is an infinite set of primes and $H_k(Y)$ is infinite for some k, certainly $\mathrm{map}_*(K(\mathbb{Z}, m), Y_{(R)})$ is not contractible for any m. Namely, as in the proof of the preceding lemma, it would follow that $\operatorname{map}_*(M(\mathbb{Q}, m), Y_{(R)})$ is contractible for big enough m, contradicting (i) of Theorem 4.4.

6 Phantom phenomena

Let X and Y be CW complexes. Say that maps $f, g\colon X \to Y$ are equivalent, $f \sim g$, if the restrictions $f|_K\colon K \to Y$ and $g|_K\colon K \to Y$ are homotopic for every finite subcomplex K of X. This is clearly an equivalence relation and each equivalence class is the union of a number of path components. If $f \sim g$ while f and g belong to different path components, f and g are called a (nontrivial) *phantom pair*. The reader is referred to the survey article of McGibbon [18] for more information on phantom maps.

Clearly we could introduce the relation on path components of $\operatorname{map}(X, Y)$: path components C and D are equivalent if their images – also path components – under restrictions to $\operatorname{map}(K, Y)$ coincide. Here K has to range over some cofinal family of finite subcomplexes of X. In case X is countable, therefore, K can range over $\{L_i\}$ where $L_1 \leqslant L_2 \leqslant \cdots$ is an ascending filtration of finite subcomplexes for X.

Denote as usual $Z_i = \operatorname{map}(L_i, Y)$ and $Z = \operatorname{map}(X, Y)$, and let C_i be the image of the path component C under $Z \to Z_i$. Then the equivalence class of the path component C is precisely the preimage of $\{C_i\}$ under the natural map $\pi_0(Z) \to \lim_i \pi_0(Z_i)$. For each i, let ζ_i be the image of $\zeta \in C$ in Z_i. By virtue of the short exact sequence (2) we can identify the equivalence class of C with $\lim^1_i \pi_1(Z_i, \zeta_i) = \lim^1_i \pi_1(C_i, \zeta_i)$. We call the equivalence class of C the *phantom class* of C, and denote it by $\operatorname{Ph}(C)$. Then C is called a *phantom component* if $\operatorname{Ph}(C) \equiv \lim^1_i \pi_1(C_i, \zeta_i)$ is nontrivial.

Hence, the space $Z = \operatorname{map}(X, Y)$ is devoid of phantom maps if and only if the $\lim^1_i \pi_1(Z_i, \zeta_i)$ vanish for all possible choices of $\zeta = \{\zeta_i\} \in Z$. If the homotopy groups of Y (and hence of Z_i) are countable this is if and only if the sequences $\{\pi_1(Z_i, \zeta_i) \mid i\}$ satisfy the Mittag-Leffler condition for all possible choices of ζ, by Lemma 3.2.

Switching now to an abstract inverse sequence of fibrations $\{Z_i\}$ with limit space Z, we can still talk about phantom classes and phantom path components of Z (with respect to the inverse sequence). Assume that the Z_i have CW homotopy type. Then if Z also has CW homotopy type, all phantom classes are trivial and Z has no phantom components by (ii) of Corollary 3.4. More than that, the $\lim^1_i \pi_k(Z_i, \zeta_i)$ are trivial for all $k \geqslant 1$.

This has a geometric interpretation when $Z = \operatorname{map}_*(X, Y)$ is the space of pointed maps. Taking $\zeta = \text{const}$, the vanishing of $\lim^1_i \pi_k(Z_i, \text{const})$ is equivalent to nonexistence of (pointed) phantom maps $\Sigma^{k-1}X \to Y$ or, equivalently, $X \to \Omega^{k-1}Y$. One could say that in this case Z is *stably* free of phantom maps.

However, note that (ii) of Corollary 3.4 is saying even more. Namely, the sequences $\{\pi_k(Z_i, \zeta_i) \mid i\}$ satisfy the Mittag-Leffler condition *uniformly* in k, i.e. for each i_0 there is $i > i_0$ such that the image of $\pi_k(Z_i, \zeta_i) \to \pi_k(Z_{i_0}, \zeta_{i_0})$ is stable for each $k \geqslant 1$.

Consider the space $Z = \operatorname{map}_*\big(M(\mathbb{Q}, m), S^n\big)$ from Example 1 on page 444. Assume that $m > n$. By our standard representation, Z is the limit of a certain sequence $\{Z_i\}$ where for each i, the space $Z_i \simeq \Omega^m S^n$ has finite homotopy groups. Hence, all sequences $\pi_k(Z_i, \zeta_i)$ satisfy the Mittag-Leffler condition. (In fact Z is weakly contractible.) However, Serre's theorem on torsion in homotopy groups of spheres as applied in the proof of Theorem 4.4

shows that the sequences $\pi_k(Z_i, \zeta_i)$ cannot satisfy the Mittag-Leffler condition uniformly with respect to k.

As a contrast, consider the tower of Eilenberg-MacLane spaces $Z_i = K(\mathbb{Z}_{p^i}, 1)$:

$$\cdots \to K(\mathbb{Z}_{p^3}, 1) \to K(\mathbb{Z}_{p^2}, 1) \to K(\mathbb{Z}_p, 1)$$

where $K(\mathbb{Z}_{p^{i+1}}, 1) \to K(\mathbb{Z}_{p^i}, 1)$ is induced by the mod p^i morphism $\mathbb{Z}_{p^{i+1}} \to \mathbb{Z}_{p^i}$. Clearly, the sequences in question satisfy the uniform Mittag-Leffler condition. However, this sequence violates (i) of Corollary 3.4 hence the limit space Z does not have CW homotopy type. Explicitly, Z is weakly equivalent to $(S^1)^\wedge_p = K(\mathbb{Z}^\wedge_p, 1)$ but not genuinely homotopy equivalent.

Definition. Let X and Y be pointed countable CW complexes, and let $* = L_1 \leqslant L_2 \leqslant \cdots$ be a pointed filtration of finite subcomplexes for X.

(i) The space $\mathrm{map}_*(X, Y)$ is *stably phantomless* if for each map ζ the sequence

$$\left\{\pi_k\left(\mathrm{map}_*(L_i, Y), \zeta_i\right) \mid i\right\} \tag{8}$$

satisfies the Mittag-Leffler condition for each $k \geqslant 1$.

(ii) The space $\mathrm{map}_*(X, Y)$ is *uniformly phantomless* if the sequences (8) satisfy the Mittag-Leffler condition uniformly with respect to $k \geqslant 1$.

By (ii) of Corollary 3.4, if $\mathrm{map}_*(X, Y)$ has CW homotopy type then it is uniformly phantomless. Clearly, if $\mathrm{map}_*(X, Y)$ is uniformly phantomless then it is stably phantomless. The examples exhibited above show that the converse is false in both cases. However, the geometric Moore conjecture can be restated as follows.

Conjecture 6.1. *Let Y be a simply connected finite complex and let p be a prime. The following are equivalent.*

(i) *The space* $\mathrm{map}_*(S^m[p^{-1}], Y)$ *has CW homotopy type for some m.*
(ii) *The space* $\mathrm{map}_*(S^m[p^{-1}], Y)$ *is uniformly phantomless for some m.*
(iii) *The space* $\mathrm{map}_*(S^m[p^{-1}], Y)$ *is stably phantomless for some m.*

An inspection upon the proof of Theorem 4.4 shows that (iii) of 6.1 is equivalent to ellipticity of Y, and that (ii) holds if and only if Y is elliptic and has a homotopy exponent at p. The details are left to the reader.

What is interesting here is the fact that in the setting of function spaces, (ii) seems to be a natural property that squeezes in between (i) and (iii), and combines both ellipticity and the existence of a homotopy exponent.

A Auxiliary results

The purpose of Appendix A is to supply proofs of a handful of results dealing with fibrations and the compact open topology. (Of the existing literature, the traditional leans on local compactness while the modern resorts to ‘convenient’ topological categories. We use neither.)

Lemmas A.2 and A.4 are most easily proved by employing the concept of a lifting function which we recall first.

Let $\mathrm{FP}\,E = \mathrm{map}(I, E)$ denote the space of free paths into E.

Definition. Let $p : E \to B$ be a map and let $\varepsilon_0 \colon \mathrm{FP}\,B \to B$ denote evaluation at 0. Let $\bar{E} = E\,{}_p\!\sqcap_{\varepsilon_0} \mathrm{FP}\,B = E \sqcap \mathrm{FP}\,B$ be the pullback of $E \xrightarrow{p} B \xleftarrow{\varepsilon_0} \mathrm{FP}\,B$. A lifting function for p is a map $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ that makes the following diagram commute.

(9)
$$\begin{tikzcd}
\bar{E} \arrow[rr] \arrow[dd] \arrow[dr, "\lambda"] & & \mathrm{FP}\,B \arrow[dd, "\varepsilon_0"] \\
& \mathrm{FP}\,E \arrow[ur, "p_*"'] \arrow[dl, "\varepsilon_0"'] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}$$

Note that since the square is a pullback, there is a natural map $\nu\colon \mathrm{FP}\,E \to \bar{E}$, and the universal property forces the pointwise identity $\nu\lambda = \mathrm{id}$. Recall that p has a lifting function if and only if it is a fibration. (See Fadell [8].)

Lemma A.1. *Let $X \times Y$ be compactly generated and let Z be a space. Then* $\mathrm{map}(X \times Y, Z)$ *and* $\mathrm{map}(X, \mathrm{map}(Y, Z))$ *are naturally homeomorphic.*

Proof. See Dugundji [7], Theorem XII.5.3. □

Lemma A.2. *Let X be a compactly generated space, and let L and A be two cofibered subspaces. Let (Y, M) be any pair. The restriction*

(10) $$\mathrm{map}\big((X, L), (Y, M)\big) \to \mathrm{map}\big((A, L \cap A), (Y, M)\big)$$

is a fibration with a lifting function that is functorial in (Y, M).

Proof. Set $E = \mathrm{map}\big((X, L), (Y, M)\big)$ and $B = \mathrm{map}\big((A, L \cap A), (Y, M)\big)$. The spaces $\mathrm{FP}\,E$ and $\mathrm{FP}\,B$ can be identified with $\mathrm{map}\big((X \times I, L \times I), (Y, M)\big)$ and $\mathrm{map}\big((A \times I, (L \cap A) \times I), (Y, M)\big)$, respectively, since X and L are compactly generated (see Lemma A.1). Evaluation $\varepsilon_0\colon \mathrm{FP}\,E \to E$ can be identified with restriction to $X \times 0 \equiv X$, and similarly for $\varepsilon_0\colon \mathrm{FP}\,B \to B$. The pullback $\bar{E} = E \sqcap \mathrm{FP}\,B$ can be identified with $\mathrm{map}\big((X \times \{0\} \cup A \times I), L \times \{0\} \cup (A \cap L) \times I, (Y, M)\big)$.

Since both L and A are cofibered in X, there exists a retraction

$$\rho\colon (X \times I, L \times I) \to (X \times \{0\} \cup A \times I, L \times \{0\} \cup (A \cap L) \times I).$$

Then $\lambda\colon \bar{E} \to \mathrm{FP}\,E$, $\lambda(f) = f \circ \rho$, is a lifting function for $E \to B$. If (Y', M') is another pair then set $E' = \mathrm{map}\big((X,L),(Y',M')\big)$ and similarly for B'. Clearly, a map of pairs $(Y,M) \to (Y',M')$ induces a morphism of restriction fibrations, and consequently also a morphism of lifting-function diagrams (9). $\square$

Lemma A.3. *Let X be any space. The functors* $\mathrm{map}(X,\_)$ *and* $\mathrm{map}_*(X,\_)$ *preserve topological pull-back squares. Precisely, if A is the pullback of $B \xrightarrow{\beta} D \xleftarrow{\gamma} C$, then* $\mathrm{map}(X,A)$ *is the pullback of* $\mathrm{map}(X,B) \xrightarrow{\beta_*} \mathrm{map}(X,D) \xleftarrow{\gamma_*} \mathrm{map}(X,C)$*. If the spaces and maps involved are pointed, then* $\mathrm{map}_*(X,A)$ *is the pullback of* $\mathrm{map}_*(X,B) \xrightarrow{\beta_*} \mathrm{map}_*(X,D) \xleftarrow{\gamma_*} \mathrm{map}_*(X,C)$*.*

Proof. The natural map $F\colon\ \mathrm{map}(X,B\times C) \to \mathrm{map}(X,B)\times\mathrm{map}(X,C)$ is a homeomorphism (see Maunder [16], Theorem 6.2.34), and it is trivial to check that $F(\mathrm{map}(X,B \sqcap C)) = \mathrm{map}(X,B) \sqcap \mathrm{map}(X,C)$. $\square$

Lemma A.4. *Let $p\colon E \to B$ be a fibration and let X be a compactly generated space. Then the induced map $p_*\colon\ \mathrm{map}(X,E) \to \mathrm{map}(X,B)$ is a fibration with a lifting function that is functorial in X.*

If, in addition, p is a pointed map of well-pointed spaces, then also the map $p_*\colon\ \mathrm{map}_*(X,E) \to \mathrm{map}_*(X,B)$ is a fibration, for any choice of base-point in X.

Proof. Let $\lambda\colon \bar{E} \to \mathrm{FP}\,E$ be a lifting function for p. We apply $\mathrm{map}(X,\_)$ to diagram (9) and denote $\Lambda = \mathrm{map}(X,\lambda)$. We use Lemma A.3 to view the domain of Λ as $\mathrm{map}(X,E) \sqcap \mathrm{map}(X,\mathrm{FP}\,B)$. As X is compactly generated, so is $X \times I$, and therefore $\mathrm{map}(X,\mathrm{FP}\,B)$ and $\mathrm{map}(X,\mathrm{FP}\,E)$ are homeomorphic to, respectively, $\mathrm{FP}\,\mathrm{map}(X,B)$ and $\mathrm{FP}\,\mathrm{map}(X,E)$ by Lemma A.1. Thus Λ is a lifting function for p_*, and it is clearly functorial in X.

If $p(e_0) = b_0$ and e_0, b_0 are nondegenerate, then $\bar{e} = (e_0, \mathrm{const}_{b_0})$ is nondegenerate in $E \sqcap \mathrm{FP}\,B$, and consequently λ may be chosen so that $\lambda(\bar{e}) = \mathrm{const}_{e_0}$. The function Λ is a lifting function for $p_*\colon\ \mathrm{map}_*(X,E) \to \mathrm{map}_*(X,B)$. $\square$

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