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Turning a self-map into a self-fibration

Jaka Smrekar [1]

Fakulteta za matematiko in fiziko, Jadranska ulica 19, SI-1111 Ljubljana, Slovenia

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ABSTRACT

Let $f : Z \to Z$ be a self-map on the topological space Z. Generalizing the well-known factorization of a map into the composite of a homotopy equivalence and a Hurewicz fibration we prove that f is homotopy equivalent to a self-fibration $g : W \to W$.

It is well-known that any continuous map $f : X \to Y$ between topological spaces can be factored as $f = \pi \circ \iota$ where $\iota : X \to W$ is an embedding as a strong deformation retract and π is a Hurewicz fibration, that is, a map with the homotopy lifting property for all spaces (see for example Spanier [5, Chapter 2, Section 8, Theorem 9]). In addition to being a very useful tool for classical homotopy theory, a refined version of this fact (see Remark 2 below) is a basic ingredient of a closed model category structure on the category of all topological spaces (see Strøm [7]).

Generalizing, we show that a self-map can similarly be 'replaced' with a self-fibration as follows.

Theorem 1. *Let $f : Z \to Z$ be a self-map. Then there exist a fibration $p : W \to Z$ which is a homotopy equivalence, a pointwise right inverse $s : Z \to W$, and a self-fibration $g : W \to W$ making the following diagram strictly commutative.*

$$\begin{array}{ccc} W & \xrightarrow{g} & W \\ {\scriptstyle s}\uparrow \simeq & & \simeq \uparrow{\scriptstyle s} \\ Z & \xrightarrow{f} & Z \end{array} \tag{1}$$

Furthermore, the composite $s \circ p$ is homotopic to the identity on W leaving Z fixed.

E-mail address: jaka.smrekar@fmf.uni-lj.si.

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Note that Theorem 1 is indeed a generalization of the classical factorization as it follows from $p \circ s = \mathrm{id}_Z$ that $f = p \circ g \circ s$ where $p \circ g$ is the composite of two fibrations and hence a fibration.

Remark 2. It is desirable (in fact requisite for a closed model category) that one can factor maps as $f = \pi\iota$ where π is a fibration and ι is a homotopy equivalence *and* a closed cofibration. This can be achieved for topological spaces and continuous maps by the following neat construction of Strøm (see [7, proof of Proposition 2]): if $Z \subset W$ is an embedding as a strong deformation retract, then $\iota : Z \equiv Z \times \{0\} \subset E := Z \times [0,1] \cup W \times (0,1]$ is a closed cofibration and the projection $q : E \to W$ is a fibration. Thus indeed, in Theorem 1, $f = (p \circ g \circ q) \circ \iota$ where $\iota : Z \to E$ is a homotopy equivalence and a closed cofibration, and $\pi = pgq$ is a fibration. However, for the problem of replacing a self-map with a self-fibration this is not wholly satisfactory as one would want the map s in diagram (1) to be a homotopy equivalence and a closed cofibration.

It would be interesting to know whether that can be achieved in general. We show that it can be achieved under additional hypotheses. We need a definition.

Definition 1. A topological space Z is called *strong Hausdorff* if the diagonal of $Z \times Z$ is a zero set, that is, there exists a continuous function $\nu : Z \times Z \to [0,1]$ such that $\nu(z,w) = 0 \Leftrightarrow z = w$.

Lemma 3.

(i) *The property of being strong Hausdorff is hereditary.*
(ii) *The property of being strong Hausdorff is countably multiplicative.*
(iii) *Let K be a compact space. If Z is strong Hausdorff then so is* $\mathrm{map}(K,Z)$.

For topological spaces X and Y, $\mathrm{map}(X,Y)$ denotes the space of continuous maps $X \to Y$ with the compact open topology.

Proof of Lemma 3. Hereditariness is tautological. Let Z_i, $i = 1,2,\ldots$ be strong Hausdorff by virtue of functions $\nu_i : Z_i \times Z_i \to [0,1]$. Then the function

$$\big(\{z_i\},\{w_i\}\big) \mapsto \sum_{i=1}^{\infty} \frac{1}{2^i}\nu_i(z_i,w_i)$$

shows that the product $\prod_{i=1}^{\infty} Z_i$ is strong Hausdorff.

Let Z be strong Hausdorff by virtue of ν. The function

$$\mathrm{map}(K,Z) \times \mathrm{map}(K,Z) \to [0,1], \quad (f,g) \mapsto \max_{k\in K} \nu\big(f(k),g(k)\big)$$

renders also $\mathrm{map}(K,Z)$ strong Hausdorff.

Addendum 4. *In Theorem 1, the following two implications hold.*

(i) *If Z has a nondegenerate (closed) base-point z_0 and $f(z_0) = z_0$, it can be arranged that in (1), $s(z_0) = w_0$ is nondegenerate and $g(w_0) = w_0$.*
(ii) *If Z is strong Hausdorff then so is W and $s : Z \to W$ is a closed cofibration.*

Proof of Theorem 1 and Addendum 4. Set $I = [0,1]$. Let Z_{n+1} be the subspace of

$$Z \times \mathrm{map}(I,Z) \times \mathrm{map}\big(I^2,Z\big) \times \cdots \times \mathrm{map}\big(I^n,Z\big)$$

consisting of those $(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(n)})$ that satisfy the relations

$$\zeta^{(i)}(s_1, \ldots, s_j, 0, t_1, t_2, \ldots, t_k) = f^{i-k}\zeta^{(k)}(t_1, t_2, \ldots, t_k) \qquad (R(i,j,k))$$

whenever $1 \leqslant i \leqslant n$, $j \geqslant 0$, $k \geqslant 0$, and $j + k = i - 1$. The symbol $\zeta^{(0)}()$ is to be interpreted as $\zeta^{(0)}$.

Alternatively, we can start by defining $Z_1 = Z$ and, given Z_n, we let Z_{n+1} be the set of those $(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(n)})$ for which $(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(n-1)}) \in Z_n$ and, in addition, the relations $R(n, j, k)$ for the possible j and k hold. This observation makes it possible to view Z_{n+1} as obtained by the following pull-back diagram:

$$\begin{array}{ccc} Z^{n+1} & \longrightarrow & \mathrm{map}(I^n, Z) \\ {\scriptstyle q_{n+1}}\downarrow & & \downarrow{\scriptstyle r} \\ Z_n & \xrightarrow{\phi_n} & \mathrm{map}(C_n, Z) \end{array} \qquad (2)$$

Here, $C_n = I^{n-1} \times \{0\} \cup I^{n-2} \times \{0\} \times I \cup \cdots \cup \{0\} \times I^{n-1}$, and r is the restriction map assigning each $\zeta^{(n)} \in \mathrm{map}(I^n, Z)$ its restriction to C_n. Note that $\mathrm{map}(C_n, Z)$ can be identified with a subspace of $\mathrm{map}(I^{n-1} \times \{0\}, Z) \times \mathrm{map}(I^{n-2} \times \{0\} \times I, Z) \times \cdots \times \mathrm{map}(\{0\} \times I^{n-1}, Z)$. By means of this identification, the map ϕ_n is given by

$$\phi_n\bigl(\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(n-1)}\bigr) = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1})$$

where $\varphi_j(s_1, \ldots, s_{n-j-1}, 0, t_1, \ldots, t_j) = f^{n-j}\zeta^{(j)}(t_1, \ldots, t_j)$. In other words, φ_j is the composite of f^{n-j}, $\zeta^{(j)}$, and the projection of I^n to the last $j-1$ coordinates. Thus it is clear that ϕ is continuous.

As r equals the composition with the cofibration $C_n \to I^n$, it is a Hurewicz fibration (see for example [5, Chapter 2, Section 8, Theorem 2]). Moreover, $C_n \to I^n$ is a homotopy equivalence, and hence so is r (see [4, Lemma 2.5]). Consequently, $q_{n+1} : Z_{n+1} \to Z_n$ is a fibration which is a homotopy equivalence. (For a proof of the latter one could use Theorem 1.1 of [1].)

Let Z_∞ denote the inverse limit space of the sequence $\{(Z_n, q_n)\}$. By Geoghegan [2], the canonical projection $Z_\infty \to Z_1 = Z$ is a homotopy equivalence. Note that by construction, Z_∞ can be viewed as the space of those $\{\zeta^{(i)}\} \in \prod_{i=0}^{\infty} \mathrm{map}(I^i, Z)$ (in the Cartesian product topology) which satisfy all possible relations $R(i, j, k)$. Here we understand $\mathrm{map}(I^0, Z) = Z$.

We define $g : Z_\infty \to Z_\infty$ by $\{\zeta^{(i)}\} \mapsto \{\omega^{(i)}\}$ where for each i,

$$\omega^{(i)}(t_1, \ldots, t_i) = \zeta^{(i+1)}(t_1, \ldots, t_i, 1).$$

Abusing notation we can write $g : \{\zeta^{(i)} \mid i \geqslant 0\} \mapsto \{\zeta^{(i+1)}|_{I^i \times \{1\}} \mid i \geqslant 0\}$.

We claim that g is a fibration. To this end, suppose given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z_\infty \\ {\scriptstyle a \mapsto (a,0)}\downarrow & & \downarrow{\scriptstyle g} \\ A \times I & \xrightarrow{h} & Z_\infty \end{array}$$

where $\alpha = \{\alpha_i : A \to \mathrm{map}(I^i, Z)\}$ and $h = \{h_i : A \times I \to \mathrm{map}(I^i, Z)\}$. We let $H_0 : A \times I \to \mathrm{map}(\{0\}, Z)$ equal $\alpha_0 \circ \mathrm{pr}_A$. For $i \geqslant 1$, we define $H_i : A \times I \to \mathrm{map}(I^i, Z)$ by

$$H_i(a, u)(t_1, \ldots, t_{i-1}, t_i) = \begin{cases} \alpha_i(a)(t_1, \ldots, t_{i-1}, \frac{2t_i}{2-u}), & 0 \leqslant 2t_i \leqslant 2 - u \leqslant 2, \\ h_{i-1}(a, 2t_i + u - 2)(t_1, \ldots, t_{i-1}), & 1 \leqslant 2 - u \leqslant 2t_i \leqslant 2. \end{cases}$$

By the characteristic property of an inverse limit, the maps H_i will define a homotopy $H : A \times I \to Z_\infty$ with the desired properties if the set-theoretic image of $\{H_i\}$ is contained in Z_∞. The verification is left to the reader.

Finally, note that the canonical projection $Z_\infty \to Z$, $\{\zeta^{(i)}\} \mapsto \zeta^{(0)}$, admits a section $s : Z \to Z_\infty$ given by $z \mapsto \{\mathrm{const}_{f^i(z)} \mid i \geqslant 0\}$. (Here, const_* denotes the constant map to $*$.) This means that automatically, s is a homotopy equivalence. Letting $W = Z_\infty$ we obtain diagram (1).

Let $H : W \times [0,1] \to W$ be the homotopy defined as $H(\{\zeta^{(i)}\}, u) = \{\omega^{(i)}\}$ where $\omega^{(0)} = \zeta^{(0)}$ and $\omega^{(i)}(\mathbf{t}) = \zeta^{(i)}((1-u)\cdot \mathbf{t})$ for $i \geqslant 1$. Then H begins in the identity and ends in $s \circ p$, and $H(s(z), u) = s(z)$, for all z and u.

For nondegenerate base-points note first that if the inclusion $\{z_0\} \hookrightarrow Z$ is a closed cofibration, so is $\{\mathrm{const}_{z_0}\} \hookrightarrow \mathrm{map}(K, Z)$ for any compact space K. (See Strøm [7, Lemma 4].) Since diagrams (2) are pull-back squares with vertical arrows fibrations, it follows by induction that for each n, the point $(z_0, \mathrm{const}_{z_0}, \ldots, \mathrm{const}_{z_0})$ is a (closed) nondegenerate base-point in Z_n. Set $w_0 = \{\mathrm{const}_{z_0} \mid i\} \in Z_\infty$.

Recall that $C \hookrightarrow Z$ is a closed cofibration if and only if there exist a neighborhood U of C which deforms in Z to C (rel C), and a function $\beta : Z \to I$ such that $C = \beta^{-1}(0)$ and $\beta|_{Z-U} \equiv 1$. (See Strøm [6, Theorem 2].) Let U and β correspond to $\{z_0\} \hookrightarrow Z$. Let $\tilde{U}$ be the preimage of U under $Z_\infty \to Z$. Arguing similarly as in the proof of Corollary 3.5 of [3] we obtain a homotopy $\tilde{U} \times I \to Z_\infty$ beginning in inclusion, ending in const_{w_0}, and leaving w_0 fixed. We can define $B : Z_\infty \to [0,1]$ as follows

$$B\big(\{\zeta^{(i)} \mid i\}\big) = \min\left\{1, \sum_{i=0}^{\infty} \frac{1}{2^i} \max_{\mathbf{t}\in I^i} \beta\big(\zeta^{(i)}(\mathbf{t})\big)\right\}.$$

Note that $B(\{\zeta^{(i)} \mid i\}) = 0$ if and only if $\zeta^{(i)} = \mathrm{const}_{z_0}$ for all i, and since $\{\zeta^{(i)} \mid i\} \in \tilde{U}$ if and only if $\zeta^{(0)} \in U$, it follows that $B|_{W-\tilde{U}} \equiv 1$. This proves (i).

To show (ii) we only have to show that $Z \equiv s(Z)$ is a zero-set in W since we already know that it is a strong deformation retract. By assumption, there is a map $\nu : Z \times Z \to [0,1]$ such that $\nu^{-1}(0)$ is precisely the diagonal. Then $N : W \to [0,1]$,

$$N\big(\{\zeta^{(i)} \mid i\}\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} \max_{\mathbf{t}\in I^i} \nu\big(f^i\big(\zeta^{(0)}\big), \zeta^{(i)}(\mathbf{t})\big)$$

has $N^{-1}(0) = s(Z)$ as claimed. Finally, W is strong Hausdorff by Lemma 3.

Problem 5. *Various parts of the above proof apparently depend on the special features of the topological category. It would be interesting to investigate additional conditions (if any) that a closed model category should satisfy in order to be able to replace a self-map with a fibration in the sense of Theorem 1.*

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